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THE CAMBRIDGE AND DUBLIN

MATHEMATICAL JOURNAL.

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The second member of equation (k), p. 206, should be 0.
The foot-note in p. 154, should be on (c).

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INDEX TO VOL. I.

GEOMETRY OF TWO DIMENSIONS.

GEOMETRY OF THREE DIMENSIONS.

ALGEBRA.

DIEFERENTIAL AND INTEGRAL CALCULUS.

3 8

[redacted]

MECHANICS.

ASTRONOMY.

OPTICS.

HEAT, ELECTRICITY, AND MAGNETISM.

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THE

CAMBRIDGE AND DUBLIN

MATHEMATICAL JOURNAL.

GENERAL THEOREMS ON MULTIPLE INTEGRALS.

By R. Leslie Ellis, M.A., Fellow of Trinity College, Cambridge.

In Liouville's theorem for the reduction of a certain class of definite multiple integrals, the integrations comprise all positive values of the variables which do not transgress a limiting inequality, which either is of, or may easily be reduced to, a linear form. Take for illustration the case of two variables, and let $mx + ny < h$ be the limiting inequality in question, m, n and h being positive. Then, geometrically, $mx + ny = h$ is the equation of a straight line which forms the base of a triangle of which the intercepts of the positive half axes of co-ordinates are the sides, and our integration extends over the whole surface of this triangle. A similar interpretation may of course be given in the case of three variables. But to return to that of two. Let $mx + ny = h$ cut the axis of x in the point M and that of y in the point N; conceive another straight line $m'x + n'y = h'$; m', n', h' being also all positive; and let it cut the axes in M', N' respectively. Let us suppose for distinctness that $\frac{m}{m'}$ is greater than $\frac{n}{n'}$. Then, if the value of $\frac{h}{h'}$ be intermediate between those of the two fractions $\frac{m}{m'}$ and $\frac{n}{n'}$, it will be easily seen that the two lines must intersect in some point A, lying in the positive quadrant of co-ordinates, and that we shall have a quadrilateral $OMAN'$, (O being the origin of co-ordinates,) formed by the axes and by the two bounding lines. If now we integrate any function of x and y for all positive values of the variables not transgressing the *two* inequalities $mx + ny \leqq h$, $m'x + n'y \leqq h'$,

we shall in effect integrate over the surface of the quadrilateral $OMAN'$. But if the two lines did not intersect within the positive quadrant, then one or other bounding inequality would be inoperative, and we should in effect integrate over the surface, not of a quadrilateral, but of a triangle, as in the case contemplated by Liouville's theorem. It is manifest that we may have, instead of two limiting inequalities, any larger number we please, and that our integrations may thus be made to extend over an irregular polygon of a greater or less number of sides. I do not believe that any writer on multiple integrals has considered the case in which the limits are given by more than one inequality, but the restriction to that of one is clearly unnecessary.

Let us suppose there are r variables $x, y \dots z$, and that we have to evaluate the integral

$$\int_0 dx \dots \int_0 dz\, e^{-ax-\dots-az}\, \phi(mx + \dots pz)\, \phi_{,}(m_{,}x + \dots p_{,}z) \dots \quad (1),$$

subject to the two inequalities

$$mx + \dots pz \leq h, \qquad m_{,}x + \dots p_{,}z \leq h_{,},$$

$m \dots p, h$; $m_{,} \dots p_{,}, h_{,}$ being all positive; and ϕ and $\phi_{,}$ any functions whose values may be represented within the limits of integration by Fourier's theorem.

Let the value of the integral in question be I; then, by considerations analogous to those of which I made use in a paper which appeared at the commencement of the last volume of the *Journal*, we shall have

$$I = \frac{1}{\pi^2}\int^{h} \phi u\, du \int^{h_{,}} \phi_{,}u_{,}\, du_{,} \int_0^\infty da \int_0^\infty da_{,}.G,$$

where

$$G = \int_0^\infty dx \dots \int_0^\infty dz\, e^{-ax-\dots-az} \cos a(mx + \dots pz - u) \cos a_{,}(m_{,}x + \dots p_{,}z - u_{,}),$$

and the lower limits of integration with respect to u and u' may be any negative quantities.

I remark, in the first place, that

$$\int_0^\infty da \int_0^\infty da_1\, G = \tfrac{1}{4}\int_{-\infty}^\infty da \int_{-\infty}^\infty da_{,}H, \quad \text{where}$$

$$H = \int_0^\infty dx \dots \int_0^\infty dz\, e^{-ax-\dots-az} \cos\{(am + a_{,}m_{,})x + \dots (ap + a_{,}p_{,})z - au - a_{,}u_{,}\},$$

and therefore

$$I = \frac{1}{4\pi^2}\int^{h} \phi u\, du \int^{h_{,}} \phi_{,}u_{,}\, du_{,} \int_{-\infty}^\infty da \int_{-\infty}^\infty da_{,}H.$$

Let $\qquad H = K \cos(au + a_{,}u_{,}) + L \sin(au + a_{,}u_{,})$.

Then it will be easily seen that

$$K = \frac{N}{D}, \quad L = \frac{N'}{D};$$

where, if we take the case of three variables,

$$N = abc\left(1 - \frac{am + a_,m_,}{a}\,\frac{an + a_,n_,}{b} - \frac{am + a_,m_,}{a}\,\frac{ap + a_,p_,}{c} - \frac{an + a_,n_,}{b}\,\frac{ap + a_,p_,}{c}\right),$$

$$N' = abc\left(\frac{am + a_,m_,}{a} + \frac{an + a_,n_,}{b} + \frac{ap + a_,p_,}{c} - \frac{am + a_,m_,}{a}\,\frac{an + a_,n_,}{b}\,\frac{ap + a_,p_,}{c}\right),$$

$$D = \{a^2 + (am + a_,m_,)^2\}\,\{b^2 + (an + a_,n_,)^2\}\,\{c^2 + (ap + a_,p_,)^2\}.$$

(Precisely the same law of formation of these quantities would obtain if we were to take any number of variables. I have taken the case of three merely for distinctness of representation.)

Putting for cos $(au + a_,u_,)$ and sin $(au + a_,u_,)$ their exponential values, we find that

$$HD = a\ldots c\left\{1 - \sqrt{(-1)}\,\frac{am + a_,m_,}{a}\right\}\ldots\left\{1 - \sqrt{(-1)}\,\frac{ap + a_,p_,}{c}\right\}e^{(au+a_,u_,)\sqrt{(-1)}}$$

$$+\, a\ldots c\left\{1 + \sqrt{(-1)}\,\frac{am + a_,m_,}{a}\right\}\ldots\left\{1 + \sqrt{(-1)}\,\frac{ap + a_,p_,}{c}\right\}e^{-(au+a_,u_,)\sqrt{(-1)}};$$

and as

$$a^2 + (am + a_,m_,)^2 = a^2\left\{1 - \sqrt{(-1)}\,\frac{am + a_,m_,}{a}\right\}\left\{1 + \sqrt{(-1)}\,\frac{am + a_,m_,}{a}\right\},$$

$$H = \frac{e^{(au+a_,u_,)\sqrt{(-1)}}}{\{a + \sqrt{(-1)}\,(am + a_,m_,)\}\ldots\{c + \sqrt{(-1)}\,(ap + a_,p_,)\}} + \frac{e^{-(au+a_,u_,)\sqrt{(-1)}}}{\{a - \sqrt{(-1)}\,(am + a_,m_,)\}\ldots\{c - \sqrt{(-1)}\,(ap + a_,p_,)\}}.$$

Now, assume that

$$\frac{1}{(a + am + a_,m_,)\ldots(c + ap + a_,p_,)}$$

$$= \frac{F_{ab}}{(a + am + a_,m_,)(b + an + a_,n_,)} + \frac{F_{ac}}{(a + am + a_,m_,)(c + ap + a_,p_,)} + \&c. \quad \ldots\ldots\ldots (2);$$

where F_{ab}, F_{ac}, &c. are independent of a and $a_,$. This assumption is justifiable because it introduces $\frac{r.r-1}{2}$ disposable quantities F, viz. as many as there are combinations two and

two of the r quantities $a, b \ldots . c$, and it will be easily seen that there are the same number of conditions to be satisfied.

Consequently as

$$e^{(au+a'u')\sqrt{(-1)}} = \cos(au + a'u') + \sqrt{(-1)} \sin(au + a'u'),$$

we shall have

$$\left.\begin{array}{l} H = F_{ab} \left\{ab - (am + a_{\prime}m_{\prime})(an + a_{\prime}n_{\prime})\right\} \cos(au + a_{\prime}u_{\prime}) \\ \qquad + \left\{a(an + a_{\prime}n_{\prime}) + b(am + a_{\prime}m_{\prime})\right\} \sin(an + a_{\prime}n_{\prime}) \\ \text{divided by} \\ \qquad \left\{a^2 + (am + a'm')^2\right\} \left\{b^2 + (an + a'n')^2\right\} \end{array}\right\} + \&c.$$

Let us next assume $u = mx + ny$, $u_{\prime} = m_{\prime}x + n_{\prime}y$, x and y being here two new variables; also $\alpha' = am + a_{\prime}m_{\prime}$, and $\beta' = an + a_{\prime}n_{\prime}$: then the coefficient of F_{ab} in the expression of H will become

$$\frac{(ab - \alpha'\beta')\cos(\alpha'x + \beta'y) + (a\beta' + b\alpha')\sin(\alpha'x + \beta'y)}{(a^2 + \alpha'^2)(b^2 + \beta'^2)}.$$

Moreover $du du_1 da da_1$ will be replaced by $dx dy d\alpha' d\beta'$; and therefore, as we have

$$I = \frac{1}{4\pi^2} \int^h \phi u du \int^k \phi_{\prime} u_{\prime} du_{\prime} \int_{-\infty}^{+\infty} da \int_{-\infty}^{+\infty} da_{\prime} H, \text{ we shall have}$$

$$I = \frac{1}{4\pi^2} \Sigma F_{ab} \iint \phi(mx + ny)\, \phi_{\prime}(m_{\prime}x + n_{\prime}y)\, dx dy M,$$

where the sign of summation extends to all the quantities F, and where

$$M = \int_{-\infty}^{+\infty} d\alpha' \int_{-\infty}^{+\infty} d\beta' \frac{(ab - \alpha'\beta')\cos(\alpha'x + \beta'y) + (a\beta' + b\alpha')\sin(\alpha'x + \beta'y)}{(a^2 + \alpha_{\prime}^2)(b^2 + \beta_{\prime}^2)}.$$

From the known integrals

$$\int_{-\infty}^{+\infty} \frac{\cos \alpha x . d\alpha}{a^2 + \alpha^2} = \frac{\pi}{a} e^{\mp ax}, \quad \int_{-\infty}^{+\infty} \frac{\alpha \sin \alpha x . d\alpha}{a^2 + \alpha^2} = \pm \pi e^{\mp ax},$$

the upper signs to be taken when x is positive, it follows that

$$M = \pi^2 e^{\mp ax \mp by} (1 \pm 1 \pm 1 \pm 1).$$

If x and y are both positive, the bracket becomes $1 + 1 + 1 + 1$ or 4; if x only be negative, it becomes $1 - 1 - 1 + 1$ or 0; if y only be negative, it becomes $1 - 1 + 1 - 1$ or 0; and similarly if both x and y are negative. Thus generally

$$M = 4\pi^2 e^{-ax-by} \text{ or } M = 0.$$

There are, indeed, exceptional cases; as if y be zero, x being positive, when $M = 2\pi^2 e^{-ax}$, and similarly if x be zero, y being

positive; and again, if x and y are both zero, when $M = \pi^2$: but of these, as we are about to multiply M by the element $dx\,dy$, it is unnecessary to take account. Therefore, in integrating for x and y, we include only positive values of the variables; and as u and u' are not to be greater than h and $h_,$ respectively, x and y must be such as not to transgress the inequalities $\quad mx + ny \leqq h, \quad m_,x + n_,y \leqq h_,$.

Thus we find that

$$I = \Sigma F_{ab} \int_0 dx \int_0 dy\, \phi(mx + ny)\, \phi_,(m_,x + n_,y)\, e^{-ax-by},$$

the limits being given by the two above-written inequalities. It appears, therefore, that the integral (1), when there are two limiting inequalities, is reducible to the sum of a series of double integrals.

This result is analogous to that which is obtained in the case of the function $\phi(mx + \dots pz)\, e^{-ax \cdots -cz}$, in the paper already referred to.

It remains to determine the form of the quantity F_{ab}. This is done at once by multiplying equation (2) by $(a + \alpha m + \alpha_, m_,)(b + \alpha n + \alpha_, n_,)$, and replacing α, $\alpha_,$ by values which make both these factors vanish. It hence appears that

$$F_{ab} = \frac{(mn_, - m_,n)^{r-2}}{\{c(mn_, - m_,n) + a(np_, - n_,p) + b(pm_, - p_,m)\} \dots},$$

the denominator being the continued product of $r - 2$ factors, each of the same form as the one written down. Of course the other quantities F are obtained in the same manner.

Let us now take the more general case in which there are s limiting inequalities, s being less than r, and in which the function to be integrated is

$$\phi_1(m_1x + \dots p_1z) \dots \phi_s(m_sx + \dots p_sz)\, e^{-ax \cdots -cz},$$

the inequalities in question being

$$m_1x + \dots p_1z \leqq h_1, \dots m_sx + \dots p_sz \leqq h_s.$$

We shall arrive at a perfectly analogous result in this more general case. In the first place the integral sought may be thus written,

$$\frac{1}{\pi^s} \int^{h_1} \phi_1 u_1 du_1 \dots \int^{h_s} \phi_s u_s du_s \int_0^\infty d\alpha_1 \dots \int_0^\infty d\alpha_s G, \text{ where}$$

$$G = \int_0^\infty dx \dots \int_0^\infty dz\, e^{-ax-\dots-cz} \cos\alpha_1(m_1x + \dots p_1z - u_1) \dots \cos\alpha_s(m_sx + \dots p_sz - u_s).$$

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Now a little consideration will convince us that

$$\int_0^\infty da_1 \ldots \int_0^\infty da_s G = \frac{1}{2^s}\int_{-\infty}^{+\infty} da_1 \ldots \int_{-\infty}^{+\infty} da_s H, \text{ where}$$

$$H = \int_0^\infty dx \ldots \int_0^\infty dz\, e^{-ax-\cdots-cz} \cos\left[x\,\Sigma am + \ldots + z\,\Sigma ap - \Sigma au\right]:$$

for if we take the expression

$$\cos\left[x\,\Sigma am + \ldots + z\,\Sigma ap - \Sigma au\right],$$

make a_1 negative, add the resulting expression to the original one: then in the two terms thus got make a_2 negative, and as before add the results, we shall, continuing this process, get in all 2^s terms, which will be found to be equal to 2^s times the continued product of the cosines involved in G.

Effecting the integrations indicated in H, we see that

$$H = \frac{N \cos \Sigma au + N' \sin \Sigma au}{D},$$

where $\quad D = \{a^2 + (\Sigma am)^2\} \ldots \{c^2 + (\Sigma ap)^2\},$

and N and N' follow the same law of formation as in the particular case already considered, except that for $\dfrac{am + a'm'}{a}$, &c. we substitute $\dfrac{\Sigma am}{a}$, &c. With this remark we perceive that

$$H = \frac{e^{\Sigma au\sqrt{(-1)}}}{\{a + \sqrt{(-1)}\,\Sigma am\}\ldots\{c + \sqrt{(-1)}\,\Sigma ap\}} + \frac{e^{-\Sigma au\sqrt{(-1)}}}{\{a - \sqrt{(-1)}\,\Sigma am\}\ldots\{c - \sqrt{(-1)}\,\Sigma ap\}} \ldots\text{(3)}.$$

The assumption now to be made is that

$$\frac{1}{(a + \Sigma am)\ldots(c + \Sigma ap)} = \Sigma \frac{F}{\Delta} \ldots\ldots(2'),$$

where Δ is the product of every set of s factors taken out of the whole number of r factors

$$a + \Sigma am, \;\ldots\; c + \Sigma ap,$$

and F is independent of $a_1 \ldots a_s$.

There will thus be $\dfrac{r \,.\, r-1 \ldots r-s+1}{1 \,.\, 2 \ldots s}$ disposable quantities F, which will be found to be the number required to make (2′) identically true. Consequently we shall have

$$H = \Sigma F \frac{\nu \cos \Sigma au + \nu' \sin \Sigma au}{\delta},$$

where δ is the product of s factors of the form $a^2 + (\Sigma am)^2$; and ν and ν' are formed just as in the case of $s = 2$: that is to say, we shall have

$$\nu = a \ldots c\,(1 - C_2 + C_4 - \&c.), \quad \nu' = a \ldots c\,(C_1 - C_3 + \&c.),$$

where C_t is the sum of the products of every combination that can be made of the s quantities $\dfrac{\Sigma am}{a}$, &c., taken t and t together.

In order to simplify the expression

$$\frac{\nu \cos \Sigma au + \nu' \sin \Sigma au}{\delta},$$

let us denote the s quantities Σam, &c., which are involved in it, by the single symbols $\beta_1 \ldots \beta_s$, and assume

$$u_1 = \Sigma m_1 x, \quad u_2 = \Sigma m_2 x, \quad \ldots . \; u_s = \Sigma m_s x,$$

the sign of summation Σ extending only to that set of s out of the r quantities $x \ldots . z$, which corresponds to the factors involved in the denominator δ. Of course x, y, &c. are here, as before, new variables. (In the case of $s = 3$, for instance, these assumptions will be of the form

$$\begin{aligned}
\beta_1 &= a_1 m_1 + a_2 m_2 + a_3 m_3, & u_1 &= m_1 x + n_1 y + p_1 z,\\
\beta_2 &= a_1 n_1 + a_2 n_2 + a_3 n_3, & u_2 &= m_2 x + n_2 y + p_2 z,\\
\beta_3 &= a_1 p_1 + a_2 p_2 + a_3 p_3, & u_3 &= m_3 x + n_3 y + p_3 z.)
\end{aligned}$$

It follows from this that $du_1 \ldots du_s\, da_1 \ldots da_s$ will be replaced by $dx \ldots dy \,.\, d\beta_1 \ldots d\beta_s$, and that the factors in δ will take the simpler form $a^2 + \beta_1^2$, $b^2 + \beta_2^2$, &c.; while Σau will become $\beta_1 x + \beta_2 y + \ldots .$

The integrations with respect to β extend, like those for a, from $-\infty$ to $+\infty$. Let

$$\int_{-\infty}^{+\infty} d\beta_1 \cdots \int_{-\infty}^{+\infty} d\beta_s \, \frac{\nu \cos \Sigma \beta x + \nu' \sin \Sigma \beta x}{\delta} = M'.$$

Then, from the obvious analogy between the forms ν and ν', and those of the developments of $\cos \Sigma\beta x$ and $\sin \Sigma\beta x$ respectively, it follows that if x, y, &c. are all positive,

$$M' = \pi^s e^{-ax-by} \cdots (1 + 1 + \ldots .),$$

there being twice as many units within the brackets as there are terms in the development of $\sin (f_1 + \ldots . f_s)$, or of $\cos (f_1 + \ldots . f_s)$, that is to say, twice 2^{s-1} or 2^s.

Moreover, if any one, as x, of the quantities x, y, &c., is negative, $M' = 0$; and this, whether it alone is negative or

any others, are so too. For if $x = -x'$, let its coefficient β_1 be assumed equal to $-\beta_1'$, when the expression of M' becomes of the same form as if x were positive, except that ν and ν' are changed by having $-\beta_1'$ wherever β_1 occurred previously. Now none of the quantities β can occur raised to any power, and therefore every term involving β_1 will change sign when β_1 is replaced by $-\beta_1'$. Hence we shall have

$$M_{,} = \pi^s e^{-ax'-by-\dots}(1 \pm 1\dots\dots),$$

there being as many negative units as positive within the brackets, since in the development of $\sin(f_1 + \dots . f_s)$ or $\cos(f_1 + \dots . f_s)$ there are 2^{s-2} terms independent of the sine of f_1 and 2^{s-2} terms which involve that quantity, and which therefore change sign when f_1 does so. Hence the quantity within the bracket, and consequently $M_{,}$, is equal to zero if x be negative; and so, of course, for the other variables $y\,.\,.z$.

M will, in particular cases analogous to those already noticed, assume exceptional or limiting values, but of these we need not take account. And thus we arrive at the following remarkable theorem:

The definite integral of r *variables* $\mathrm{x}\dots\dots\mathrm{z}$

$$\int_0 d\mathrm{x}\dots\dots\int_0 d\mathrm{z}\,\phi_1(\mathrm{m}_1\mathrm{x} + \dots\dots \mathrm{p}_1\mathrm{z})\dots\dots\phi_s(\mathrm{m}_s\mathrm{x} + \dots\dots \mathrm{p}_s\mathrm{z})\,e^{-a\mathrm{x}-\dots c\mathrm{z}},$$

whose limits are given by s *inequalities*

$$\mathrm{m}_1\mathrm{x} + \dots\dots \mathrm{p}_1\mathrm{z} \leqq \mathrm{h}_1, \dots\dots \mathrm{m}_s\mathrm{x} + \dots\dots \mathrm{p}_s\mathrm{z} \leqq \mathrm{h}_s,$$

can generally be expressed as a linear function of

$$\frac{\mathrm{r}\,.(\mathrm{r}-1)\dots(\mathrm{r}-\mathrm{s}+1)}{1.2\dots\mathrm{s}}$$

integrals of s *variables each. The form of each of these integrals may be deduced from the original integral by omitting from it any set of* r − s *of the variables, and similarly the form of the limiting inequalities may be got by omitting the same set of variables from the original inequalities* (r > s).

In certain cases, however, when the constants a, m, &c. have particular values, the theorem fails because the assumption (2′) becomes illegitimate. This failure is indicated by certain of the quantities F becoming infinite. To determine the form of F, we have merely to multiply (2′) by Δ, and then to equate to zero all the s factors of which Δ is composed. All the quantities F, except the particular one under consideration, will then disappear, and we have s equations determining the s quantities a. Hence it will appear that

F is equal to a fraction whose numerator is unity, and denominator equal to the value assumed by the product of the remaining $r - s$ factors, when the values already assigned for the quantities a, &c. have been substituted for them; a result which it is obvious can be immediately expressed in the notation of determinants. F will therefore become infinite if our equating the s factors by which it is divided in (2′) to zero will make one or more of the remaining $r - s$ factors vanish. Let it make t of these factors vanish; then equating these t factors also to zero, we get in all $s + t$ equations, which are equivalent to s independent ones. Therefore any set of s out of these $s + t$ equations will satisfy the remaining t equations. Hence

$$\frac{(s+t)(s+t-1)\ldots\ldots(t+1)}{1.2.\ldots.t}$$

of the quantities F will become infinite, and therefore the second side of (2′) will consist of finite terms and of a finite quantity expressed in the form of the sum of that number of infinite terms. This indetermination of course indicates a change in the form of the function, the general character of which the reader will have little difficulty in perceiving. But the consideration of these particular cases, some of which are interesting, must be deferred to another occasion.

I am inclined to believe that the processes developed in this paper will admit both of simplification and extension. For the exponential function we may substitute with certain modifications any function of $ax + \ldots cz$, in accordance with a result given by Mr. Boole in his very interesting Memoir on a new Method in Analysis, which is published in the *Transactions of the Royal Society*. (This result would include the one which I obtained in the last volume of the *Journal*, from which however it might be deduced.)

Thus, if in the theorem established in this paper we replace $a, b \ldots c$ by $ka, kb \ldots kc$, k being a wholly arbitrary quantity, we may, comparing the coefficients of its powers, deduce new theorems from the given one. Developing the first side of the equation, the coefficient of k^n will be

$$\frac{1}{1.2\ldots n}\int_0 dx\ldots\int_0 dz\,\phi_1(m_1x+\ldots p_1z)\ldots\phi_s(m_sx+\ldots p_sz)\,(ax+\ldots cz)^n,$$

and in the second it will be the sum of a series of terms of the form

$$\frac{F}{1.2\ldots(n+r-s)}\int_0 dx\int_0 dy\ldots\phi(m_1x+n_1y+\ldots)$$
$$\ldots\phi_s(m_sx+n_sy+\ldots)\,(ax+by+\ldots)^{n+r-s},$$

as it is manifest that F will become $\dfrac{F}{k^{r-s}}$. Hence, if $\psi(ax + \ldots cz)$ be such a function that its development may be substituted for it in the integrations, we shall have

$$\int_0 dx \ldots \int_0 dz\, \phi_1(m_1 x + \ldots p_1 z) \ldots \phi_s(m_s x + \ldots p_s z)\, \psi(ax + \ldots cz)$$
$$= \Sigma F \int_0 dx \int_0 dy \ldots \phi_1(m_1 x + n_1 y + \ldots) \ldots \phi_s(m_s x + n_s y + \ldots)$$
$$\psi_1(ax + by + \ldots),$$

where $\dfrac{d^{r-s}}{dt^{r-s}}\psi_1 t = \psi t$ and all the differential coefficients of $\psi_1 t$ of an order lower than the $(r - s)^{\text{th}}$ vanish for $t = 0$. This is, I believe, in the case of s equal to unity, precisely equivalent to one of Mr. Boole's results. It might also, I imagine, be obtained without having recourse to developments.

ON THE EQUATION OF LAPLACE'S FUNCTIONS.

By George Boole.

The partial differential equation of the second order, known as the Equation of Laplace's Functions, and usually expressed in the form

$$\frac{d}{d\mu}(1 - \mu^2)\frac{du}{d\mu} + \frac{1}{1 - \mu^2}\frac{d^2u}{d\phi^2} + n(n+1)u = 0 \ldots\ldots (1),$$

is not more remarkable for the importance of its physical applications, than for the difficulties which it presents in a purely mathematical point of view. Mr. Hargreave, in the *Philosophical Transactions* for 1841, first succeeded in obtaining an expression for the complete integral. His analysis is original and most ingenious. Assuming two new variables, x and y, connected with the former ones by the relations

$$x = \phi + \tfrac{1}{2}\sqrt{-1}\log\frac{1+\mu}{1-\mu}, \quad y = \phi - \tfrac{1}{2}\sqrt{-1}\log\frac{1+\mu}{1-\mu} \ldots (2),$$

he reduces the equation to the form

$$\frac{d^2u}{dx\,dy} + \frac{n(n+1)u}{4\cos^2\dfrac{x-y}{2}} = 0 \ldots\ldots\ldots\ldots (3),$$

and, by a process of reduction which it is not necessary here to explain, he ultimately finds

$$u = \ldots\int \cos^{-2}\frac{y-x}{2}\int \cos^{-2}\frac{y-x}{2}\left\{\int \cos^{2n}\frac{x-y}{2}\,\chi(y)\,dy + \psi(x)\right\} dy\,dy$$

$$\ldots n \text{ times} \ldots . (4),$$

in which $\chi(y)$, $\psi(x)$, denote arbitrary functions of y and x.

The correctness of this solution I have since verified by a different analysis. The result is, however, so complicated by signs of integration, that the determination of the arbitrary functions is extremely difficult, and the particular deductions in Mr. Hargreave's paper are, I conceive, erroneous. Under these circumstances, it becomes an object of interest to inquire, whether the equation does not admit of a better form of solution. Such is the question which I propose to consider in this paper. I shall shew that the complete integral may be expressed in a form at once symmetrical and free from signs of integration, and shall, by a proper determination of the arbitrary functions, deduce from it the actual forms of Laplace's coefficients. As the method to be employed is not perhaps generally known, it may be proper to state first the preliminary theorems on which it depends, referring the reader, for a more particular account of them, to a paper on a "General Method in Analysis," published in the *Philosophical Transactions* for 1844, Part II.

1. It is known that the symbols $\frac{d}{d\theta}$ and ϵ^{θ}, or as for convenience we may write D and ϵ^{θ}, satisfy the following relations:

$$f(D)\,\epsilon^{m\theta} = f(m)\,\epsilon^{m\theta} \ldots\ldots\ldots\ldots (5),$$

$$f(D)\,\epsilon^{m\theta}u = \epsilon^{m\theta}f(D+m)\,u \ldots\ldots (6).$$

PROP. 1. Assuming these properties, it may be shewn that *linear differential equations of the form*

$$(a + bx + cx^2 \ldots)\frac{d^n u}{dx^n} + (a' + b'x + c'x^2 \ldots \frac{d^{n-1}u}{dx^{n-1}} + \&c. = X,$$

may always be reduced to the form

$$u + \phi_1(D)\,\epsilon^{\theta}u + \phi_2(D)\,\epsilon^{2\theta}u + \&c. = U \ldots . (7);$$

in which $\epsilon^{\theta} = x$, U *is a function of* ϵ^{θ}, *and* $\phi_1(\mathrm{D})$, $\phi_2(\mathrm{D})$, *&c. are functions of the symbol* D *or* $\frac{\mathrm{d}}{\mathrm{d}\theta}$. This I have designated as the symbolical form of the linear differential equation: and it may be remarked, that there exists a similar form, admitting of similar treatment, in equations of finite differences.

[redacted]

As an example, let us take the equation

$$x^2\frac{d^2u}{dx^2}+q^2x^2u-6u=0\ \ldots\ldots\ldots\ (8),$$

which occurs in the theory of the earth's figure.

Let $x=\epsilon^\theta$. Now (Gregory's *Examples*, pp. 31, 32,)

$$x^2\frac{d^2}{dx^2}=D(D-1).$$

Hence
$$D(D-1)u+q^2\epsilon^{2\theta}u-6u=0,$$
$$\therefore\ (D+2)(D-3)u+q^2\epsilon^{2\theta}u=0,$$
$$u+\frac{q^2}{(D+2)(D-3)}\epsilon^{2\theta}u=0,$$

which is the symbolical form required.

From the symbolical form (7) we may at once deduce a theory of the solution of differential equations, in series extending to those cases in which the ordinary methods fail; but we shall here confine ourselves to two theorems, on which the *finite* solution of such equations chiefly depends.

PROP. 2. *When the equation* (7) *is of the form*

$$u+af(D)\epsilon^\theta u+bf(D)f(D-1)\epsilon^{2\theta}u+\&c.=0\ \ldots\ (9),$$

it may be resolved into a system of equations of the form

$$\left.\begin{aligned}u-q_1f(D)\epsilon^\theta u&=0,\\ u-q_2f(D)\epsilon^\theta u&=0,\end{aligned}\right\}\ \ldots\ldots\ldots(10),$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot$$

q_1, q_2, *&c. being the roots of the equation*

$$q^n+aq^{n-1}+bq^{n-2}+\&c.=0\ \ldots\ldots\ (11).$$

To prove this, we observe, that if $f(D)\epsilon^\theta u=\rho u$, then

$$f(D)f(D-1)\epsilon^{2\theta}u=f(D)\epsilon^\theta f(D)\epsilon^\theta u=\rho^2u,\ \text{by (6)},$$

and so on; whence (9) gives

$$(1+a\rho+b\rho^2+\&c.)u=0,$$

or
$$(1-q_1\rho)(1-q_2\rho)\ldots u=0,$$

and the theorem is manifest. The reader will easily extend it to the case in which the equation has a second member. (*Phil. Trans.* 1844, p. 245.)

This case corresponds, in the present theory, to the case of

equations with constant coefficients in the received one, but is far more general. The differential equations

$$\frac{d^n u}{dx^n} + a\,\frac{d^{n-1}u}{dx^{n-1}} + b\,\frac{d^{n-2}u}{dx^{n-2}} + \&\text{c.} = 0,$$

when reduced to the symbolical form, becomes

$$u + \frac{a}{D}\,\epsilon^{\theta}u + \frac{b}{D\,(D-1)}\,\epsilon^{2\theta}u + \&\text{c.} = 0,$$

which is seen to be only a particular form of (9).

3. When an equation of the symbolical form has but two terms in its first member, we can always determine whether it is integrable or not. All the integrable forms occurring in physical inquiries, with which I am acquainted, are of this class. The method of integration depends on the following theorem.

Prop. 3. *The equation* $\mathrm{u} + \phi\,(\mathrm{D})\,\epsilon^{r\theta}\mathrm{u} = \mathrm{U}$, *will be converted into the form* $\mathrm{v} + \psi\,(\mathrm{D})\,\epsilon^{r\theta}\mathrm{v} = \mathrm{V}$, *by the relations*

$$u = P_r\,\frac{\phi\,(D)}{\psi(D)}\,v, \qquad U = P_r\,\frac{\phi\,(D)}{\psi(D)}\,V \;\ldots\ldots\; (12);$$

in which $\mathrm{P}_r\,\dfrac{\phi\,(\mathrm{D})}{\psi(\mathrm{D})}$ *denotes the indefinite symbolical product*

$$\frac{\phi\,(D)\;\phi\,(D-r)\;\phi\,(D-2r)\;\ldots\ldots}{\psi\,(D)\;\psi\,(D-r)\;\psi\,(D-2r)\ldots\ldots}.$$

(*Phil. Trans.* p. 247.)

To prove this theorem, let $u = f(D)\,v$; and substituting in the given equation, we have

$$f(D)\,v + \phi\,(D)\;\epsilon^{r\theta} f(D)\,v = U,$$

$$\therefore f(D)\,v + \phi\,(D) f(D-r)\;\epsilon^{r\theta} v = U, \quad \text{by (6)},$$

or $$v + \frac{\phi\,(D) f(D-r)}{f(D)}\,\epsilon^{r\theta}v = \{f(D)\}^{-1}U;$$

and equating coefficients,

$$\frac{\phi\,(D) f(D-r)}{f(D)} = \psi\,(D), \quad \{f(D)\}^{-1}U = V.$$

The former of these equations gives

$$f(D) = \frac{\phi\,(D)}{\psi(D)} f(D-r),$$

an equation of differences of the first order relative to $f(D)$, of which the solution is

$$f(D) = P_r \frac{\phi(D)}{\psi(D)};$$

whence $\quad u = P_r \dfrac{\phi(D)}{\psi(D)} v, \qquad U = P_r \dfrac{\phi(D)}{\psi(D)} V.$

We proceed to exemplify our theory in the solution of the equation of Laplace's Functions.

We have

$$\frac{d}{d\mu}(1-\mu^2)\frac{du}{d\mu} + \frac{1}{1-\mu^2}\frac{d^2u}{d\phi^2} + n(n+1)u = 0..(13).$$

Now ϕ only enters into this equation through the symbol of differentiation $\dfrac{d}{d\phi}$, which is commutative with respect to μ and $\dfrac{d}{d\mu}$. Let $\dfrac{d}{d\phi}\sqrt{-1} = a$, then

$$\frac{d}{d\mu}(1-\mu^2)\frac{du}{d\mu} - \frac{a^2}{1-\mu^2}u + n(n+1)u = 0...(14).$$

If we can integrate this equation regarding a as a constant, and afterwards in the most general manner interpret our result, when for a we write $\dfrac{d}{d\phi}\sqrt{-1}$, we shall evidently be in possession of the complete integral required.

Now if in (14) we write $\mu = \epsilon^\theta$, and pass to the symbolical form, we shall have an equation involving three terms in the first member, and our analysis does not in its existing state possess any *general* method of treating such equations. We must therefore endeavour, by transforming the original equation, to obviate this difficulty.

The expanded form of (14) is

$$(1-\mu^2)\frac{d^2u}{d\mu^2} - 2\mu\frac{du}{d\mu} - \frac{a^2}{1-\mu^2}u + n(n+1)u = 0...(15).$$

Assume $u = (1-\mu^2)^r v$, then

$$\frac{du}{d\mu} = (1-\mu^2)^r\frac{dv}{d\mu} - 2r\mu(1-\mu^2)^{r-1}v.$$

$$\frac{d^2u}{d\mu^2} = (1-\mu^2)^r\frac{d^2v}{d\mu^2} - 2r(1-\mu^2)^{r-1}\left(2\mu\frac{dv}{d\mu} + v\right)$$
$$+ 4r(r-1)\mu^2(1-\mu^2)^{r-2}v.$$

Substituting and effecting some reductions, we have

$$(1-\mu^2)^{r+1}\frac{d^2v}{d\mu^2}-(4r+2)\,\mu\,(1-\mu^2)^r\frac{dv}{d\mu}$$
$$+\{n(n+1)-2r-4r^2)\,(1-\mu^2)^r\,v+(4r^2-a^2)\,(1-\mu^2)^{r-1}v=0.$$

Let $4r^2-a^2=0$, then $r=\pm\frac{a}{2}$. Either sign may be taken; we choose the negative one, and suppose $r=-\frac{a}{2}$. Our equation now becomes, on dividing both sides by $(1-\mu^2)^r$,

$$(1-\mu^2)\frac{d^2v}{d\mu^2}+2(a-1)\,\mu\frac{dv}{d\mu}+\{n(n+1)-a(a-1)\}\,v=0\ldots(16).$$

In order to reduce this equation to the symbolical form, multiply by μ^2, then

$$(1-\mu^2)\,\mu^2\frac{d^2v}{d\mu^2}+2(a-1)\,\mu^3\frac{dv}{d\mu}+\{n(n+1)-a(a-1)\}\,\mu^2v=0.$$

Let $\mu=\varepsilon^\theta$, then $\mu\frac{d}{d\mu}=\frac{d}{d\theta}=D$, $\mu^2\frac{d^2}{d\mu^2}=D(D-1)$, whence

$$(1-\varepsilon^{2\theta})\,D(D-1)\,v+2(a-1)\varepsilon^{2\theta}Dv$$
$$+\{n(n+1)-a(a-1)\}\varepsilon^{2\theta}v=0\ldots\ldots(17).$$

Now $\varepsilon^{2\theta}D(D-1)\,v=(D-2)(D-3)\,\varepsilon^{2\theta}v$ by (6),
and $\varepsilon^{2\theta}Dv=(D-2)\,\varepsilon^{2\theta}v$.

Substituting these forms in (17), we have

$$D(D-1)\,v-\{(D-2)(D-3)-2(a-1)(D-2)-n(n+1)$$
$$+a(a-1)\}\,\varepsilon^{2\theta}v=0.$$

Or $D(D-1)\,v-\{D^2-(2a+3)\,D+a^2+3a-n^2-n+2\}\varepsilon^{2\theta}v=0$.

Or $\quad D(D-1)\,v-(D-a+n-1)(D-a-n-2)\,\varepsilon^{2\theta}v=0$,

on resolving the coefficient of $\varepsilon^{2\theta}v$ into its factors. Hence

$$v-\frac{(D-a+n-1)(D-a-n-2)}{D(D-1)}\,\varepsilon^{2\theta}v=0\ldots(18)$$

is the symbolical form required, and it is seen that the first member involves only two terms.

Now, let us assume

$$w-\frac{(D-a-n-1)(D-a-n-2)}{D(D-1)}\,\varepsilon^{2\theta}w=W\ldots\ldots(19).$$

In these two equations v and w respectively stand for u and v of Prop. 3.

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Hence $$\phi(D) = \frac{(D-a+n-1)(D-a-n-2)}{D(D-1)},$$

$$\psi(D) = \frac{(D-a-n-1)(D-a-n-2)}{D(D-1)};$$

$$\therefore P_r \frac{\phi(D)}{\psi(D)} = P_s \frac{D-a+n-1}{D-a-n-1}$$
$$= (D-a+n-1)(D-a+n-3)\ldots(D-a-n+1);$$
$$\therefore v = (D-a+n-1)(D-a+n-3)\ldots\ldots(D-a-n+1)\,w,$$
$$o = (D-a+n-1)(D-a+n-3)\ldots\ldots(D-a-n+1)\,W.$$

The complete value of W determined from the last equation would be

$$W = c_1 \varepsilon^{(a-n+1)\theta} + c_2 \varepsilon^{(a-n+3)\theta} + \&\text{c.};$$

but inasmuch as the transformed equation (19) is of the same degree as the original one (18), and will therefore introduce the requisite number of arbitrary constants, it is not necessary to retain any in W, so that we have simply $W = 0$: and it is remarkable that, were we to retain all the constants which the complete value of W involves, the final value of v would not be at all affected: the unnecessary constants would disappear. We have thus to consider the system,

$$v = (D-a+n-1)(D-a+n-3)\ldots(D-a-n+1)\,w \ldots (20),$$

$$w - \frac{(D-a-n-1)(D-a-n-2)}{D(D-1)}\,\varepsilon^{2\theta} w = 0 \ldots\ldots\ldots\ldots (21).$$

The last equation may, by Prop. 2, be resolved into the system

$$\left.\begin{aligned} w + \frac{D-a-n-1}{D}\,\varepsilon^{\theta} w = 0 \\ w - \frac{D-a-n-1}{D}\,\varepsilon^{\theta} w = 0 \end{aligned}\right\} \ldots\ldots (22).$$

From the former of these equations we have

$$Dw + (D-a-n-1)\,\varepsilon^{\theta} w = 0;$$

or $$Dw + \varepsilon^{\theta}(D-a-n)\,w = 0.$$

$$\therefore \frac{dw}{d\theta} - \frac{(a+n)\,\varepsilon^{\theta}}{1+\varepsilon^{\theta}}\,w = 0,$$

$$w = (1+\varepsilon^{\theta})^{n+a}\,\psi(\phi),$$

$\psi(\phi)$ denoting an arbitrary function of ϕ. In like manner, from the second equation of the system (22), we have

$$w = (1-\varepsilon^{\theta})^{n-a}\,\chi(\phi).$$

Hence the complete value of w is

$$w = (1+\epsilon^{\theta})^{n+a}\psi(\phi) + (1-\epsilon^{\theta})^{n+a}\chi(\phi)$$
$$= (1+\mu)^{n+a}\psi(\phi) + (1-\mu)^{n+a}\chi(\phi) \ldots\ldots (23).$$

To simplify the expression for v we observe that, in general,

$$f(D)\,U = f(D)\,\epsilon^{r\theta}\epsilon^{-r\theta}U$$
$$= \epsilon^{r\theta}f(D+r)\,\epsilon^{-r\theta}U \text{ by (6).}$$

Now inverting the order of the factors in the right-hand member of (20),

$$v = (D-a-n+1)(D-a-n+3)\ldots(D-a+n-1)\,w$$
$$= \epsilon^{(a+n-1)\theta}D(D+2)\ldots(D+2n-2)\,\epsilon^{-(a+n-1)\theta}w.$$

But $D+2 = \epsilon^{-2\theta}D\epsilon^{2\theta}$, $D+4 = \epsilon^{-4\theta}D\epsilon^{4\theta}$, and so on. Substituting, we have

$$v = \epsilon^{(a+n-1)\theta}D\epsilon^{-2\theta}D\epsilon^{-2\theta}\ldots D\epsilon^{(2n-2)\theta}\epsilon^{-(a+n-1)\theta}w.$$

Making $\epsilon^{\theta}=\mu$, $D=\mu\dfrac{d}{d\mu}$, the above equation assumes the form

$$v = \mu^{n+a}\left(\frac{d}{d\mu}\,\frac{1}{\mu}\right)^{n}\mu^{n-a}w \ldots\ldots\ldots\ldots (24).$$

Now $u = (1-\mu^2)^{-\frac{a}{2}}v$; hence, writing in full the values of v and w, we have

$$u = (1-\mu^2)^{-\frac{a}{2}}\mu^{n+a}\left(\frac{d}{d\mu}\,\frac{1}{\mu}\right)^{n}\mu^{n-a}\left\{(1+\mu)^{n+a}\psi(\phi)+(1-\mu)^{n+a}\chi(\phi)\right\}$$
$$\ldots\ldots\ldots (25).$$

It remains to interpret this remarkable expression.

As $\psi(\phi)$, $\chi(\phi)$, are perfectly arbitrary, it is obvious that we may, in place of them, write $\psi(\epsilon^{\phi\sqrt{-1}})$, $\chi(\epsilon^{\phi\sqrt{-1}})$. The interpretation of our formula does not require this transformation, but the result is thereby simplified. We are thus at liberty to express the integral in the following form:

$$u = \mu^{n}\left(\frac{\mu}{\sqrt{(1-\mu^2)}}\right)^{a}\left(\frac{d}{d\mu}\,\frac{1}{\mu}\right)^{n}\left\{(\mu+\mu^2)^{n}\left(\frac{1+\mu}{\mu}\right)^{a}\psi(\epsilon^{\phi\sqrt{+1}})\right.$$
$$\left. + (\mu-\mu^2)^{n}\left(\frac{1-\mu}{\mu}\right)^{a}\chi(\epsilon^{\phi\sqrt{-1}})\right\} \ldots\ldots (26),$$

or by two equations thus,

$$u = \left\{\frac{\mu}{\sqrt{(1-\mu^2)}}\right\}^{a}F(\mu,\,\epsilon^{\phi\sqrt{-1}}),$$

where $F(\mu,\ \epsilon^{\phi\sqrt{-1}}) = \mu^n \left(\dfrac{d}{d\mu}\ \dfrac{1}{\mu}\right)^n \left\{(\mu + \mu^2)^n \left(\dfrac{1+\mu}{\mu}\right)^a \psi(\epsilon^{\phi\sqrt{-1}})\right.$

$$\left. + (\mu - \mu^2)^n \left(\frac{1-\mu}{\mu}\right)^a \chi(\epsilon^{\phi\sqrt{-1}})\right\} \ldots\ldots (27).$$

Now t being any quantity independent of ϕ, we have

$$t^a f(\epsilon^{\phi\sqrt{-1}}) = t^{\frac{d}{d\phi}\sqrt{-1}} f(\epsilon^{\phi\sqrt{-1}})$$

$$= \epsilon^{\sqrt{-1}\log t\,\frac{d}{d\phi}} f(\epsilon^{\phi\sqrt{-1}})$$

$$= f(\epsilon^{(\phi+\sqrt{-1}\log t)\sqrt{-1}}),\ \text{by Taylor's theorem,}$$

$$= f\left(\frac{\epsilon^{\phi\sqrt{-1}}}{t}\right).$$

Hence
$$\left(\frac{1+\mu}{\mu}\right)^a \psi(\epsilon^{\phi\sqrt{-1}}) = \psi\left(\frac{\mu\,\epsilon^{\phi\sqrt{-1}}}{1+\mu}\right),$$

and so on; whence finally

$$u = F\left(\mu,\ \frac{\sqrt{(1-\mu^2)}}{\mu}\,\epsilon^{\phi\sqrt{-1}}\right),$$

where $F(\mu,\ \epsilon^{\phi\sqrt{-1}})$

$$= \mu^n \left(\frac{d}{d\mu}\ \frac{1}{\mu}\right)^n \left\{(\mu+\mu^2)^n \psi\left(\frac{\mu\,\epsilon^{\phi\sqrt{-1}}}{1+\mu}\right) + (\mu-\mu^2)^n \chi\left(\frac{\mu\,\epsilon^{\phi\sqrt{-1}}}{1-\mu}\right)\right\} \ldots (28),$$

which is the complete integral required. The symbol $\left(\dfrac{d}{d\mu}\ \dfrac{1}{\mu}\right)^n$, of course, implies the performance of n successive operations, each of which is effected by dividing the subject by μ, and then taking the differential coefficient with respect to that variable.

Discussion of the Integral.

If we represent the two particular integrals in the above general solution by U and V respectively, then in U, supposing $\psi(\epsilon^{\phi\sqrt{-1}}) = \epsilon^{r\phi\sqrt{-1}}$, we shall have

$$U = \left\{\frac{\sqrt{(1-\mu^2)}\,\epsilon^{\phi\sqrt{-1}}}{\mu}\right\}^r \mu^n \left(\frac{d}{d\mu}\ \frac{1}{\mu}\right)^n (\mu + \mu^2)^n \left(\frac{\mu}{1+\mu}\right)^r$$

$$= (1-\mu^2)^{\frac{r}{2}} \mu^{n-r} \left(\frac{d}{d\mu}\ \frac{1}{\mu}\right)^n \mu^{n+r} (1+\mu)^{n-r} \epsilon^{r\phi\sqrt{-1}}$$

$$= f(\mu)\,\epsilon^{r\phi\sqrt{-1}},\ \text{for abbreviation}. \ldots\ldots\ldots\ldots (29).$$

Again, in U let $\psi(\epsilon^{\phi\sqrt{-1}}) = \epsilon^{-r\phi\sqrt{-1}}$, we have

$$U = \left\{\frac{\sqrt{(1-\mu^2)}\,\epsilon^{\phi\sqrt{-1}}}{\mu}\right\}^{-r} \mu^n \left(\frac{d}{d\mu}\,\frac{1}{\mu}\right)^n (\mu+\mu^2)^n \left(\frac{\mu}{1+\mu}\right)^{-r}$$

$$= (1-\mu^2)^{-\frac{r}{2}} \mu^{n+r} \left(\frac{d}{d\mu}\,\frac{1}{\mu}\right)^n \mu^{n-r} (1+\mu)^{n+r} \epsilon^{-r\phi\sqrt{-1}}$$

$$= f_1(\mu)\,\epsilon^{-r\phi\sqrt{-1}}, \text{ for abbreviation. } \ldots\ldots\ldots (30).$$

The functions $f_1(\mu)$, $f(\mu)$, have a very singular relation, viz.

$$f_1(\mu) = (-)^n f(-\mu) \ldots\ldots\ldots\ldots (31),$$

which may be proved thus. The two first terms of the expansion of $f(\mu)$, in ascending powers of μ, are

$$(r+n-1)(r+n-3)\ldots(r-n+1) - (r+n)(r+n-2)\ldots(r-n)\,\mu \ldots (32);$$

and the two first terms in the expansion of $f_1(\mu)$ are

$$(-)^n \{(r+n-1)(r+n-3)\ldots(r-n+1)$$
$$+ (r+n)(r+n-2)\ldots(r-n)\,\mu\} \ldots . (33).$$

But $f(\mu)\,\epsilon^{r\phi\sqrt{-1}}$, and $f_1(\mu)\,\epsilon^{-r\phi\sqrt{-1}}$ are both solutions of the differential equation given. If we substitute them in that equation in the forms

$$u = \Sigma u_m \mu^m \epsilon^{r\phi\sqrt{-1}}, \quad u = \Sigma u_m \mu^m \epsilon^{-r\phi\sqrt{-1}},$$

we shall find that the successive values of u_m are in each case connected by the relation

$$u_m - \frac{2(m-2)^2 - n(n+1) + 2r^2}{m(m-1)} u_{m-2}$$
$$+ \frac{(m+n-3)(m-n-4)}{m(m-1)} u_{m-4} = 0 \ldots . (34),$$

the values u_0 and u_1 being arbitrary. From the relation just given it appears, that all the even coefficients, u_2, u_4, &c., will be formed from u_0, and all the odd ones, u_3, u_5, &c., from u_1. Hence, on comparing (32) and (33), we see that the even terms in the expansion of $f_1(\mu)$ will differ only from those in the expansion of $f(\mu)$ by the sign $(-)^n$, and that the odd terms will only differ by the sign $(-)^{n+1}$; and as the odd terms change sign with μ, the relation between $f_1(\mu)$ and $f(\mu)$ will be such as we have assigned in (31).

From the above it appears that the assumption

$$\psi(\epsilon^{\phi\sqrt{-1}}) = a\,\epsilon^{r\phi\sqrt{-1}} + b\,\epsilon^{-r\phi\sqrt{-1}},$$

will give

$$U = a f(\mu)\,\epsilon^{r\phi\sqrt{-1}} + (-)^n b f(-\mu)\,\epsilon^{-r\phi\sqrt{-1}} \ldots . (35);$$

[redacted]

and, by an exactly similar process of reasoning and comparison, we shall find that if, in the second integral V, we assume

$$\chi(\epsilon^{\phi\sqrt{-1}}) = a'\,\epsilon^{r\phi\sqrt{-1}} + b'\,\epsilon^{-r\phi\sqrt{-1}},$$

then $$V = a'f(-\mu)\,\epsilon^{r\phi\sqrt{-1}} + (-)^n\,b'f(\mu)\,\epsilon^{-r\phi\sqrt{-1}} \ldots\ldots (36).$$

Adding these values,

$$\begin{aligned} u &= \{af(\mu) + a'f(-\mu)\}\,\epsilon^{2\phi\sqrt{-1}} + (-)^n\,\{b'f(\mu) + bf(-\mu)\}\,\epsilon^{-r\phi\sqrt{-1}} \\ &\left.\begin{aligned} &= [\{a + (-)^n b'\}f(\mu) + \{a' + (-)^n b\}f(-\mu)]\cos r\phi \\ &\quad + [\{a - (-)^n b'\}f(\mu) + \{a' - (-)^n b\}f(-\mu)]\sqrt{-1}\,\sin r\phi \end{aligned}\right\} \ldots (37). \end{aligned}$$

If we assume

$$a - (-)^n b' = 0, \qquad a' - (-)^n b = 0,$$

we have $$u = 2\,\{af(\mu) + a'f(-\mu)\}\cos r\phi \ldots\ldots (38):$$

and as this is true for all values of r, it is clear that, had we assumed $\psi(\epsilon^{\phi\sqrt{-1}}) = \Sigma\,(a\,\epsilon^{r\phi\sqrt{-1}} + b\,\epsilon^{-r\phi\sqrt{-1}})$, &c., we should have had, in place of (38),

$$u = \Sigma\,\{2af(\mu) + 2a'f(-\mu)\}\cos r\phi;$$

or, putting $2a = c,\ 2a' = c'$,

$$u = \Sigma\,\{cf(\mu) + c'f(-\mu)\}\cos r\phi \ldots\ldots\ldots (39),$$

the values of c and c' differing for different values of r.

This is the most general form, *of the kind*, which the integral can assume. It is remarkable that the coefficient of $\cos r\phi$ will be always a *finite algebraic* function of μ, whether r be integral or not, its expression being

$$f(\mu) = (1 - \mu^2)^{\frac{r}{2}}\,\mu^{n-r}\left(\frac{d}{d\mu}\,\frac{1}{\mu}\right)^n \mu^{n+r}\,(1 + \mu)^{n-r} \ldots\ldots (40).$$

It is also to be remarked that, by attributing other forms to ψ and χ, and especially *logarithmic* forms, we can obtain an infinite variety of finite solutions of a character altogether different from the above.

Forms of Laplace's Coefficients.

In the case we have now to consider, u, or as it is commonly written, P_n, is the coefficient of t^n in the development of the function

$$[1 - 2\,\{\mu\mu' + \sqrt{(1 - \mu^2)(1 - \mu'^2)}\cos\phi\}\,t + t^2]^{-\frac{1}{2}} \ldots (41),$$

ϕ standing for $\phi - \phi'$ in the ordinary treatises.

Here it is evident, that the values of r in $\cos r\phi$ are all integral, and that its limits will be 0 and n. Hence (39) gives

$$P_n = \Sigma_{r=0}^{r=n} \{cf(\mu) + c'f(-\mu)\} \cos r\phi.$$

To determine the coefficients, we must examine the first term of the expanded coefficient of t^n cos $r\phi$ in the expansion of (41). This is easily found to be either

$$2\,\frac{1.3 \ldots n+r-1}{2.4 \ldots n+r} \times \frac{1.3 \ldots n-r-1}{2.4 \ldots n-r},$$

$$\text{or}\ \ 2\,\frac{1.3 \ldots n+r}{2.4 \ldots n+r-1} \times \frac{3.5 \ldots n-r}{2.4 \ldots n-r-1}\,\mu\mu' \ldots. (42),$$

according as $n-r$ is even or odd.

Again, the first term of the expansion of $f(\mu)$ is, in this case,

$$(r+n-1)(r+n-3)\ldots(r-n+1),$$

$$\text{or}\ \ -(r+n)(r+n-2)\ldots(r-n)\,\mu \ldots\ldots (43),$$

according as $n-r$ is even or odd, the remaining term vanishing. If $n-r$ be even, it is clear from (34) that, as the first term involving an odd index in $f(\mu)$ vanishes, all such terms will vanish. Here then $f(-\mu) = f(\mu)$. In the case of $n-r$ being odd, it is evident that all the terms in the expansion of $f(\mu)$ will have odd indices, and $f(-\mu) = -f(\mu)$. In either case we have

$$P_n = \Sigma_{r=0}^{r=n}\, a f(\mu) \cos r\phi,$$

a being a constant, *i.e.* a quantity independent of μ and ϕ. Now, from the form of (41), it is evident that μ and μ' are symmetrically involved in P_n. Hence $a = a_r f(\mu')$, and

$$P_n = \Sigma_{r=0}^{r=n}\, a_r f(\mu) f(\mu') \cos r\phi.$$

If $n-r$ be even, the first term of the expansion of $a_r f(\mu) f(\mu')$ will be

$$a_r \{(r+n-1)(r+n-3)\ldots(r-n+1)\}^2;$$

which, equated with the corresponding term in (42), gives

$$a_r = \frac{2}{1.2 \ldots n+r\ \ 1.2 \ldots n-r},$$

except when $r=0$; in which case

$$a_0 = \frac{1}{(1.2 \ldots n)^2}.$$

[redacted]

If $n-r$ be odd, the first term of the expansion of $a_r f(\mu) f(\mu')$ will be $a_r \{(r+n)(r+n-2)\dots(r-n)\}^2$; which, equated with the corresponding term in (42), leads to the same result.

Hence, finally, we have

$$P_n = A_0 + 2\,(A_1 \cos\phi + A_2 \cos 2\phi \dots + A_n \cos n\phi) \dots (44),$$

where any coefficient A_r is of the form

$$A_r = \frac{f(\mu) f(\mu')}{1.2\dots n+r \;\; 1.2 \dots n-r},$$

and in general

$$f(\mu) = (1-\mu^2)^{\frac{r}{2}} \mu^{n-r} \left(\frac{d}{d\mu}\,\frac{1}{\mu}\right)^n \mu^{n+r} (1+\mu)^{n-r}.$$

I have entered with more particularity into the details of the above solution, than to some might have appeared necessary; but it was my object in this paper, not only to integrate the Equation of Laplace, but also to illustrate, and in so doing, if it might be, to recommend a method in Analysis.

Lincoln, July 28, 1845.

ON SOME ANALYTICAL FORMULÆ, AND THEIR APPLICATION TO THE THEORY OF SPHERICAL CO-ORDINATES.

By Arthur Cayley, M.A., Fellow of Trinity College, Cambridge.

Section 1.

The formulæ in question are only very particular cases of some relating to the theory of the transformation of functions of the second order, which will be given in a following paper. But the case of three variables, here as elsewhere, admits of a symmetrical notation so much simpler than in any other case (on the principle that with three quantities a, b, c, functions of b, c; of c, a; and a, b, may symmetrically be denoted by A, B, C, which is not possible with a greater number of variables) that it will be convenient to employ here a notation entirely different from that made use of in the general case, and by means of which the results will be exhibited in a more compact form. There is no difficulty in verifying by actual multiplication, any of the equations here obtained.

It will be expedient to employ the abbreviation of making a single letter stand for a system of quantities. Thus for instance, if $\aleph = \theta, \phi, \psi$, this merely means that $\Phi(\aleph)$ is to stand for $\Phi(\theta, \phi, \psi)$, $k\aleph$ for $k\theta, k\phi, k\psi$, &c.

[redacted]

Suppose then $\qquad \omega = \xi, \eta, \zeta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$

$$\omega' = \xi', \eta', \zeta',$$

$$\vdots$$

$$Q = A, B, C, F, G, H. \quad . \quad . \quad . \quad (2),$$

$$W(\omega, \omega', Q) = A\xi\xi' + B\eta\eta' + C\zeta\zeta'$$
$$+ F(\eta\zeta' + \eta'\zeta) + G(\zeta\xi' + \zeta'\xi) + H(\xi\eta' + \xi'\eta) \ldots (3).$$

The function W satisfies a remarkable equation, as follows.

Write $\qquad \mathfrak{A} = BC - F^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$

$$\mathfrak{B} = CA - G^2,$$
$$\mathfrak{C} = AB - H^2.$$
$$\mathfrak{F} = GH - AF,$$
$$\mathfrak{G} = HF - BG,$$
$$\mathfrak{H} = FG - CH.$$

$$\mathfrak{Q} = \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H} \quad . \quad . \quad . \quad . \quad (5).$$

$$\overline{\omega\omega'} = \eta\zeta' - \eta'\zeta, \ \zeta\xi' - \zeta'\xi, \ \xi\eta' - \xi'\eta \quad . \quad . \quad (6).$$

We have

$$W(\omega_1, \omega_2, Q)\ W(\omega_3, \omega_4, Q) - W(\omega_1, \omega_3, Q)\ W(\omega_2, \omega_4, Q)$$
$$= W(\overline{\omega_1\omega_4}, \overline{\omega_2\omega_3}, \mathfrak{Q}) \ldots (7),$$

of which we may notice also the particular cases

$$W(\omega_1\omega_2, Q)\ W(\omega_3\omega_3, Q) - W(\omega_1\omega_3, Q)\ W(\omega_2, \omega_3, Q)$$
$$= W(\overline{\omega_1\omega_3}, \overline{\omega_2\omega_3}, \mathfrak{Q}). \ .(8),$$

$$W(\omega_1, \omega_1, Q)\ W(\omega_2, \omega_2, Q) - \{W(\omega_1\omega_2, Q)\}^2$$
$$= W(\overline{\omega_1\omega_2}, \overline{\omega_1\omega_2}, \mathfrak{Q}). \ .(9).$$

To which we may join the following formulæ, for the transformation of the function W.

Suppose

$$\omega_1 = ax_1 + a'y_1 + a''z_1, \ bx_1 + b'y_1 + b''z_1, \ cx_1 + c'y_1 + c''z_1 \ . \ .(10).$$
$$\omega_2 = ax_2 + a'y_2 + a''z_2, \ bx_2 + b'y_2 + b''z_2, \ cx_2 + c'y_2 + c''z_2.$$
$$\vdots$$

Then, writing $\qquad g = a, b, c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$

$$g' = a', b', c',$$
$$g'' = a'', b'', c''.$$

$$p_1 = x_1, y_1, z_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$
$$p_2 = x_2, y_2, z_2,$$
$$\vdots$$

$$\Theta = W(g, g, Q), \ W(g', g', Q), \ W(g'', g'', Q),$$
$$W(g', g'', Q), \ W(g'', g, Q), \ W(g, g', Q). \ .(13).$$

We have $$W(\omega_1, \omega_2, Q) = W(p_1, p_2, \Theta) \quad . \quad . \quad . \quad (14).$$
Similarly, writing
$$\Psi = W(\overline{g'g''}, \overline{g'g''}, \Theta),\ W(\overline{g''g}, \overline{g''g}, \Theta),\ W(\overline{gg'}, \overline{gg'}, \Theta) \ .\ .\ (15).$$
$$W(\overline{gg''}, \overline{g'g}, \Theta),\quad W(\overline{g''g'}, \overline{gg'}, \Theta),\quad W(\overline{g''g}, \overline{g'g''}, \Theta).$$
we have $$W(\overline{\omega_1\omega_4}, \overline{\omega_2\omega_3}, \Theta) = W(\overline{p_1p_4}, \overline{p_2p_3}, \Psi) \ldots .(16),$$
in which equations Θ may obviously be changed into Q.

Section 2.—Geometrical Applications.

Consider any three axes Ax, Ay, Az, and let λ, μ, ν be the cosines of the inclinations of these lines to each other.

Let Λ, M, N be the inclinations of the co-ordinate planes to each other; l, m, n, the inclination of the axes to the co-ordinate planes. Suppose, besides,

$$\begin{aligned}
\mathfrak{a} &= 1 - \lambda^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17).\\
\mathfrak{b} &= 1 - \mu^2,\\
\mathfrak{c} &= 1 - \nu^2,\\
\mathfrak{f} &= \mu\nu - \lambda,\\
\mathfrak{g} &= \nu\lambda - \mu,\\
\mathfrak{h} &= \lambda\mu - \nu.
\end{aligned}$$

$$k = 1 - \lambda^2 - \mu^2 - \nu^2 + 2\lambda\mu\nu \quad . \quad . \quad . \quad (18).$$

We have the following systems of equations:

$$\begin{aligned}
&\sqrt{(\mathfrak{bc})} \cos \Lambda = -\mathfrak{f}, && \sqrt{(\mathfrak{bc})} \sin \Lambda = \sqrt{(k)}, && \sqrt{(\mathfrak{a})} \sin l = \sqrt{(k)} \ldots (19).\\
&\sqrt{(\mathfrak{ca})} \cos \mathrm{M} = -\mathfrak{g}, && \sqrt{(\mathfrak{ca})} \sin \mathrm{M} = \sqrt{(k)}, && \sqrt{(\mathfrak{b})} \sin m = \sqrt{(k)}\\
&\sqrt{(\mathfrak{ab})} \cos \mathrm{N} = -\mathfrak{h}, && \sqrt{(\mathfrak{ab})} \sin \mathrm{N} = \sqrt{(k)}, && \sqrt{(\mathfrak{c})} \sin n = \sqrt{(k)}.
\end{aligned}$$

$$\begin{aligned}
\mathfrak{a} + \nu\mathfrak{h} + \mu\mathfrak{g} &= k, \quad . \quad . \quad . \quad . \quad . \quad (20).\\
\nu\mathfrak{a} + \mathfrak{h} + \lambda\mathfrak{g} &= 0,\\
\mu\mathfrak{a} + \lambda\mathfrak{h} + \mathfrak{g} &= 0,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{h} + \nu\mathfrak{b} + \mu\mathfrak{f} &= 0, \quad . \quad . \quad . \quad . \quad . \quad (21).\\
\nu\mathfrak{h} + \mathfrak{b} + \lambda\mathfrak{f} &= k,\\
\mu\mathfrak{h} + \lambda\mathfrak{b} + \mathfrak{f} &= 0.
\end{aligned}$$

$$\begin{aligned}
\mathfrak{g} + \nu\mathfrak{f} + \mu\mathfrak{c} &= 0, \quad . \quad . \quad . \quad . \quad . \quad (22).\\
\nu\mathfrak{g} + \mathfrak{f} + \lambda\mathfrak{c} &= 0,\\
\mu\mathfrak{g} + \lambda\mathfrak{f} + \mathfrak{c} &= k.
\end{aligned}$$

$$
\begin{aligned}
\mathfrak{b}\mathfrak{c} - \mathfrak{f}^2 &= ka \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23).\\
\mathfrak{c}\mathfrak{a} - \mathfrak{g}^2 &= kb,\\
\mathfrak{a}\mathfrak{b} - \mathfrak{h}^2 &= kc,\\
\mathfrak{g}\mathfrak{h} - \mathfrak{a}\mathfrak{f} &= kf,\\
\mathfrak{h}\mathfrak{f} - \mathfrak{b}\mathfrak{g} &= kg,\\
\mathfrak{f}\mathfrak{g} - \mathfrak{c}\mathfrak{h} &= kh,\\
\mathfrak{a}\mathfrak{b}\mathfrak{c} - \mathfrak{a}\mathfrak{f}^2 - \mathfrak{b}\mathfrak{g}^2 - \mathfrak{c}\mathfrak{h}^2 + 2\mathfrak{f}\mathfrak{g}\mathfrak{h} &= k^2 \quad . \quad . \quad . \quad (24).
\end{aligned}
$$

Imagine now a line AO, and let α, β, γ be the cosines of its inclinations to the three axes. Suppose also, θ, ϕ, χ being its inclinations to the co-ordinate planes, we write

$$a = \frac{\sin\theta}{\sqrt{(\mathfrak{a})}}, \quad b = \frac{\sin\phi}{\sqrt{(\mathfrak{b})}}, \quad c = \frac{\sin\chi}{\sqrt{(\mathfrak{c})}} \quad . \quad . \quad . \quad (25).$$

If we consider a point P on the line AO, at a distance unity from the origin, we see immediately, by considering the projections in the directions perpendicular to the co-ordinate planes, that the co-ordinates of this point are a, b, c. By projecting on the three axes and on the line AO, we then obtain the equations

$$
\begin{aligned}
\alpha &= \ a + \nu b + \mu c \quad . \quad . \quad . \quad . \quad . \quad . \quad (26).\\
\beta &= \nu a + \ b + \lambda c,\\
\gamma &= \mu a + \lambda b + c,\\
1 &= \alpha a + \beta b + \gamma c \quad . \quad . \quad . \quad . \quad . \quad . \quad (27).
\end{aligned}
$$

From which we obtain

$$
\begin{aligned}
ka &= \mathfrak{a}\alpha + \mathfrak{h}\beta + \mathfrak{g}\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (28).\\
kb &= \mathfrak{h}\alpha + \mathfrak{b}\beta + \mathfrak{f}\gamma,\\
kc &= \mathfrak{g}\alpha + \mathfrak{f}\beta + \mathfrak{c}\gamma,\\
1 &= \alpha a + \beta b + \gamma c \quad . \quad . \quad . \quad . \quad . \quad . \quad (29).
\end{aligned}
$$

And hence

$$1 = a^2 + b^2 + c^2 + 2\lambda bc + 2\mu ac + 2\nu ab \quad . \quad . \quad (30).$$

$$k = \mathfrak{a}\alpha^2 + \mathfrak{b}\beta^2 + \mathfrak{c}\gamma^2 + 2\mathfrak{f}\beta\gamma + 2\mathfrak{g}\alpha\gamma + 2\mathfrak{h}\alpha\beta \quad . \quad . \quad (31).$$

Or writing

$$
\begin{aligned}
a, b, c &= t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32).\\
\alpha, \beta, \gamma &= \tau \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33).\\
1, 1, 1, \lambda, \mu, \nu &= \mathrm{q} \quad . \quad . \quad . \quad . \quad . \quad . \quad (34).\\
\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{f}, \mathfrak{g}, \mathfrak{h} &= \mathfrak{q} \quad . \quad . \quad . \quad . \quad . \quad . \quad (35).
\end{aligned}
$$

we have the equations

$$
\begin{aligned}
1 &= W(t, t, \mathrm{q}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (36).\\
k &= W(\tau, \tau, \mathfrak{q}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (37).
\end{aligned}
$$

Let AO' be any other line, and δ its inclination to AO: a', β', γ', a', b', c', the quantities corresponding to a, β, γ, a, b, c, and similarly t', τ' to t, τ. We have of course

$$1 = W(t', t', \mathfrak{q}) \quad . \quad . \quad . \quad . \quad (38),$$

$$k = W(\tau', \tau', \mathfrak{q}) \quad . \quad . \quad . \quad . \quad (39).$$

We have besides, by projecting on the line AO', the equation

$$\cos\delta = a'a + \beta'b + \gamma'c \quad . \quad . \quad . \quad (40),$$

or the analogous one

$$\cos\delta = a'a + \beta'b + \gamma'c \quad . \quad . \quad . \quad (41).$$

From either of which we deduce

$$\cos\delta = aa'+bb'+cc'+\lambda.(bc'+b'c)+\mu.(ca'+c'a)+\nu.(ab'+a'b) \dots (42),$$

$$k\cos\delta = \mathfrak{a}aa'+\mathfrak{b}\beta\beta'+\mathfrak{c}\gamma\gamma'+\mathfrak{f}(\beta\gamma'+\beta'\gamma)+\mathfrak{g}(\gamma a'+\gamma'a)+\mathfrak{h}(a\beta+a'\beta) \dots (43);$$

which may otherwise be written

$$\cos\delta = W(t, t', \mathfrak{q}) \quad . \quad . \quad . \quad . \quad (44),$$

$$k\cos\delta = W(\tau, \tau', \mathfrak{q}) \quad . \quad . \quad . \quad . \quad (45).$$

Or again, observing the equations which connect the quantities t, τ,

$$\cos\delta = \frac{W(t, t', \mathfrak{q})}{\sqrt{\{W(t, t, \mathfrak{q}).W(t', t', \mathfrak{q})\}}} \quad . \quad . \quad (46),$$

$$\cos\delta = \frac{W(\tau, \tau', \mathfrak{q})}{\sqrt{\{W(\tau, \tau, \mathfrak{q}).W\tau', \tau', \mathfrak{q}\}}} \quad . \quad . \quad (47),$$

forms which, though more complicated, have certain advantages; for instance, we derive immediately from them the new equations

$$\sin\delta = \frac{\sqrt{\{W(\overline{tt'}, \overline{tt'}, \mathfrak{q})\}}}{\sqrt{\{W(t, t, \mathfrak{q})\,W(t', t', \mathfrak{q})\}}} \quad . \quad . \quad (48),$$

$$\sin\delta = \frac{\sqrt{\{kW(\overline{\tau\tau'}, \overline{\tau\tau'}, \mathfrak{q})\}}}{\sqrt{\{W(\tau, \tau, \mathfrak{q})\,W(\tau', \tau', \mathfrak{q})\}}} \quad . \quad . \quad (49).$$

Written more simply thus

$$\sin\delta = W(\overline{tt'}, \overline{tt'}, \mathfrak{q}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50),$$

$$\sqrt{k}\sin\delta = \sqrt{\{W(\overline{\tau\tau'}, \overline{\tau\tau'}, \mathfrak{q})\}} \quad . \quad . \quad . \quad . \quad (51),$$

to which we may join

$$\cot\delta = \frac{W(t, t', \mathfrak{q})}{\sqrt{\{W(\overline{tt'}, \overline{tt'}, \mathfrak{q})\}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (52),$$

$$\sqrt{k} \cot \delta = \frac{W(\tau, \tau', \mathfrak{q})}{\sqrt{\{W(\overline{\underline{\tau\tau'}}, \overline{\underline{\tau\tau'}}, \mathfrak{q})\}}} \quad . \quad . \quad . \quad . \quad (53).$$

Section 3.—On Spherical Coordinates.

Consider the points X, Y, Z, on the surface of a sphere, as the intersections of the three axes of the preceding section, with a sphere having its centre in the origin. It is evident that λ, μ, ν are the cosines of the sides of the spherical triangle XYZ, $\Lambda, \mathrm{M}, \mathrm{N}$ are its sides, l, m, n are the perpendiculars from the angles upon the opposite sides. Let P be the point where the line AO intersects the sphere: the position of the point P may be determined by means of the ratios $\xi : \eta : \zeta$, supposing ξ, η, ζ denote quantities proportional to the α, β, γ of the preceding section, i.e.

$$\xi : \eta : \zeta = \cos PX : \cos PY : \cos PZ \quad . \quad . \quad (54).$$

Or again, by means of the ratios $x : y : z$, supposing x, y, z denote quantities proportional to the a, b, c of the preceding section, i.e.

$$x : y : z = \frac{\sin Px}{\sin X} : \frac{\sin Py}{\sin Y} : \frac{\sin Pz}{\sin Z} \quad . \quad . \quad (55),$$

(Px, Py, Pz are the perpendiculars from P on the sides of the spherical triangle XYZ).

Which equations may be otherwise written,

$$\frac{x \sin X}{y \sin Y} = \frac{\sin PZY}{\sin PZX} \quad . \quad . \quad . \quad . \quad (56).$$

$$\frac{y \sin Y}{z \sin Z} = \frac{\sin PXZ}{\sin PXY},$$

$$\frac{z \sin Z}{x \sin X} = \frac{\sin PYX}{\sin PYZ}.$$

The ratios $\xi : \eta : \zeta$, or $x : y : z$, are termed the spherical coordinate ratios of the point P. The two together may be termed conjoint systems: the first may be termed the cosine system, and the second the sine system. The coordinates of the two systems are evidently connected by

$$\xi : \eta : \zeta = x + \nu y + \mu z : \nu x + y + \lambda z : \mu x + \lambda y + z \; \ldots \; (57),$$

or $\quad x : y : z = \mathfrak{a}\xi + \mathfrak{h}\eta + \mathfrak{g}\zeta : \mathfrak{h}\xi + \mathfrak{b}\eta + \mathfrak{f}\zeta : \mathfrak{g}\xi + \mathfrak{f}\eta + \mathfrak{c}\zeta \; \ldots \; (58).$

The systems may conveniently be represented by the single letters

$$\varpi = \xi, \eta, \zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad (59),$$

$$p = x, y, z \quad . \quad . \quad . \quad . \quad . \quad . \quad (60).$$

[redacted]

Fundamental formula of spherical coordinates; distance of two points.

Let P, P' be the points, δ their distance, ω, p the conjoint coordinate systems of the first point, ω', p' of the second; we have obviously

$$\cos\delta = \frac{W(p, p', q)}{\sqrt{\{W(p, p, q)\, W(p', p', q)\}}} \quad \ldots\ldots (61),$$

$$\sin\delta = \frac{\sqrt{\{W(\overline{pp'}, \overline{pp'}, \mathfrak{q})\}}}{\sqrt{\{W(p, p, q)\, W(p', p', q)\}}},$$

$$\cot\delta = \frac{W(p, p', q)}{\sqrt{\{W(\overline{pp'}, \overline{pp'}, \mathfrak{q})\}}};$$

or
$$\cos\delta = \frac{W(\omega, \omega', \mathfrak{q})}{\sqrt{\{W(\omega, \omega, \mathfrak{q})\, W(\omega', \omega', \mathfrak{q})\}}} \quad \ldots\ldots (62).$$

$$\frac{1}{\sqrt{k}}\sin\delta = \frac{\sqrt{\{W(\overline{\omega\omega'}, \overline{\omega\omega'}, q)\}}}{\sqrt{\{W(\omega, \omega, \mathfrak{q})\, W(\omega', \omega', \mathfrak{q})\}}},$$

$$\sqrt{k}\cot\delta = \frac{W(\omega, \omega')\, \mathfrak{q})}{\sqrt{\{W(\overline{\omega\omega'}, \overline{\omega\omega'}, q)\}}}.$$

Equation of a great Circle.

Let the conjoint coordinate systems of the pole be

$$e = a, b, c \quad \ldots\ldots\ldots\ldots (63),$$
$$\epsilon = \alpha, \beta, \gamma \quad \ldots\ldots\ldots\ldots (64).$$

Then, expressing that the distance of any point P in the locus from the pole is equal to 90°, we have immediately the equations

$$W(p, e, q) = 0 \quad \ldots\ldots\ldots\ldots (65),$$
$$W(\omega, \epsilon, \mathfrak{q}) = 0 \quad \ldots\ldots\ldots\ldots (66),$$

which may otherwise be written in the forms

$$a\xi + b\eta + c\zeta = 0 \quad \ldots\ldots\ldots\ldots (67),$$
$$\alpha x + \beta y + \gamma\zeta = 0 \quad \ldots\ldots\ldots\ldots (68),$$

or the equation of a great circle is linear in either coordinate system. Conversely, any linear equation belongs to a great circle.

Suppose the equation given in the form

$$A\xi + B\eta + C\zeta = 0 \quad \ldots\ldots\ldots\ldots (69);$$

or by an equation between cosine coordinate ratios:—The sine system for the pole is given by

$$e = A, B, C \quad \ldots\ldots\ldots\ldots (70),$$

and the cosine system by

$$\epsilon = A + \nu B + \mu C,\quad \nu A + B + \lambda C,\quad \mu A + \lambda B + C \ldots (71).$$

Suppose the circle given by an equation between sine coordinates, or in the form

$$\mathrm{A}x + \mathrm{B}y + \mathrm{C}z = 0 \;\ldots\ldots\ldots\ldots\; (72).$$

The cosine system of coordinates for the pole is given by

$$\epsilon = \mathrm{A},\ \mathrm{B},\ \mathrm{C} \ldots\ldots\ldots\ldots\ldots (73),$$

and the sine system by

$$e = \mathfrak{a}\mathrm{A} + \mathfrak{h}\mathrm{B} + \mathfrak{g}\mathrm{C},\quad \mathfrak{h}\mathrm{A} + \mathfrak{b}\mathrm{B} + \mathfrak{f}\mathrm{C},\quad \mathfrak{g}\mathrm{A} + \mathfrak{f}\mathrm{B} + \mathfrak{c}\mathrm{C} \ldots (74).$$

It is hardly necessary to observe, that if

$$A\xi + B\eta + C\zeta = 0 \quad \ldots\ldots \quad (75),$$

$$Ax + By + Cz = 0 \quad \ldots\ldots \quad (76),$$

represent the same great circle,

$$\mathrm{A} : \mathrm{B} : \mathrm{C} = A + \nu B + \mu C : \nu A + B + \lambda C : \mu A + \lambda B + C \ldots (77),$$

$$A : B : C = \mathfrak{a}\mathrm{A} + \mathfrak{h}\mathrm{B} + \mathfrak{g}\mathrm{C} : \mathfrak{h}\mathrm{A} + \mathfrak{b}\mathrm{B} + \mathfrak{f}\mathrm{C} : \mathfrak{g}\mathrm{A} + \mathfrak{f}\mathrm{B} + \mathfrak{c}\mathrm{C} \ldots (78).$$

Inclination of two great Circles.

Let the equations of these be

$$\begin{cases} A\xi + B\eta + C\zeta = 0 & \ldots\quad (79), \\ \text{or}\ \ \mathrm{A}x + \mathrm{B}y + \mathrm{C}z = 0 & \ldots\quad (80), \end{cases}$$

$$\begin{cases} A'\xi + B'\eta + C'\zeta = 0 & \ldots\quad (81), \\ \text{or}\ \ \mathrm{A}'x + \mathrm{B}'y + \mathrm{C}'z = 0 & \ldots\quad (82), \end{cases}$$

and let e, ϵ, have the same values as above, and e', ϵ', corresponding ones. To obtain the inclination of the two circles, we have only, in the formulæ given above for the distance of two points, to change p, p', ω, ω', into $e, e', \epsilon, \epsilon'$.

The distance of a point from a given circle may be found with equal facility; for this is evidently the compliment of the distance of the point from the pole of the circle. In like manner we may find the condition that two great circles intersect at right angles, &c.

There are evidently a whole class of formulæ, not by any means peculiar to the present system of coordinates, such as

$$Ax + By + Cz - s\,.\,(A'x + B'y + C'z) \quad .\ .\ (83),$$

for the equation of a great circle subjected to pass through the points of intersection of

$$Ax + By + Cz = 0,\qquad A'x + B'y + C'z = 0.$$

Again,

$$\begin{vmatrix} x, & y, & z \\ a, & b, & c \\ a', & b', & c' \end{vmatrix} = 0 \;.\;.\;.\;.\;.\; (84),$$

for the equation of the great circle which passes through the points given by the sine systems $a : b : c$ and $a' : b' : c'$, &c., and which are obtained so easily that it is not worth while writing down any more of them.

Transformation of Coordinates.

Let X_1, Y_1, Z_1, be the new points of reference, and suppose X_1 is given by the conjoint systems $e = a, b, c$, $\epsilon = \alpha, \beta, \gamma$; and similarly Y_1, Z_1, by the analogous systems $e', \epsilon' - e'', \epsilon''$.

Suppose, as before, P is given by one of the systems ω, p; and let ω_1, p_1 be the new systems which determine the position of P with reference to X_1, Y_1, Z_1.

In the first place, λ_1, μ_1, ν_1, are given by the formulæ

$$\lambda_1 = \frac{W(e', e'', q)}{\sqrt{\{W(e', e', q)\, W(e'', e'', q)\}}} = \frac{W(\epsilon', \epsilon'', \mathfrak{q})}{\sqrt{\{W(\epsilon', \epsilon', \mathfrak{q})\, W(\epsilon'', \epsilon'', \mathfrak{q})\}}} \quad \ldots\ldots (85),$$

$$\mu_1 = \frac{W(e'', e, q)}{\sqrt{\{W(e'', e'', q)\, W(e, e, q)\}}} = \frac{W(\epsilon'', \epsilon, \mathfrak{q})}{\sqrt{\{W(\epsilon'', \epsilon'', \mathfrak{q})\, W(\epsilon, \epsilon, \mathfrak{q})\}}},$$

$$\nu_1 = \frac{W(e, e', q)}{\sqrt{\{W(e, e, q)\, W(e', e', q)\}}} = \frac{W(\epsilon, \epsilon', \mathfrak{q})}{\sqrt{\{W(\epsilon, \epsilon, \mathfrak{q})\, W(\epsilon', \epsilon', \mathfrak{q})\}}}.$$

The system ω_1 is evidently given immediately by

$$\xi_1 : \eta_1 : \zeta_1 = \frac{W(e, p, q)}{\sqrt{\{W(e, e, q)\}}} : \frac{W(e', p, q)}{\sqrt{\{W(e', e', q)\}}} : \frac{W(e'', p, q)}{\sqrt{\{W(e'', e'', q)\}}} \ldots (86)$$

$$= \frac{W(\epsilon, \omega, \mathfrak{q})}{\sqrt{\{W(\epsilon, \epsilon, \mathfrak{q})\}}} : \frac{W(\epsilon', \omega, \mathfrak{q})}{\sqrt{\{W(\epsilon', \epsilon', \mathfrak{q})\}}} : \frac{W(\epsilon'', \omega, \mathfrak{q})}{\sqrt{\{W(\epsilon'', \epsilon'', \mathfrak{q})\}}} \ldots (87).$$

And from these we may obtain the system p_1, by means of the formulæ

$$x_1 : y_1 : z_1 = \mathfrak{a}_1\xi_1 + \mathfrak{h}_1\eta_1 + \mathfrak{g}_1\zeta_1 : \mathfrak{h}_1\xi_1 + \mathfrak{b}_1\eta_1 + \mathfrak{f}_1\zeta_1 : \mathfrak{g}_1\xi_1 + \mathfrak{f}_1\eta_1 + \mathfrak{c}_1\zeta_1 \ldots (88).$$

This requires some further development however. We must in the first place form the system $\mathfrak{a}_1$, $\mathfrak{b}_1$, $\mathfrak{c}_1$, $\mathfrak{f}_1$, $\mathfrak{g}_1$, $\mathfrak{h}_1$: this is done immediately from the formulæ of Sect. 2, and we have

$$\mathfrak{a}_1 = \frac{W(\overline{e'e''}, \overline{e'e''}, \mathfrak{q})}{W(e', e', q)\, W(e'', e'', q)} = \frac{k\, W(\overline{\epsilon'\epsilon''}, \overline{\epsilon'\epsilon''}, q)}{W(\epsilon', \epsilon', \mathfrak{q})\, W(\epsilon'', \epsilon'', q)} \ldots (89),$$

$$\mathfrak{b}_1 = \frac{W(\overline{e''e}, \overline{e''e}, \mathfrak{q})}{W(e'', e'', q)\, W(e'', e, q)} = \frac{k\, W(\overline{\epsilon''\epsilon}, \overline{\epsilon''\epsilon}, q)}{W(\epsilon'', \epsilon'', \mathfrak{q})\, W(\epsilon, \epsilon, \mathfrak{q})},$$

[redacted]

$$\mathfrak{e}_1 = \frac{W(\overline{ee'}, \overline{ee'}, \mathfrak{q})}{W(e, e, \mathrm{q})\, W(e', e', \mathrm{q})} = \frac{kW(\overline{\epsilon\epsilon'}, \overline{\epsilon\epsilon'}, \mathrm{q})}{W(\epsilon, \epsilon, \mathfrak{q})\, W(\epsilon', \epsilon', \mathfrak{q})},$$

$$\mathfrak{f}_1 = \frac{W(\overline{e''e}, \overline{ee'}, \mathfrak{q})}{W(e, e, \mathrm{q})\sqrt{\{W(e', e', \mathrm{q})\, W(e'', e'', \mathrm{q})\}}}$$

$$= \frac{kW(\overline{\epsilon''\epsilon}, \overline{\epsilon\epsilon'}, \mathrm{q})}{W(\epsilon, \epsilon, \mathfrak{q})\sqrt{\{W(\epsilon', \epsilon', \mathfrak{q})\, W(\epsilon'', \epsilon'', \mathfrak{q})\}}},$$

$$\mathfrak{g}_1 = \frac{W(\overline{ee'}, \overline{e'e''}, \mathfrak{q})}{W(e', e', \mathrm{q})\sqrt{\{W(e'', e'', \mathrm{q})\, W(e, e, \mathrm{q})\}}}$$

$$= \frac{kW(\overline{\epsilon\epsilon'}, \overline{\epsilon'\epsilon''}, \mathrm{q})}{W(\epsilon', \epsilon', \mathfrak{q})\sqrt{\{W(\epsilon'', \epsilon'', \mathfrak{q})\, W(\epsilon, \epsilon, \mathfrak{q})\}}},$$

$$\mathfrak{h}_1 = \frac{W(\overline{e'e''}, \overline{e''e}, \mathfrak{q})}{W(e'', e'', \mathrm{q})\sqrt{\{W(e, e, \mathrm{q})\, W(e', e', \mathrm{q})\}}}$$

$$= \frac{kW(\overline{\epsilon'\epsilon''}, \overline{\epsilon''\epsilon}, \mathrm{q})}{W(\epsilon'', \epsilon'', \mathfrak{q})\sqrt{\{W(\epsilon, \epsilon, \mathfrak{q})\, W(\epsilon', \epsilon', \mathfrak{q})\}}}.$$

$$x_1 : y_1 : z_1 = \sqrt{\{W(e, e, \mathrm{q})\}}\,\{W(e, p, \mathrm{q})\, W(\overline{e'e''}, \overline{e'e''}, \mathfrak{q})$$
$$+ W(e', p, \mathrm{q})\, W(\overline{e'e''}, \overline{e''e}, \mathfrak{q}) + W(e'', p, \mathfrak{q})\, W(\overline{e'e''}, \overline{ee'}, \mathfrak{q})\} \ldots (90),$$
$$: \sqrt{\{W(e', e', \mathrm{q})\}}\,\{W(e, p, \mathrm{q})\, W(\overline{e''e}, \overline{e'e''}, \mathfrak{q})$$
$$+ W(e', p, \mathrm{q})\, W(\overline{e''e}, \overline{e''e}, \mathfrak{q}) + W(e'', p, \mathrm{q})\, W(\overline{e''e}, \overline{ee'}, \mathrm{q})\}$$
$$: \sqrt{\{W(e'', e'', \mathrm{q})\}}\,\{W(e, p, \mathrm{q})\, W(\overline{ee'}, \overline{e'e''}, \mathfrak{q})$$
$$+ W(e', p, \mathrm{q})\, W(\overline{ee'}, \overline{e''e}, \mathfrak{q}) + W(e'', p, \mathrm{q})\, W(\overline{ee'}, \overline{ee'}, \mathfrak{q})\};$$

which may be reduced to the very simple form

$$x_1 : y_1 : z_1 = \sqrt{\{W(e, e, \mathrm{q})\}}\, W(\overline{e'e''}, \omega, \mathfrak{q}) \ldots . (91),$$
$$: \sqrt{\{W(e', e', \mathrm{q})\}}\, W(\overline{e''e}, \omega, \mathfrak{q}),$$
$$: \sqrt{\{W(e'', e'', \mathrm{q})\}}\, W(\overline{ee'}, \omega, \mathfrak{q}).$$

And in like manner we obtain

$$x_1 : y_1 : z_1 = \sqrt{\{W(\epsilon, \epsilon, \mathfrak{q})\}}\, W(\overline{\epsilon'\epsilon''}, p, \mathrm{q}) \ldots . (92),$$
$$: \sqrt{\{W(\epsilon', \epsilon', \mathfrak{q})\}}\, W(\overline{\epsilon''\epsilon}, p, \mathrm{q}),$$
$$: \sqrt{\{W(\epsilon'', \epsilon'', \mathfrak{q})\}}\, W(\overline{\epsilon\epsilon'}, p, \mathrm{q}).$$

It will be as well to indicate the steps of this reduction. Consider the quantity in { } in the first line of the equation which gives the ratios $x_1 : y_1 : z_1$; and suppose for a moment $\overline{e'e''} = l, m, n$, &c., selecting the portion of the expression which is multiplied by $\mathfrak{a}$, this is

$$\mathfrak{a}l \,.\, \{l(a\xi + b\eta + c\zeta) + l'(a'\xi + b'\eta + c'\zeta) + l''(a''\xi + b''\eta + c''\zeta)\}.$$

Or, since

$$la + l'a' + l''a'' = \overline{ee'e''}, \quad lb + l'b' + l''b'' = 0, \quad lc + l'c' + l''c'' = 0,$$

this reduces itself to $\overline{ee'e''}$, $\mathfrak{a}l\xi$, which is a term of

$$\overline{ee'e''}\, W(\overline{e'e''}, \omega, \mathfrak{q});$$

and by comparing the remaining terms in the same manner, it would be seen that the whole reduces itself to

$$\overline{ee'e''}\, W(\overline{e'e''}, \omega, \mathfrak{q});$$

whence the formulæ in question.

The formulæ (86), (87), (91), (92), completely resolve the problem of the transformation of coordinates; they determine respectively p_1 from p or ω, ω_1 from p or ω.

To complete the present part of the subject we may add the following formulæ.

Suppose
$$\begin{aligned} x_1 : y_1 : z_1 &= \mathrm{a}x_1 + \mathrm{a}'y_1 + \mathrm{a}''z_1 \quad \ldots\ldots\ldots\ldots (93),\\ &: \mathrm{b}x_1 + \mathrm{b}'y_1 + \mathrm{b}''y_1,\\ &: \mathrm{c}x_1 + \mathrm{c}'y_1 + \mathrm{c}''z_1, \end{aligned}$$

which we see from the preceding formulæ is the form of the relation between the systems p_1 and p. And suppose, as before, λ_1, μ_1, ν_1 are the cosines of the distances of the new points of reference $\mathrm{X}_1, \mathrm{Y}_1, \mathrm{Z}_1$.

We can immediately determine the relations that must exist between these coefficients, in order that they may actually denote such a transformation. For this purpose write

$$\begin{aligned} \mathrm{a}, \mathrm{b}, \mathrm{c} &= j \quad \ldots\ldots\ldots\ldots (94),\\ \mathrm{a}', \mathrm{b}', \mathrm{c}' &= j',\\ \mathrm{a}'', \mathrm{b}'', \mathrm{c}'' &= j''. \end{aligned}$$

Then the distance between the point P and any other point P' is given by the formula

$$\cos\delta = \frac{W(p, p', \mathrm{q})}{\sqrt{\{W(p, p, \mathrm{q})\, W(p', p', \mathrm{q})\}}}$$

$$= \frac{W(p_1, p_1', \Theta)}{\sqrt{\{W(p_1, p_1, \Theta)\, W(p_1', p_1', \Theta)\}}} \quad \ldots . \ (95),$$

where $\Theta = W(j, j, \text{q}),\; W(j', j', \text{q}),\; W(j'', j'', \text{q}),$

$$W(j', j'', \text{q}),\; W(j'', j, \text{q}),\; W(j, j', \text{q}) \;\ldots\ldots\; (96).$$

But we must evidently have

$$\cos\delta = \frac{W(p_1, p_1', \text{q}_1)}{\sqrt{\{W(p_1, p_1, \text{q}_1)\, W(p_1', p_1', \text{q}_1)\}}} \;\ldots\ldots\; (97),$$

or the quantities Θ must be proportional to the quantities q, i.e.

$$W(j, j, \text{q}) : W(j', j', \text{q}) : W(j'', j'', \text{q}) : W(j', j'', \text{q})$$
$$: W(j'', j, \text{q}) : W(j'', j, \text{q}) \;\ldots\ldots\; (98),$$
$$= 1 : 1 : 1 : \lambda_1 : \mu_1 : \nu_1.$$

And in precisely the same manner, if instead of x, y, z, x_1, y_1, z_1, in the above formulæ, we had had $\xi, \eta, \zeta : \xi_1, \eta_1, \zeta_1$, the result would have been

$$W(j, j, \mathfrak{q}) : W(j', j', \mathfrak{q}) : W(j'', j'', \mathfrak{q}) : W(j', j'', \mathfrak{q})$$
$$: W(j'', j, \mathfrak{q}) : W(j'', j, \mathfrak{q})$$
$$= \mathfrak{a} : \mathfrak{b} : \mathfrak{c} : \mathfrak{f} : \mathfrak{g} : \mathfrak{h} \;\ldots\ldots\ldots\ldots\; (99).$$

It is hardly necessary to remark, that throughout the preceding formulæ an expression, such as $W(p, p', \text{q})$, is proportional to either of the quantities

$$x\xi' + y\eta' + z\zeta' \text{ or } x'\xi + y'\eta + z'\zeta,$$

and may be changed into one of these multiplied by an arbitrary constant; which may be always made to disappear by a corresponding change in another quantity of the same form: thus, for instance,

$$\frac{W(p, p', \text{q})}{W(p, p, \text{q})} = \frac{x'\xi + y'\eta + z'\zeta}{x\xi + y\eta + z\zeta} \;\ldots\ldots\; (100);$$

but these forms being unsymmetrical, it is better in general not to introduce them.

All the preceding expressions simplify exceedingly, reducing themselves in fact to the ordinary formulæ for the transformation of rectangular coordinates in Geometry of three dimensions, for the case where the triangle XYZ has its sides and angles right angles. As this presents no difficulty, I shall not enter upon it at present.

NOTE ON INDUCED MAGNETISM IN A PLATE.

By William Thomson, B.A., Fellow of St Peter's College.

If a plate of soft iron be submitted to the action of a magnet of any kind, it immediately becomes magnetized "by induction;" and the effects of this are exhibited in the attraction or repulsion it exercises upon small magnetic bodies in its neighbourhood. The determination of these effects, from the elementary laws of magnetic induction, is a problem of considerable practical interest. In the case of a plate bounded by infinite parallel planes, I have succeeded in obtaining a complete solution of a very simple nature, by means of a principle which will be developed in a future paper. The object of the present note is to compare this solution with a formula given by Green in his *Essay on Electricity and Magnetism*, as an approximate result, but which appears to be inadmissible.

Let the influencing magnet, which may be of any form and size, and magnetized in any manner, be denoted by Q; and let us suppose it to be held *behind* the plate of soft iron. The solution which I have obtained enables us to find the total magnetic action on a point, P, situated in any position, either within or without the plate; but at present I shall only state the result when P is *before* the plate. In this case the actual magnetic effect on P may be produced by supposing Q and the plate to be removed, and a certain imaginary series of magnets Q', Q_1, Q_2, &c. to be substituted, the system being constructed thus. Each of the imaginary magnets is equal and similar to Q, and similarly magnetized; Q' occupies the place of Q, and the others are similarly placed behind it, along a line perpendicular to the plate, the distance between corresponding points of each consecutive pair being equal to twice the thickness of the plate. The intensities of the successive magnets decrease in a geometrical progression, of which the common ratio is m^2, (a quantity measuring the inductive capacity for magnetism of the plate,) commencing with that of Q', which is equal to $1 - m^2$, if the intensity of Q be unity. It is hardly necessary to point out the analogy between this and the corresponding result in optics, in which the illumination produced through a plate of glass, by a candle, is found to be due to the candle itself, with diminished brightness, and to a row of images behind it, with intensities decreasing in a geometrical progression, which arise from successive internal reflections.

If the iron plate be infinitely thin, all the *images*, Q_1, Q_2, &c. will coincide with Q'; and, since the sum of their intensities is unity, the total effect will be the same as that of Q, which will therefore be unaffected by the interposition of the screen. The same will be the case if the distance of Q be infinitely great, and the thickness of the screen finite; but in this case, at least as far as the present result can shew us, the dimensions of the planes which bound the plate must be infinitely great compared with the distance of Q.

The result which I have stated is applicable also to the imaginary case in which, instead of being a magnet, Q is a mass of positive or negative magnetism.* Thus, let Q be a unit of positive magnetism collected in a point, which case is investigated by Green. To express the action analytically, let Q be taken as origin of coordinates, a line perpendicular to the plate as axis of x, and the plane through this line, and P, as plane of (x, y). Then denoting by a the thickness of the plate, and considering Q as a positive unit of matter, we shall have, for the total potential at P, due to Q and the plate,

$$F=(1-m^2)\left\{\frac{1}{(x^2+y^2)^{\frac{1}{2}}}+\frac{m^2}{\{(x+2a)^2+y^2\}^{\frac{1}{2}}}+\frac{m^4}{\{(x+4a)^2+y^2\}^{\frac{1}{2}}}+\text{\&c.}\right\} \quad \ldots\ldots (1).$$

For all magnetic bodies m is between 0 and 1, the former limit being its value when the inductive capacity for magnetism is nothing, and the latter being never attained, though it is approached in such bodies as iron, of which the inductive capacity is great. In the extreme case of $m=1$, the laws of induction in a magnetic body degenerate into those of electrical equilibrium on the surface of a conductor of electricity. If in the expression for F we put $m=1$, one of the factors vanishes and the other becomes infinite, but the ultimate value of the product is nothing, which shews that the effect of the plate is to destroy all action behind it. This we know to be the case when an infinite conducting screen of any form is placed before an electrified body.

* This expression does not imply any hypothesis of a magnetic matter or of a fluid or fluids, but it is merely used for brevity in consequence of the principle established by Coulomb, Poisson, and Ampère, that the action of a magnetized body of any kind, or of a collection of electric "closed currents," may always be represented by an imaginary positive and negative distribution of matter, of which the whole mass is algebraically nothing. By an element of positive or negative magnitude, we merely mean a portion of this imagined matter.

In the case when the plate is of iron, the value of m is nearly unity. Hence, as the series is multiplied by $1 - m^2$, it might be imagined that, if we "neglect small quantities of the order $(1 - g)$ compared with those which are retained," $(1 - g$ being, in Green's notation, a quantity of the same order as $1 - m)$, an approximate result would be obtained by putting $m = 1$ in the successive terms of the series within the vinculum. And it is thus that Green, having, in the investigation, neglected quantities multiplied by $(1 - g)^2$, arrives at the result,

$$F = \frac{4(1-g)}{3}\left\{\frac{1}{(x^2+y^2)^{\frac{1}{2}}} + \frac{1}{\{(x+2a)^2+y^2\}^{\frac{1}{2}}} + \frac{1}{\{(x+4a)^2+y^2\}^{\frac{1}{2}}} + \&c.\right\}$$

As, however, this series has an infinite sum, it is clear that no value of m can be sufficiently near to unity to render the approximation admissible. If instead of Q we were to substitute a magnet, or any collection of positive and negative particles, such that the sum of the masses is zero, the series for the potential, deduced from Green's expression, would converge: and the same remark is applicable to the series which would be found for the *attraction* of the system on a point beyond the screen, even when Q is a positive point, by differentiating the expression for F. Notwithstanding this, the approximation is still inadmissible; since, if we expand the rigorous expression in either case in ascending powers $(1 - m)$, we find that, though the first term is finite, the coefficients of all the terms which follow it are infinite.

Although the method by which I obtained the rigorous solution is quite distinct from that followed by Green, being independent of any mathematical process, it may be satisfactory to shew that the result can be deduced from his own analysis, and even with greater ease than his solution is obtained, after making unnecessary approximation.

By a very remarkable investigation, in which he extends Laplace's well-known analysis for spherical coordinates to the case when the radius of the sphere becomes infinite, Green arrives (*Essay on Electricity*, p. 64) at the following expression for the total potential at P, due to the positive unit of matter Q, and to the interposed plate, before making any approximation:

$$F = \frac{8}{\pi}(1-g)(1+2g)\int_0^\infty \frac{d\gamma\,\epsilon^{-\gamma x}}{(2+g)^2 - 9g^2\epsilon^{-2\gamma a}}\int_0^1 \frac{d\beta}{(1-\beta^2)^{\frac{1}{2}}}\cos(\beta\gamma y).$$

Let $m = \dfrac{3g}{2+g}$. Then we have, by expansion, and by changing the order of the integration,

$$F = \frac{2}{\pi}(1-m^2)\int_0^1 \frac{d\beta}{(1-\beta^2)^{\frac{1}{2}}}$$

$$\int_0^\infty d\gamma \, . \, \epsilon^{-\gamma x}(1 + m^2\epsilon^{-2\gamma a} + m^4\epsilon^{-4\gamma a} + \&c.) \cos(\beta\gamma y)$$

$$= \frac{2}{\pi}(1-m^2)\int_0^1 \frac{d\beta}{(1-\beta^2)^{\frac{1}{2}}}$$

$$\left\{\frac{x}{x^2+\beta^2y^2} + \frac{m^2(x+2a)}{(x+2a)^2+\beta^2y^2} + \frac{m^4(x+4a)}{(x+4a)^2+\beta^2y^2} + \&c.\right\}$$

$$= \frac{2}{\pi}(1-m^2)\,\Sigma\int_0^{\frac{1}{2}\pi} \frac{m^{2i}x_i\,d\theta}{x_i^2+y^2\sin^2\theta}, \quad \text{where } x_i = x+2ia,$$

$$= (1-m^2)\,\Sigma\,\frac{m^{2i}}{(x_i^2+y^2)^{\frac{1}{2}}},$$

which agrees with the expression given above.

St. Peter's College, Oct. 14th, 1845.

ON THE VARIATION OF ELEMENTS IN THE PLANETARY THEORY.

By Hugh Blackburn, B.A. Trinity College.

The six quantities which determine the form and position of a planet's elliptic orbit and the planet's place at the period from which time is measured, are called the elements of its orbit. When they are known, Kepler's second and third laws are sufficient to determine the planet's place at any subsequent period. In nature, owing to the action of the planets on one another, their orbits are not accurately fixed ellipses; but the disturbing force being small, they may conveniently be represented as moving in variable ellipses, whose elements at any instant are determined by the condition, that the actual motion shall be the same as if they moved undisturbed in those ellipses. This method is naturally suggested by observation, but was first treated analytically by Lagrange.

Let xyz, $x'y'z'$, be the coordinates, with reference to three rectangular axes fixed in space, of the disturbed and disturbing planets respectively: and let

$$R = \frac{m'}{\{(x-x')^2+(y-y')^2+(z-z')^2\}^{\frac{1}{2}}} - \frac{m'(xx'+yy'+zz')}{(x'^2+y'^2+z'^2)^{\frac{3}{2}}}.$$

Then the equations of the disturbed planet's motion are

$$\left.\begin{aligned}\frac{d^2x}{dt^2}+\frac{\mu x}{r^3}&=\frac{dR}{dx}\\ \frac{d^2y}{dt^2}+\frac{\mu y}{r^3}&=\frac{dR}{dy}\\ \frac{d^2z}{dt^2}+\frac{\mu z}{r^3}&=\frac{dR}{dz}\end{aligned}\right\}\cdots\cdots\cdots\cdots\text{(I)},$$

which coincide with the equations of undisturbed motion when we put $R=0$. The six arbitrary constants introduced by the integration of these equations, when $R=0$, are the elements or functions of them, and the variable elements in the case of disturbed motion are to be found from the six relations given by making the expressions for $x, y, z, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, the same as if the disturbing force were to cease. If the values of x, y, z, x', y', z', in terms of t and the elements, be substituted in R, we have the following remarkable theorem. The variations (or differential coefficients with respect to t) of all the elements may be rigorously expressed as linear functions of the partial differential coefficients of R with respect to the elements, multiplied by functions of the elements themselves not involving t explicitly. This was proved from general considerations by Lagrange, and afterwards extended to the case of any mechanical problem, nearly at the same time that Laplace independently deduced similar expressions for the variations from the consideration of elliptic motion.*

It is well, for the sake of clearness, to distinguish three kinds of change which the orbit of the planet may undergo: the form and size of the ellipse may be altered; the orbit (or rather the lines of perihelion and epoch) may revolve in its plane; and the plane of the orbit may revolve about the radius vector of the planet as instantaneous axis. The two former are due to the resolved part of the disturbing force in the plane of the orbit, and the values of the variations of the elements corresponding will be the same, whether the force perpendicular to the plane of the orbit be taken account of or not. This will apply to the axis major and eccentricity, and also to the longitude of perihelion and epoch, if measured from a line which remains fixed relatively to the plane

* See Lagrange in the *Memoirs of the Institute* for 1808, 1809, and *Mech. Analyt.* last edit. sect. V. and VII. Part. II.; also Poisson in the *Journal de l'Ec. Polytech.* Cahier XVI.

of the orbit. In this case the expressions deduced in Airy's Tracts* will still be rigorously true, when the motion of the plane of the orbit is considered. It is more convenient, however, to measure these angles from the line of nodes, when they will be affected by the motion of the plane, and consequently the same expressions will not apply. The necessary modifications for this way of measuring the angles will be given hereafter. The motion of the plane of the orbit, and the corresponding variations of node and inclination, which are due to the disturbing force perpendicular to the plane of the orbit, are not considered in Airy's Tract. In Pratt's *Mechanics* the rigorous expressions are given, but the investigation of them is only approximate, and seems no simpler than a rigorous method.

These few pages will, I hope, be found useful as a supplement to Airy's Tract, and a substitute for the investigation in Pratt.

For distinctness, the plane of the disturbed planet's orbit is supposed to intersect the planes of xy and yz between the positive axes, and the positive direction of measuring angles in the plane of the orbit is from the intersection with xy to that with yz.

Let l, m, n, be the direction-cosines of the plane of the orbit, and Ω the longitude of node measured from the axis of x in the plane of xy, i the inclination of the plane of the orbit to that of xy, so that we have

$$l = \sin i \sin \Omega, \quad m = -\sin i \cos \Omega, \quad n = \cos i.$$

Let H be twice the area described on the plane of the orbit in a unit of time, and therefore lH, mH, nH, the corresponding areas on the coordinate planes. Let also ξ, η, ζ, be the coordinates of the planet referred to three rectangular axes, so chosen, that the plane of ξ, η, coincides with the plane of the orbit at the time t, so that $\zeta = 0$ at that instant. Then the equation to the plane of the orbit is

$$lx + my + nz = 0,$$

where $$l^2 + m^2 + n^2 = 1.$$

Differentiating these two equations, and remembering that $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, are to be expressed in the same way as if the motion were not disturbed, we have

* The investigation of the variation of perihelion given by Mr. O'Brien, vol. I., p. 165, of this Journal, is preferable to that in Airy's *Tracts*.

$$l\frac{dx}{dt}+m\frac{dy}{dt}+n\frac{dz}{dt}=0\ \ldots\ldots\ldots\ldots(1),$$

$$x\frac{dl}{dt}+y\frac{dm}{dt}+z\frac{dn}{dt}=0\ \ldots\ldots\ldots\ldots(2),$$

$$l\frac{dl}{dt}+m\frac{dm}{dt}+n\frac{dn}{dt}=0\ \ldots\ldots\ldots\ldots(3).$$

And differentiating (1) again, we have by equations (I)

$$\frac{dl}{dt}\frac{dx}{dt}+\frac{dm}{dt}\frac{dy}{dt}+\frac{dx}{dt}\frac{dz}{dt}=-\left(l\frac{d^2x}{dt^2}+m\frac{d^2y}{dt^2}+n\frac{d^2z}{dt^2}\right)$$

$$=-\left(l\frac{dR}{dx}+m\frac{dR}{dy}+n\frac{dR}{dz}\right)=-\frac{dR}{d\zeta}=-S\ \ldots\ (4),$$

where S is the force perpendicular to the plane of the orbit.

Combining (2) and (3) with this last, we get

$$\frac{\dfrac{dl}{dt}}{mz-ny}=\frac{\dfrac{dm}{dt}}{nx-lz}=\frac{\dfrac{dn}{dt}}{ly-mx}=\frac{-\dfrac{dR}{d\zeta}}{H}=\frac{-S}{H}\ \ldots\ (5),$$

since $y\dfrac{dz}{dt}-z\dfrac{dy}{dt}=lH,\ z\dfrac{dx}{dt}-x\dfrac{dz}{dt}=mH,\ x\dfrac{dy}{dt}-y\dfrac{dx}{dt}=nH.$

Also, since the velocity perpendicular to the plane of the instantaneous orbit is always zero,

$$\frac{d\zeta}{dt}=\frac{d\zeta}{di}\frac{di}{dt}+\frac{d\zeta}{d\Omega}\frac{d\Omega}{dt}=0\ \ldots\ldots\ldots\ (6);$$

and $$\frac{d\zeta}{d\Omega}=l\frac{dx}{d\Omega}+m\frac{dy}{d\Omega}+n\frac{dz}{d\Omega}=-\left(x\frac{dl}{d\Omega}+y\frac{dm}{d\Omega}+z\frac{dn}{d\Omega}\right)$$

$$=-(x\cos\Omega+y\sin\Omega)\sin i\ \ldots\ (7),$$

$$\frac{d\zeta}{di}=-\left(x\frac{dl}{di}+y\frac{dm}{di}+z\frac{dn}{di}\right)=-\frac{x}{n}\left(n\frac{dl}{di}-l\frac{dn}{di}\right)-\frac{y}{n}\left(n\frac{dm}{di}-m\frac{dn}{di}\right)$$

$$=-(x\sin\Omega-y\cos\Omega)\sec i\ \ldots\ (8).$$

Now, from (5) and (7),

$$\frac{dn}{dt}=-\sin i\,\frac{di}{dt}=-\frac{S}{H}(x\cos\Omega+y\sin\Omega)\sin i,$$

$$\therefore\ \frac{di}{dt}=\frac{S}{H}(x\cos\Omega+y\sin\Omega)=-\frac{1}{H\sin i}\,\frac{dR}{d\zeta}\cdot\frac{d\zeta}{d\Omega}.$$

And, from (6) and (8),

$$\frac{d\Omega}{dt} = \frac{1}{H \sin i} \frac{dR}{d\zeta} \frac{d\zeta}{di} = -\frac{S}{H \sin i \cos i} (x \sin \Omega - y \cos \Omega).$$

Hence, if the axis of ξ be made to coincide with the line of nodes at time t, and if r and θ be the polar co-ordinates in the plane of the orbit, we shall have the various expressions,

$$\left.\begin{aligned} \frac{di}{dt} &= \frac{S.\xi}{H} = \frac{S.r \cos\theta}{H} = -\frac{1}{H \sin i} \frac{dR}{d\zeta} \frac{d\zeta}{d\Omega} \\ \frac{d\Omega}{dt} &= \frac{S.\eta}{H \sin i} = \frac{S.r \sin\theta}{H \sin i} = \frac{1}{H \sin i} \frac{dR}{d\zeta} \frac{d\zeta}{di} \end{aligned}\right\} \dots \text{(A)}.$$

As R is to be expressed in terms of t and the elements, the latter part of these expressions must be transformed before they can be of much use, but the method of measuring the angles in the plane of the orbit (the longitude of perihelion and epoch) must first be fixed. One way is to suppose the angles measured from a *fixed line* in the plane of the orbit, which implies that the angles in that plane are not affected by its rotation about any line in itself, but only by rotation of the orbit about the normal to its plane. On this supposition, if the longitude of node is to be varied alone, the plane of the orbit must revolve about a line in itself at right angles to the line of nodes, and, if the inclination alone, about the line of nodes. The variation therefore of R by a variation of Ω or i alone is produced by a motion of the planet perpendicular to the plane of its orbit, and we shall therefore have

$$\frac{dR}{di} = \frac{dR}{d\zeta} \frac{d\zeta}{di}, \quad \frac{dR}{d\Omega} = \frac{dR}{d\zeta} \frac{d\zeta}{d\Omega},$$

and the expressions (A) become

$$\frac{di}{dt} = -\frac{1}{H \sin i} \frac{dR}{d\Omega}, \quad \frac{d\Omega}{dt} = \frac{1}{H \sin i} \frac{dR}{di} \dots . (a).$$

These are the simplest expressions for the variations of Ω and i; but for the purpose of determining the planet's place the mode of measuring the angles implied is not the best, for, when the plane of the orbit changes, it becomes necessary to have some means of finding the position of the fixed line, before the angles can be measured from it. For this purpose let ϑ be the angle between the fixed line and the line of nodes. Then to find ϑ we have the relation

$$d\vartheta = -\cos i \, d\Omega,$$

or

$$\vartheta = C - \int \cos i \, d\Omega.$$

[redacted]

The angle ϑ is sometimes called the longitude of the node in the orbit. When it is determined, the position of the fixed line is known, and then the planet's place can be found by measuring the angles from it.

The better method is at once to measure the angles from the line of nodes, or, which gives a simpler result, to measure the longitude of perihelion from the line of nodes, and the epoch, which will in this case be the value of the mean anomaly (instead of the mean longitude) when $t = 0$, from perihelion. Let ω be the longitude of perihelion, and c the epoch thus measured.

If now Ω is to be made to vary alone, the orbit must turn about an axis perpendicular to the plane xy, so as to preserve i, ω, and c unchanged. This motion is the same as rotation about an axis in its plane perpendicular to the line of nodes through an angle $d\Omega \sin i$, combined with rotation about the normal to its plane through an angle $d\Omega \cos i$.

Now $$\frac{dR}{d\Omega} = \frac{dR}{d\xi}\frac{d\xi}{d\Omega} + \frac{dR}{d\eta}\frac{d\eta}{d\Omega} + \frac{dR}{d\zeta}\frac{d\zeta}{d\Omega},$$

and $\left(\dfrac{dR}{d\xi}\dfrac{d\xi}{d\Omega} + \dfrac{dR}{d\eta}\dfrac{d\eta}{d\Omega}\right) d\Omega$ is the variation of R in consequence of the rotation of the orbit through the angle $d\Omega \cos i$ about the normal to its plane. But the only element affected by rotation about the normal is ω; therefore the change of R by rotation through the angle $d\Omega \cos i$ is $= \dfrac{dR}{d\omega} d\Omega \cos i$;

therefore $$\frac{dR}{d\xi}\frac{d\xi}{d\Omega} + \frac{dR}{d\eta}\frac{d\eta}{d\Omega} = \frac{dR}{d\omega}\cos i;$$

and therefore $$\frac{dR}{d\zeta}\frac{d\zeta}{d\Omega} = \frac{dR}{d\Omega} - \frac{dR}{d\omega}\cos i.$$

The value of $\dfrac{dR}{d\zeta}\dfrac{d\zeta}{di}$ is the same as before, since the motion of the plane of the orbit, when i alone varies, is the same. The expressions (A) in this case will therefore become

$$\left.\begin{aligned} \frac{di}{dt} &= -\frac{1}{H\sin i}\frac{dR}{d\Omega} + \frac{\cos i}{H\sin i}\frac{dR}{d\omega} \\ \frac{d\Omega}{dt} &= \frac{1}{H\sin i}\frac{dR}{di} \end{aligned}\right\} \quad \ldots\ldots (b).$$

If the epoch be measured from the line of nodes, it is at once

evident that (calling this epoch ε) $\frac{dR}{d\omega}$ must be replaced by $\frac{dR}{d\varepsilon} + \frac{dR}{d\omega}$, both ε and ω being changed by rotation about the normal to the plane of the orbit. We have also the relation $\varepsilon = c + \omega$, which would give the same result by transformation. The expressions (b) will thus become

$$\left.\begin{aligned} \frac{di}{dt} &= -\frac{1}{H \sin i}\frac{dR}{d\Omega} + \frac{\cos i}{H \sin i}\left(\frac{dR}{d\varepsilon} + \frac{dR}{d\omega}\right) \\ \frac{d\Omega}{dt} &= \frac{1}{H \sin i}\frac{dR}{di} \end{aligned}\right\} \ldots . (b').$$

It should be observed, that the variations of longitude of perihelion and epoch will also be differently expressed in the two cases above given. If ϖ and ϵ be the longitude of perihelion and epoch in the first case, the values of $\frac{d\varpi}{dt}$ and $\frac{d\epsilon}{dt}$ will be those given in Airy and Pratt,* since the angles are altered only by rotation of the orbit in its plane.

If we take the epoch c, measured from perihelion, the value of $\frac{dc}{dt}$, which $= \frac{d\epsilon}{dt} - \frac{d\varpi}{dt}$, will be the same, whichever way of measuring the longtitude of perihelion we choose. The relations between ω, ε, and ϖ, ϵ, are these,

$$\left.\begin{aligned} d\omega &= d\varpi - \cos i \, d\Omega \\ d\varepsilon &= d\epsilon - \cos i \, d\Omega \end{aligned}\right\}.$$

As for $\frac{da}{dt}$ and $\frac{de}{dt}$ they are the same whether we use ϖ and ϵ or ω and ε, since $\frac{dR}{d\varpi} = \frac{dR}{d\omega}$, and $\frac{dR}{d\epsilon} = \frac{dR}{d\varepsilon}$.

If we take the epoch c, $\left(\frac{dR}{d\epsilon} + \frac{dR}{d\varpi}\right)$ in the expression for $\frac{de}{dt}$ must be replaced by $\frac{dR}{d\varpi}$ or $\frac{dR}{d\omega}$.

In combining the above formulæ with the expressions given in Airy's *Tracts* or Pratt's *Mechanics*, the sign of R must be changed, as I have followed Lagrange in taking the disturbing function positive, when the disturbing force tends to increase the coordinates.

* The term containing t as a factor should be omitted in $\frac{d\epsilon}{dt}$, as it is introduced by the variation of n. See Lagrange, *Mem. Inst.* 1808, p. 64.

[redacted]

Throughout, H may be replaced by the equivalent expressions $na^2\sqrt{(1-e^2)}$ or $\sqrt{\{\mu a(1-e^2)\}}$. The complete series of the variations in the different cases is subjoined, for facility of reference:

$$\frac{da}{dt}=\frac{2}{na}\frac{dR}{d\epsilon},$$

$$\frac{de}{dt}=\frac{1-e^2}{na^2e}\cdot\frac{dR}{d\epsilon}-\frac{\sqrt{(1-e^2)}}{na^2e}\left(\frac{dR}{d\epsilon}+\frac{dR}{d\varpi}\right),$$

$$\frac{d\varpi}{dt}=\frac{\sqrt{(1-e^2)}}{na^2e}\cdot\frac{dR}{de},$$

$$\frac{d\epsilon}{dt}=-\frac{2}{na}\frac{dR}{da}+\frac{1}{na^2e}\{\sqrt{(1-e^2)}-(1-e^2)\}\frac{dR}{de},$$

$$\frac{d\Omega}{dt}=\frac{1}{na^2\sqrt{(1-e^2)}\,.\,\sin i}\frac{dR}{di},$$

$$\frac{di}{dt}=-\frac{1}{na^2\sqrt{(1-e^2)}\,.\,\sin i}\frac{dR}{d\Omega},$$

when the angles are measured from a fixed line in the plane. When they are measured from the line of nodes, the only change in the expressions for $\frac{da}{dt}$ and $\frac{de}{dt}$ is to put ε and ω for ϵ and ϖ. Instead of $\frac{d\epsilon}{dt}$ and $\frac{d\varpi}{dt}$, we have

$$\frac{d\omega}{dt}=\frac{\sqrt{(1-e^2)}}{na^2e}\frac{dR}{de}-\frac{\cos i}{na^2\sqrt{(1-e^2)}\,.\,\sin i}\frac{dR}{di},$$

$$\frac{d\varepsilon}{dt}=-\frac{2}{na}\frac{dR}{da}$$

$$+\frac{1}{na^2e}\{\sqrt{(1-e^2)}-(1-e^2)\}\frac{dR}{de}-\frac{\cos i}{na^2\sqrt{(1-e^2)}\sin i}\frac{dR}{di},$$

and the expression for $\frac{di}{dt}$ becomes

$$=-\frac{1}{na^2\sqrt{(1-e^2)}\sin i}\frac{dR}{d\Omega}+\frac{\cos i}{na^2\sqrt{(1-e^2)}\sin i}\left(\frac{dR}{d\omega}+\frac{dR}{d\varepsilon}\right).$$

The change in this expression, it must be remembered, arises, not from any change in the *value* of $\frac{di}{dt}$, but from the different manner in which Ω is involved in R.

When we use the epoch c instead of ϵ or ε in all the above expressions, $\frac{dR}{d\epsilon}$ or $\frac{dR}{d\varepsilon}$ is replaced by $\frac{dR}{dc}$, and $\frac{dR}{d\varpi}+\frac{dR}{d\epsilon}$ or $\frac{dR}{d\omega}+\frac{dR}{d\varepsilon}$ by $\frac{dR}{d\varpi}$ or $\frac{dR}{d\omega}$; and instead of $\frac{d\epsilon}{dt}$ or $\frac{d\varepsilon}{dt}$ we have

$$\frac{dc}{dt} = -\frac{2}{na}\frac{dR}{da} - \frac{1-e^2}{na^2e}\frac{dR}{de}.$$

ON SYMBOLICAL GEOMETRY.

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INTRODUCTORY REMARKS.

THE present paper is an attempt towards constructing a symbolical geometry, analogous in several important respects to what is known as symbolical algebra, but not identical therewith; since it starts from other suggestions, and employs, in many cases, other rules of combination of symbols. One object aimed at by the writer has been (he confesses) to illustrate, and to exhibit under a new point of view, his own theory, which has in part been elsewhere published, of algebraic quaternions. Another object, which interests even him much more, and will probably be regarded by the readers of this Journal as being much less unimportant, has been to furnish some new materials towards judging of the general applicability and usefulness of some of those principles respecting symbolical language which have been put forward in modern times. In connexion with this latter object he would gladly receive from his readers some indulgence, while offering the few following remarks.

An opinion has been formerly published* by the writer of the present paper, that it is possible to regard Algebra as a *science*, (or more precisely speaking) as a *contemplation*, in some degree *analogous to Geometry*, although not to be confounded therewith; and to separate it, as such, in our conception, from its own *rules of art* and *systems of expression*: and that when so regarded, and so separated, its ultimate subject-matter is found in what a great metaphy-

* *Trans. Royal Irish Acad.*, vol. XVII. Dublin, 1835.

sician has called the inner intuition of *time*. On which account, the writer ventured to characterise Algebra as being the *Science of Pure Time;* a phrase which he also expanded into this other: that it is (ultimately) the Science of *Order and Progression.* Without having as yet seen cause to abandon that former view, however obscurely expressed and imperfectly developed it may have been, he hopes that he has since profited by a study, frequently resumed, of some of the works of Professor Ohm, Dr. Peacock, Mr. Gregory, and some other authors; and imagines that he has come to seize their meaning, and appreciate their value, more fully than he was prepared to do, at the date of that former publication of his own to which he has referred. The whole theory of the laws and logic of symbols is indeed one of no small subtlety; insomuch that (as is well known to the readers of the *Cambridge Mathematical Journal*, in which periodical many papers of great interest and importance on this very subject have appeared) it requires a close and long-continued attention, in order to be able to form a judgment of any value respecting it: nor does the present writer venture to regard his own opinions on this head as being by any means sufficiently matured; much less does he desire to provoke a controversy with any of those who may perceive that he has not yet been able to adopt, in all respects, their views. That he has adopted *some* of the views of the authors above referred to, though in a way which does not seem to himself to be contradictory to the results of his former reflexions; and especially that he feels himself to be under important obligations to the works of Dr. Peacock upon Symbolical Algebra, are things which he desires to record, or mark, in some degree, by the very *title* of the present communication; in the course of which there will occur opportunities for acknowledging part of what he owes to other works, particularly to Mr. Warren's Treatise on the Geometrical Representation of the Square Roots of Negative Quantities.

Observatory of Trinity College, Dublin, Oct. 16, 1845.

Uniliteral and Biliteral Symbols.

1. In the following pages of an attempt towards constructing a symbolical geometry, it is proposed to employ (as usual) the roman capital letters A, B, &c., with or without accents, as symbols of *points* in space; and to make use (at first) of binary combinations of those letters, as symbols of straight *lines:* the symbol of the beginning

of the line being written (for the sake of some analogies*) towards the right hand, and the symbol of the end towards the left. Thus BA will denote the line *to* B *from* A; and is not to be confounded with the symbol AB, which denotes a line having indeed the same extremities, but drawn in the opposite direction. A biliteral symbol, of which the two component letters denote determined and different points, will thus denote a finite straight line, having a determined length, direction, and situation in space. But a biliteral symbol of the particular form AA may be said to be a *null* line, regarded as the limit to which a line tends, when its extremities tend to coincide: the conception or at least the name and symbol of such a line being required for symbolic generality. All lines BA which are not null, may be called by contrast *actual;* and the two lines AB and BA may be said to be the *opposites* of each other. It will then follow that a null line is its own opposite, but that the opposites of two actual lines are always to be distinguished from each other.

On the mark =.

2. An *equation* such as $\mathrm{B} = \mathrm{A}$ ....(1),

between two uniliteral symbols, may be interpreted as denoting that A and B are *two names for one common point;* or that a point B, determined by one geometrical process, coincides with a point A determined by another process. When a formula of the kind (1) holds good, in any calculation, it is allowed to *substitute*, in any other part of that calculation, either of the two equated symbols for the other; and every other equation between two symbols of one common class must be interpreted so as to allow a similar substitution. We shall not violate this principle of symbolical language by interpreting, as we shall interpret, an equation such as

$$\mathrm{DC} = \mathrm{BA} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$

between two biliteral symbols, as denoting that the two lines,†

* The writer regards the line to B from A as being in some sense an interpretation or construction of the symbol $\mathrm{B} - \mathrm{A}$; and the evident possibility of reaching the point B, by going along that line from the point A, may, as he thinks, be symbolized by the formula $\mathrm{B} - \mathrm{A} + \mathrm{A} = \mathrm{B}$.

† The writer regards the relation between two lines, mentioned in the text, as a sort of interpretation of the following symbolic equation, $\mathrm{D} - \mathrm{C} = \mathrm{B} - \mathrm{A}$; which may also denote that the point D is ordinarily related (in space) to the point C as B is to A, and may in that view be also expressed by writing the *ordinal analogy*, D..C :: B..A; which admits of *inversion* and *alternation*. The same relation between four points may, as he thinks, be thus symbolically expressed, $\mathrm{D} = \mathrm{B} - \mathrm{A} + \mathrm{C}$. But by writing it as an equation between lines, he deviates less from received notation.

of which the symbols are equated, have *equal lengths and similar directions*, though they may have different situations in space: for if we call such lines *symbolically equal*, it will be allowed, in *this* sense of equality, which has indeed been already proposed by Mr. Warren, Dr. Peacock, and probably by some of the foreign writers referred to in Dr. Peacock's Report, as well as in that narrower sense which relates to magnitudes only, and for lines in space as well as for those which are in one plane, to assert that lines *equal* to the same line are equal to each other. (Compare *Euclid*, XI. 9.) It will also be true, that

$$D = B, \quad \text{if } DA = BA \;\ldots\ldots\ldots\ldots\; (3),$$

or in words, that the ends of two symbolically equal lines coincide if the beginnings do so; a consequence which it is very desirable and almost necessary that we should be able to draw, for the purposes of symbolical geometry, but which would not have followed, if an equation of the form (2) had been interpreted so as to denote *only* equality of lengths, or *only* similarity of directions. The opposites of equal lines are equal in the sense above explained; therefore the equation (2) gives also this *inverse* equation,

$$CD = AB \;\ldots\ldots\ldots\ldots\ldots\ldots\; (4).$$

Lines joining the similar extremities of symbolically equal lines are themselves symbolically equal (*Euc.* I. 33); therefore the equation (2) gives also this *alternate* equation,

$$DB = CA \;\ldots\ldots\ldots\ldots\ldots\ldots\; (5).$$

The *identity* BA = BA gives, as its alternate equation,

$$BB = AA \;\ldots\ldots\ldots\ldots\ldots\ldots\; (6),$$

which symbolic result may be expressed in words by saying that any two null lines are to be regarded as equal to each other. Lines equal to opposite lines may be said to be themselves opposite lines.

On the mark +.

3. The equation* $CB + BA = CA \;\ldots\ldots\ldots\ldots\ldots\; (7)$

is true in the most elementary sense of the notation, when B is any point upon the finite straight line CA; but we propose now to *remove this restriction for the purposes of symbolical*

* On the plan mentioned in former notes, this equation would be written as follows:

$$(C - B) + (B - A) = C - A.$$

It might also be thus expressed: the ordinal relation of the point C to the point A is compounded of the relations of C to B and of B to A.

geometry, and to regard the formula (7) as being universally *valid, by definition, whatever three points of space may be denoted* by the three letters ABC. The equation (7) will then *express nothing about those points*, but will serve to *fix the interpretation of the mark + when inserted between any two symbols of lines*; for if we meet any symbol formed by such insertion, suppose the symbol HG + FE, we have only to draw, or conceive drawn, from any assumed point A, a line BA = FE, and from the end B of the line so drawn, a new line CB = HG; and then the proposed symbol HG + FE will be interpreted by (7) as denoting the line CA, or at least a line equal thereto. In like manner, by defining that

$$DC + CB + BA = DA \ldots\ldots\ldots\ldots (8),$$

we shall be able to interpret any symbol of the form

$$KI + HG + FE,$$

as denoting a determined (actual or null) line; at least if we now regard a line as *determined* when it is *equal* to a determined line: and similarly for any number of biliteral symbols, connected by marks + interposed. Calling *this* act of connection of symbols, the operation of *addition*; the added symbols, *summands*; and the resulting symbol, a *sum*; we may therefore now say, that the sum of any number of symbols of given lines is itself a symbol of a determined line; and that this symbolic sum of lines represents the *total* (or final) *effect* of all those successive rectilinear *motions*, or translations of a point in space, which are represented by the several summands. This *interpretation of a symbolic sum of lines* agrees with the conclusions already published by the authors above alluded to; though the modes of symbolically obtaining and expressing it, here given, may possibly be found to be new. The same interpretation satisfies, as it ought to do, the condition that the sums of equals shall be equal (compare the demonstration of *Euclid*, XI. 10); and also this other condition, almost as much required for the advantageous employment of symbolical language, that those lines which, when added to equal lines, give equal sums, shall be themselves equal lines: or that

$$FE = DC, \quad \text{if } FE + BA = DC + BA \ldots\ldots (9).$$

It shews too that the sum of two opposite lines, and generally that the sum of all the successive sides of any closed polygon, or of lines respectively equal to those sides, is a null line: thus

$$AA = AB + BA = AC + CB + BA = \&c. \ldots\ldots (10).$$

The symbolic sum of any two lines is found to be *independent of their order*, in virtue of the same interpretation; so that the equation

$$FE + HG = HG + FE \dots\dots\dots\dots (11),$$

is true, in the present system, *not as an independent definition*, but rather as one of the modes of *symbolically expressing that elementary theory of geometry*, (*Euclid*, I. 33), on which was founded the rule for deducing, from any equation (2) between lines, the *alternate* equation (5). For if we assume, as we may, that three points A, B, C, have been so chosen as to satisfy the equations $FE = BA$, $HG = CA$; and that a fourth point D is chosen so as to satisfy the equation $DC = BA$; the same points will then, by the theorem just referred to, satisfy also the equation $DB = CA$; and the truth of the formula (11) will be proved, by observing that each of the two symbols which are equated in that formula is equal to the symbol DA, in virtue of the definition (7) of +, without any new definition: since

$$FE + HG = DC + CA = DA = DB + BA = HG + FE.$$

A like result is easily shown to hold good, for any number of summands; thus

$$FE + HG + KI = KI + HG + FE \dots\dots (12);$$

since the first member of this last equation may be put successively under the forms

$$(FE + HG) + KI, \quad KI + (FE + HG), \quad KI + (HG + FE),$$

and finally under the form of the second member; the stages of this successive transformation of symbols admitting easily of geometrical interpretations: and similarly in other cases. *Addition of lines in space* is therefore generally (as Mr. Warren has shewn it to be for lines in a single plane) a *commutative operation;* in the sense that the summands may interchange their places, without the sum being changed. It is also an *associative* operation, in the sense that any number of successive summands may be associated into one group, and collected into one partial sum (denoted by enclosing these summands in parentheses); and that then this partial sum may be added, as a single summand, to the rest: thus

$$(KI + HG) + FE = KI + (HG + FE) = KI + HG + FE \dots (13).$$

On the mark −.

4. The equation* $CA - BA = CB \dots\dots\dots\dots (14)$

* On the plan mentioned in some former notes, this equation would take the form

$$(C - A) - (B - A) = C - B.$$

is true, in the most elementary sense of the notation, when B is on CA; but we may remove this restriction by a *definitional extension* of the formula (14), for the purposes of symbolical geometry, as has been done in the foregoing article with respect to the formula (7); and then the equation (14), so extended, will express *nothing about the points* A, B, C, but will serve to fix the *interpretation of any symbol*, such as KI – FE, formed by *inserting the mark – between the symbols of any two lines.* This general meaning of the effect of the mark –, so inserted, is consistent with the particular interpretation which suggested the formula (14); it is also consistent with the usual symbolical opposition between the effects of + and –; since the comparison of (14) with (7) gives the equations

$$(\mathrm{CA} - \mathrm{BA}) + \mathrm{BA} = \mathrm{CA} \quad \ldots\ldots\ldots (15),$$

and

$$(\mathrm{CB} + \mathrm{BA}) - \mathrm{BA} = \mathrm{CB} \quad \ldots\ldots\ldots (16),$$

either of which two equations, if regarded as a general formula, and combined with the formula (7), would include, reciprocally, the definition (14) of –, and might be substituted for it.

Symbolical *subtraction* of one line from another is thus equivalent to the *decomposition* of a given rectilinear *motion* (CA) into two others, of which one (BA) is given; or to the *addition of the opposite* (AB) of the line which was to be subtracted: so that we may write the symbolical equation

$$-\mathrm{BA} = +\mathrm{AB} \quad \ldots\ldots\ldots\ldots (17),$$

because the second member of (14) may be changed by (7) to CA + AB. These conclusions respecting symbolical subtraction of lines, differ only in their notation, and in the manner of arriving at them, from the results of the authors already referred to, so far as the present writer is acquainted with them. In the present notation, when an isolated biliteral symbol is preceded by + or –, we may still interpret it as denoting a line, if we agree to prefix to it, for the purpose of such interpretation, the symbol of a null line; thus we may write

$$+\mathrm{AB} = \mathrm{AA} + \mathrm{AB} = \mathrm{AB}, \quad -\mathrm{AB} = \mathrm{BB} - \mathrm{AB} = \mathrm{BA} \ldots (18);$$

+ AB will, therefore, on this plan, be another symbol for the line AB itself, and – AB will be a symbol for the opposite line BA.

Abridged Symbols for Lines.

5. Some of the foregoing formulæ may be presented more concisely, and also in a way more resembling ordinary Algebra, by using now some new *uniliteral* symbols, such

[redacted]

as the small roman letters a, b, &c., with or without accents, as symbols of lines, instead of binary combinations of the roman capitals, in cases where the lines which are compared are not supposed to have necessarily any common point, and generally when the *situations* of lines are disregarded, but not their lengths nor their directions. Thus we shall have, instead of (11) and (12), (13), (15) and (16), these other formulæ of the present Symbolical Geometry, which agree in all respect with those used in Symbolical Algebra:

$$\mathrm{a} + \mathrm{b} = \mathrm{b} + \mathrm{a}, \quad \mathrm{a} + \mathrm{b} + \mathrm{c} = \mathrm{c} + \mathrm{b} + \mathrm{a} \ldots.(19);$$

$$(\mathrm{c} + \mathrm{b}) + \mathrm{a} = \mathrm{c} + (\mathrm{b} + \mathrm{a}) = \mathrm{c} + \mathrm{b} + \mathrm{a} \ \ldots.(20);$$

$$(\mathrm{b} - \mathrm{a}) + \mathrm{a} = \mathrm{b}, \quad (\mathrm{b} + \mathrm{a}) - \mathrm{a} = \mathrm{b} \ldots\ldots..(21);$$

and because the isolated but *affected* symbols + a, − a, may denote, by (18), the line a itself, and the opposite of that line, we have also here the usual *rule of the signs*,

$$+ (+\mathrm{a}) = -(-\mathrm{a}) = +\mathrm{a}, \quad +(-\mathrm{a}) = -(+\mathrm{a}) = -\mathrm{a} \ldots(22).$$

Introduction of the marks × *and* ÷.

6. Continuing to denote lines by letters, the formula

$$(\mathrm{b} \div \mathrm{a}) \times \mathrm{a} = \mathrm{b} \ldots\ldots\ldots\ldots..(23),$$

which is, for the relation between multiplication and division, what the first of the two formulæ (21) is for the relation between addition and subtraction, will be true, in the most elementary sense of the multiplication of a length by a number, for the case when the line b is the sum of several summands, each equal to the line a, and when the number of those summands is denoted by the quotient b ÷ a. And we shall now, for the purposes of symbolical generality, *extend* this formula (23), so as to make it be valid, *by definition, whatever two lines* may be denoted by a and b. The formula will then *express nothing respecting those lines* themselves, which can serve to distinguish them from any other lines in space; but will furnish a *symbolic condition*, which we must satisfy by the *general interpretation* of a *geometrical quotient*, and of the *operation of multiplying a line* by such a quotient.

To make such general interpretation consistent with the particular case where a quotient becomes a *quotity*, we are led to write

$$\mathrm{a} \div \mathrm{a} = 1, \quad (\mathrm{a} + \mathrm{a}) \div \mathrm{a} = 2, \quad \&\mathrm{c}. \ldots\ldots(24),$$

and conversely

$$1 \times \mathrm{a} = \mathrm{a}, \quad 2 \times \mathrm{a} = \mathrm{a} + \mathrm{a}, \quad \&\mathrm{c}. \ldots\ldots..(25);$$

and because, when quotients can be thus interpreted as quotities, the four equations

$$(c \div a) + (b \div a) = (c + b) \div a \ldots\ldots\ldots (26),$$
$$(c \div a) - (b \div a) = (c - b) \div a \ldots\ldots\ldots (27),$$
$$(c \div a) \times (a \div b) = c \div b \ldots\ldots\ldots\ldots (28),$$
$$(c \div a) \div (b \div a) = c \div b \ldots\ldots\ldots\ldots (29),$$

are true in the most elementary sense of arithmetical operations on whole numbers, we shall now *define* that these four equations are valid, *whatever three lines* may be denoted by a, b, c; and thus shall have conditions for the general *interpretations of the four operations* $+ - \times \div$ *performed on geometrical quotients.*

We shall in this way be led to interpret a quotient of which the divisor is an actual line, but the dividend a null one, as being equivalent to the symbol $1 - 1$, or *zero*; so that

$$(a - a) \div a = 0, \quad 0 \times a = a - a \ldots\ldots (30).$$

Negative numbers will present themselves in the consideration of such quotients and products as

$$(-a) \div a = 0 - 1 = -1, \quad (-1) \times a = -a, \ \&c. \ldots (31);$$

fractional numbers in such formulæ as

$$a \div (a + a) = 1 \div 2 = \tfrac{1}{2}, \quad \tfrac{1}{2} \times (a + a) = a, \ \&c. \ldots\ldots (32);$$

and *incommensurable* numbers, by the conception of the connected *limits* of quotients and products, and by the formula, which symbolical language leads us to assume,

$$\left(\lim \frac{n}{m}\right) \times a = \lim \left(\frac{n}{m} \times a\right) \ldots\ldots\ldots (33).$$

If then we give the name of SCALARS to all numbers of the kind called usually *real*, because they are all contained on the one *scale* of progression of number from negative to positive infinity; and if we agree, for the present, to denote such numbers generally by small italic letters, a, b, c, &c.; and to insert the mark $\parallel$ between the symbols of two lines when we wish to express that the directions of those lines are either exactly similar or exactly opposite to each other, in each of which two cases the lines may be said to be *symbolically parallel;* we shall have generally two equations of the forms

$$b \div a = a, \ a \times a = b, \text{ when } b \parallel a \ldots\ldots (34).$$

That is to say, the *quotient of two parallel lines* is generally a *scalar number;* and, conversely, to multiply a given line (a) by a given scalar (or real) number a, is to determine a new

[redacted]

line (b) parallel to the given line (a), the direction of the one being similar or opposite to that of the other, according as the number is positive or negative, while the length of the new line bears to the length of the given line a ratio which is marked by the same given number. So that if A_0 A_1 A_a denote any three points on one common axis of rectilinear progression, which are related to each other, upon that axis, as to their order and their intervals, in the same manner as the three scalar numbers 0, 1, a, regarded as ordinals, are related to each other on the scale of numerical progression from $-\infty$ to $+\infty$, then the equations

$$A_aA_0 \div A_1A_0 = a, \quad a \times A_1A_0 = A_aA_0, \ldots\ldots (35)$$

will be true by the foregoing interpretations.

It is easy to see that this mode of interpreting a quotient of parallel lines renders the formulæ (26) (27) (28) (29) consistent with the received rules for performing the operations $+ - \times \div$ on what are called real numbers, whether they be positive or negative, and whether commensurable or incommensurable; or rather reproduces those rules as consequences of those formulæ.

On Vectors, and Geometrical Quotients in general.

7. The other chief relation of directions of lines in space, besides parallelism, is perpendicularity; which it is not unusual to denote by writing the mark $\perp$ between the symbols of two perpendicular lines. And the other chief class of geometrical quotients which it is important to study, as preparatory to a general theory of such quotients, is the class in which the dividend is a line perpendicular to the divisor. A quotient of this latter class we shall call a VECTOR, to mark its connection (which is closer than that of a *scalar*) with the conception of *space*, and for other reasons which will afterwards appear: and if we agree to denote, for the present, such vector quotients (of perpendicular lines) by small Greek letters, in contrast to the scalar class of quotients (of parallel lines) which we have proposed to denote by small italic letters, we shall then have generally two equations of the forms

$$c \div a = \alpha, \quad c = \alpha \times a, \quad \text{if } c \perp a \ldots\ldots\ldots (36).$$

Any line e may be put under the form c + b, in which b $\parallel$ a, and c $\perp$ a; a *general geometrical quotient* may therefore, by (26) (34) (36), be considered as the *symbolic sum of a scalar and a vector*, zero being regarded as a common limit of quotients of these two classes; and consequently, if we

adopt the notation just now mentioned, we have generally an equation of the form

$$e \div a = a + \boldsymbol{a} \ldots\ldots\ldots\ldots\ldots (37).$$

This *separation of the scalar and vector parts* of a general geometrical quotient corresponds (as we see) to the decomposition, by *two separate projections*, of the dividend line into two other lines of which it is the symbolic sum, and of which one is parallel to the divisor line, while the other is perpendicular thereto. To be able to mark on some occasions more distinctly, in writing, than by the use of two different alphabets, the conception of such separation, we shall here introduce two new symbols of operation, namely the abridged words Scal. and Vect., which, where no confusion seems likely to arise from such farther abridgment, we shall also denote more shortly still by the letters S and V, prefixing them to the symbol of a general geometrical quotient in order to form separate symbols of its scalar and vector parts: so that we shall now write generally, for any two lines a and e,

$$e \div a = \text{Vect.}\,(e \div a) + \text{Scal.}\,(e \div a) \ldots\ldots (38);$$

or more concisely,

$$e \div a = V(e \div a) + S(e \div a) \ldots\ldots\ldots (39);$$

in which expression the order of the two summands may be changed, in virtue of the definition (26) of addition of geometrical quotients, because the order of the two partial dividends may be changed without preventing the dividend line e from being still their symbolic sum. A scalar cannot become equal to a vector, except by each becoming zero; for if the divisor of the vector quotient be multiplied separately by the scalar and the vector, the products of these two multiplications will be (by what has been already shown) respectively lines parallel and perpendicular to that divisor, and therefore not symbolically equal to each other, except it be at the limit where both become null lines, and are on that account regarded as equal. A scalar quotient $b \div a = a$, $(b \parallel a)$, has been seen to denote the relative length and relative direction (as similar or opposite) of two parallel lines a, b: and in like manner a vector quotient $c \div a = \boldsymbol{a}$, $(c \perp a)$, may be regarded as denoting the *relative length and relative direction* (depending on *plane* and *hand*) *of two perpendicular lines* a, c; or as indicating at once *in what ratio* the length of one line a must be altered (if at all) in order to become equal to the length of another line c, and also *round what axis*, perpendicular to both these two rectangular lines, the direction of the divisor line a must be caused or conceived to

turn, right-handedly, through a right angle, in order to attain the original direction of the dividend line c. A line drawn in the direction of this *axis of* (what is here regarded as) *positive rotation*, and having its length in the same ratio to some assumed *unit* of length as the length of the dividend to that of the divisor, may be called the INDEX of the vector. We shall thus be led to substitute, for any equation between two vector quotients, an equation between two lines, namely between their indices; for if we define that two vector quotients, such as $c \div a$ and $c' \div a'$ if $c \perp a$ and $c' \perp a'$, are *equal* when they have *equal indices*, we shall satisfy all conditions of symbolical equality, of the kinds already considered in connection with other definitions; we shall also be able to say that in every case of two such equal quotients, the two dividend lines (c and c′) bear to their own divisor lines (a and a′), respectively, one common ratio of lengths, and one common relation of directions. We shall thus also, by (23), be able to *interpret the multiplication* of any given line a′ by any given vector $c \div a$, *provided that the one is perpendicular to the index of the other*, as the operation of deducing from a′ another line c′, by altering (generally) its length in a given ratio, and by turning (always) its direction round a given axis of rotation, namely round the index of the vector, right-handedly, through a right angle. And we can now *interpret an equation between two general geometrical quotients*, such as

$$e' \div a' = e \div a \ldots\ldots\ldots\ldots\ldots (40),$$

as being equivalent to a *system of two separate equations*, one between the scalar and another between the vector parts, namely the two following:

$$S(e' \div a') = S(e \div a);\ V(e' \div a') = V(e \div a) \ldots (41);$$

of which each separately is to be interpreted on the principles already laid down; and which are easily seen (by considerations of similar triangles) to imply, when taken jointly, that the length of e′ is to that of a′ in the same ratio as the length of e to that of a; and also that the same rotation, round the index of either of the two equal vectors, which would cause the direction of a to attain the original direction of e, would also bring the direction of a′ into that originally occupied by e′. At the same time we see how to interpret the operation of multiplying any given line a′ by any given geometrical quotient $e \div a$ of two other lines, *whenever the three given lines* a, e, a′, *are parallel to one common plane;* namely as being the complex operation of altering (generally) a given length in a given ratio, and of turning a given line

round a given axis, through a given amount of right-handed rotation, in order to obtain a certain new line e′, which may be thus denoted, in conformity with the definition (23),

$$e' = (e \div a) \times a' \quad \ldots\ldots\ldots\ldots \quad (42).$$

The relation between the four lines a, e, a′, e′, may also be called a *symbolic analogy*, and may be thus denoted:

$$e' : a' :: e : a. \ldots\ldots\ldots\ldots\ldots (43);$$

a′ and e being the *means*, and e′ and a the *extremes* of the analogy. An analogy or equation of this sort admits (as it is easy to prove) of *inversion* and *alternation*; thus (43) or (42) gives, *inversely*,

$$a' : e' :: a : e, \quad a' \div e' = a \div e \ldots\ldots\ldots (44),$$

and *alternately*,

$$e' : e :: a' : a, \quad e' \div e = a' \div a. \ldots\ldots\ldots (45).$$

These results respecting analogies between *co-planal lines*, that is, between lines which are in or parallel to one common plane, agree with, and were suggested by, the results of Mr. Warren. But it will be necessary to introduce other principles, or at least to pursue farther the track already entered on, before we can arrive at an interpretation of a *fourth proportional to three lines which are not parallel to any common plane*: or can interpret the multiplication of a line by a quotient of two others, when it is not perpendicular to what has been lately called the index of the vector part of that quotient.

(*To be continued.*)

ON THE QUADRATURE OF SURFACES OF THE SECOND ORDER.

By John H. Jellett, M.A., Fellow and Tutor of Trinity College, Dublin.

The object of the present memoir is to furnish a complete discussion of a question hitherto but partially investigated, viz. the quadrature of the five principal surfaces of the second order. The writers who have hitherto treated of this subject have, as far as I am aware, confined their attention to the ellipsoid, partly perhaps as being the most familiar, and partly because being a finite surface it might seem to admit the most complete solution of the question: but, as we shall see, some highly interesting results are to be derived from an investigation of the superficial area of the paraboloids, inasmuch as they only, among surfaces of the second order (excepting of course surfaces of revolution), admit of an algebraic expression for their superficial area. Before entering

into the question with regard to surfaces of the second order, it may be well to make a few observations on the problem of quadrature with regard to surfaces generally, for the purpose of pointing out some important differences which exist between it and the rectification of curves, to which it is supposed to be analogous. In the problem of rectification it is immaterial, as far as the possibility of the solution is concerned, what origin or what coordinates we choose; and that because this problem being of a nature perfectly definite, the possibility of solving it cannot be changed but by a change in the curve itself, and not in the lines or angles by which we measure it. In fact, for each curve there is but one problem of rectification, the possibility of solving which depends solely on the nature of the curve itself. But the problem of quadrature is essentially different. Here, the object being to find an expression for a portion of a surface enclosed within one or more curves, the possibility of attaining it will evidently depend as much on the nature of these curves as upon that of the surface. Now let $Vdudv$ denote the element of a surface, and it is evident that $du\int Vdv$, or the expression once integrated, will denote an element finite in one direction, comprised within the four curves $v = \phi(u)$, $v = \psi(u)$, $u = c$, $u = c + dc$, of which the first two are arbitrary, but the second two depend absolutely on the nature of the coordinate u. Thus, for example, if the element be $Vdxdy$, it is plain that $dx \,.\int Vdy$ will denote an element two of whose bounding curves are sections perpendicular to the axis of x. Now it is perfectly possible that, although the surface itself may be one of those ordinarily said to be susceptible of quadrature, this element may, from the nature of the bounding curves $u = c$, $u = c + dc$, be wholly insusceptible of it. Hence we see both the importance to the solution of this problem of a proper selection of coordinates, and the impossibility of pronouncing that a given surface is insusceptible of quadrature, with the same certainty with which we pronounce a given curve to be insusceptible of rectification. Having premised so much with regard to the general problem, I propose to shew, with regard to surfaces of the second order, (1) That the quadrature of the ellipsoid depends on the rectification of the focal conics of the reciprocal surface. (2) That the quadrature of the hyperboloids may to a certain extent be effected by the same means. (3) That the quadrature of the paraboloids may be effected algebraically, *i.e.* that it is possible to divide the surface by curves, such that the space intercepted between any two admits of an algebraic expression. For this purpose I must premise the following Lemmas:

LEMMA 1. Let $x_1 y_1 z_1$, $x_2 y_2 z_2$, &c., $X_1 Y_1 Z_1$, $X_2 Y_2 Z_2$, &c., be the coordinates of the angular points of two polygons having the same number of sides, and let it be supposed that

$$\frac{x_1}{X_1} = \frac{x_2}{X_2} = \&c. = \frac{a}{A},$$

$$\frac{y_1}{Y_1} = \frac{y_2}{Y_2} = \&c. = \frac{b}{B},$$

$$\frac{z_1}{Z_1} = \frac{z_2}{Z_2} = \&c. = \frac{c}{C},$$

then the solid contents of the pyramids whose common vertex is at the origin of coordinates, and whose bases are these polygons, are to each other in the ratio $abc : ABC$. It is evidently sufficient to prove this for pyramids with triangular bases; and since * the solid contents of these are

$$\tfrac{1}{6}.\{x_1 (y_2 z_3 - y_3 z_2) + x_2 (y_3 z_1 - y_1 z_3) + x_3 (y_1 z_2 - y_2 z_1)\},$$

and $\tfrac{1}{6}.\{X_1 (Y_2 Z_3 - Y_3 Z_2) + \&c. \quad\quad \}$,

the proposition is evident.

LEMMA 2. If on the surface of one of two ellipsoids having the same centre and coincident axes, there be described any closed curve, and upon the other the *corresponding* curve, *i. e.* the locus of *corresponding* points,† the solid sectors which have their vertex at the common centre, and whose bases are these curves, are to each other in the ratio of the solid contents of the ellipsoids themselves.

Let the semiaxes of the ellipsoids be a, b, c, A, B, C, and take an indefinitely small element of one surface and the corresponding element of the other, and it appears from Lemma 1, that the sectors whose vertex is at the common centre and of which these elements are the oases, will be to each other in the ratio $abc : ABC$, i. e. as the whole ellipsoids. As therefore the proposition is true of these elements of the solid sectors, and as to each element of one sector corresponds an element of the other, it is evidently true of the whole sectors.

LEMMA 3. Given two reciprocal‡ ellipsoids; the central radius vector to any point on one is coincident with and

* Vide Monge. *Journal de l'Ecole Polytechnique*, tom. VIII. p. 92. (Or Gregory's *Solid Geometry*, p. 17.)

† Corresponding points are those whose coordinates are proportional to the parallel axes.

‡ It is perhaps unnecessary to observe that two ellipsoids are said to be reciprocal when their axes are coincident and reciprocally proportional.

reciprocally proportional to the perpendicular on the tangent plane at the corresponding point on the other.

For if r, p, be the radius vector and perpendicular on the tangent plane at one point, and R, P, at the other, xyz, XYZ the coordinates of these points respectively,

$$r^2 = x^2 + y^2 + z^2$$
$$= \frac{a^2}{A^2} X^2 + \frac{b^2}{B^2} Y^2 + \frac{c^2}{C^2} Z^2 = \frac{X^2}{A^4} + \frac{Y^2}{B^4} + \frac{Z^2}{C^4} = \frac{1}{P^2}.$$

Similarly $R = \frac{1}{p}$, and the cosines of the angles which r makes with the axes are $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$, or $\frac{PX}{A^2}, \frac{PY}{B^2}, \frac{PZ}{C^2}$, which are the cosines of the angles made by P with the axes. Hence it appears that r is coincident with and reciprocally proportional to P.

Prop. I. If p be the perpendicular from the centre of an ellipsoid on the tangent plane, θ and ϕ the angles which determine its position, and a, b, c the semiaxes, the element of the surface is $\frac{a^2 b^2 c^2}{p^4} . \sin\theta \, d\theta \, d\phi$.*

For, let $d\alpha$ represent this element, and ds the solid sector of which it is the base, then $d\alpha = \frac{3ds}{p}$. Now let dS be the corresponding sector of the reciprocal surface, and (Lemma 2) $ds = \frac{abc}{ABC} . dS = a^2 b^2 c^2 dS$; and since

$$dS = \tfrac{1}{3} R^3 \sin\theta \, d\theta \, d\phi = \tfrac{1}{3} \frac{\sin\theta \, d\theta \, d\phi}{p^3},$$
$$d\alpha = \frac{a^2 b^2 c^2}{p^4} . \sin\theta \, d\theta \, d\phi. \quad \text{Q. E. D.}$$

Prop. II. To integrate the above expression for the ellipsoid.

Assume
$$m^2 = a^2 \sin^2\theta + c^2 \cos^2\theta,$$
$$n^2 = b^2 \sin^2\theta + c^2 \cos^2\theta;$$
then
$$p^2 = a^2 \sin^2\theta \cos^2\phi + b^2 \sin^2\theta \sin^2\phi + c^2 \cos^2\theta$$
$$= m^2 \cos^2\phi + n^2 \sin^2\phi,$$

* Since arriving at the above expression I found that it had been previously given, with a different demonstration, by Jacobi. He has, however, only applied it to the ellipsoid, and his memoir has very little in common with the present. The theorem in Prop. IX. of the present paper is, I believe, the first attempt to compare different portions of the ellipsoidal surface.

$$\text{and} \quad d\alpha = a^2 b^2 c^2 \; \frac{\sin\theta \, d\theta \, d\phi}{(m^2 \cos^2\phi + n^2 \sin^2\phi)^2} :$$

or, if we assume $\tan w = \frac{n}{m} . \tan\phi$,

$$d\alpha = a^2 b^2 c^2 \sin\theta \, d\theta \; \left\{\frac{1}{mn^3} \sin^2 w + \frac{1}{m^3 n} \cos^2 w\right\} . dw.$$

The limits of ϕ and therefore of w, are o and 2π or o and $\frac{\pi}{2}$, provided that the result be multiplied by 4. Now

$$\int_0^{\frac{\pi}{2}} \cos^2 w \, dw = \int_0^{\frac{\pi}{2}} \sin^2 w \, dw = \frac{\pi}{4},$$

$$\text{therefore} \qquad \alpha = \pi a^2 b^2 c^2 \left\{\int \frac{\sin\theta \, d\theta}{m^3 n} + \int \frac{\sin\theta \, d\theta}{mn^3}\right\}.$$

Before proceeding further it may be well to shew the geometrical meaning of the first integration, or, in other words, to give a geometrical method of constructing the curves which bound the element denoted by the expression $\pi a^2 b^2 c^2 \sin\theta d\theta \left(\frac{1}{m^3 n} + \frac{1}{mn^3}\right)$. It is evident, from what has been said at the commencement of this paper, that the equations of these curves are $\theta = \text{const} = k$ and $\theta = k + dk$. Let x, y, z be the coordinates of a point on either of these curves; and since

$$\cos\theta = \frac{pz}{c^2} = \frac{1}{c^2} . \frac{z}{\sqrt{\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)}},$$

$$\text{we shall have} \qquad \frac{x^2}{a^4} + \frac{y^2}{b^4} - \frac{z^2 . \tan^2\theta}{c^4} = 0,$$

the equation of a cone whose intersection with the ellipsoid will give the required curve. Hence these curves may be constructed as follows. In any plane perpendicular to the axis of z describe a number of ellipses whose common centre is at the point where their plane cuts the axis, and whose axes are parallel to, and in the duplicate ratio of, those of a and b. The cones whose common vertex is at the centre of the ellipsoid, and whose bases are the ellipses so constructed, will cut the ellipsoid in the required curves.*

* [It may be readily shown that, when the three axes are unequal, the bounding curve never coincides with a line of curvature. Legendre has expressed, by means of *products* of elliptic functions, the area of a portion of an ellipsoid

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PROP. III. The portion of the superficial area of the ellipsoid included between any two of the curves described in the foregoing proposition, may be expressed by means of arcs of the focal conics of the reciprocal ellipsoid.

We have seen that if dS be the elementary area included between two consecutive curves of the foregoing species,

$$dS = \pi a^2 b^2 c^2 \sin\theta d\theta \left(\frac{1}{nm^3} + \frac{1}{n^3 m}\right).$$

Hence, if any two curves of this species be described, for one of which the value of θ is a, and for the other a', the value of the superficial area which they include is

$$S = \pi a^2 b^2 c^2 (I + I'), \text{ where}$$

$$I = \int_a^{a'} \frac{\sin\theta d\theta}{(b^2\sin^2\theta + c^2\cos^2\theta)^{\frac{3}{2}}(a^2\sin^2\theta + c^2\cos^2\theta)^{\frac{1}{2}}},$$

$$I' = \int_a^{a'} \frac{\sin\theta d\theta}{(a^2\sin^2\theta + c^2\cos^2\theta)^{\frac{3}{2}}(b^2\sin^2\theta + c^2\cos^2\theta)^{\frac{1}{2}}},$$

The first of these integrals denotes the arc of an ellipse, and the second that of a hyperbola, it being supposed that $a > b > c$.

Assume $a^2 - c^2 = a^2e^2$, $b^2 - c^2 = b^2e'^2$, $e'\cos\theta = \sin u$,

and $I = \dfrac{1}{e'ab^3} \cdot \displaystyle\int \frac{\sec^2 u\, du}{\sqrt{\left(1 - \frac{e^2}{e'^2} \cdot \sin^2 u\right)}}$, or putting $x = \tan u$,

$$I = \frac{1}{e'ab^3} \cdot \int \frac{dx\sqrt{(1 + x^2)}}{\sqrt{\left\{1 - \left(\frac{e^2}{e'^2} - 1\right) \cdot x^2\right\}}},$$

the limits of x being $\dfrac{e'\cos a}{\sqrt{(1 - e'^2\cos^2 a)}}$, $\dfrac{e'\cos a'}{\sqrt{(1 - e'^2\cos^2 a')}}$.

Now let σ be the arc of an ellipse whose semiaxes are A and B, and it is known that $d\sigma = \dfrac{dy\sqrt{\{B^4 + (A^2 - B^2)y^2\}}}{B\sqrt{(B^2 - y^2)}}$, or if we put $A^2 - B^2 = \dfrac{B^2}{K^2}$, and $y = BKx$,

$$d\sigma = BKdx\sqrt{\left(\frac{1 + x^2}{1 - K^2x^2}\right)}.$$

bounded by four lines of curvature; but he has not found that his expression may be reduced to a single definite integral, which Mr. Jellett's investigations shew to be possible. See Legendre, *Traité des Fonctions Elliptiques*, vol. I. p. 350. Also Catalan, *Sur la Transformation des Variables dans les Intégrales Multiples*. *Mémoires Couronnés par l'Académie de Bruxelles*, 1839-40.]

To make this coincide with the foregoing expression for I, assume $\frac{e^2}{e'^2} - 1 = K^2$, and we shall find $\frac{A^2}{B^2} = \frac{\frac{1}{c^2} - \frac{1}{a^2}}{\frac{1}{b^2} - \frac{1}{a^2}}$. The ellipse therefore by means of which I is represented is similar to the focal ellipse of the reciprocal surface; and since there is nothing to determine its absolute magnitude, it may be taken to be the focal ellipse itself. If then σ be the arc of this ellipse, whose ordinates measured parallel to the minor axis are $y = \frac{BKe' \cos \alpha}{\sqrt{(1 - e'^2 \cos^2 \alpha)}}$, and $y' = \frac{BKe' \cos \alpha'}{\sqrt{(1 - e'^2 \cos^2 \alpha')}}$, we shall have $I = \frac{\sigma}{BKe'ab^3} = \frac{\sigma}{ab^3 B \sqrt{(e^2 - e'^2)}}$.

Now let a', b', c', be the semiaxes of the reciprocal ellipsoid, and it is evident that $e^2 - e'^2 = \frac{b'^2 - a'^2}{c'^2} = \frac{B^2}{C'^2}$,

therefore $$I = \frac{c'\sigma}{ab^3 B^2}.$$

By using a reduction precisely similar, we find $I' = \frac{c'\sigma'}{a^3 b B^2}$, where σ' is an arc of the focal hyperbola, whose extreme ordinates are $y = \frac{BK'e \cos \alpha}{\sqrt{(1 - e^2) \cos^2 \alpha}}$ and $y' = \frac{BK'e \cos \alpha'}{\sqrt{(1 - e^2 \cos^2 \alpha')}}$. Hence we find ultimately for the required superficial area the expression

$$S = \frac{\pi c^2 c'}{B^2} \cdot \left(\frac{a}{b} \sigma + \frac{b}{a} \sigma'\right),$$

or if we assume the reciprocal ellipsoid such that

$$aa' = bb' = cc' = B^2,$$

$$S = \pi c \,.\, \left(\frac{a}{b} \,.\, \sigma + \frac{b}{a} \,.\, \sigma'\right).$$

Prop. iv. To construct geometrically the limits of the integrals I and I'.

As the expressions for the limiting values of the ordinates are precisely similar for the two integrals, it will be sufficient to consider one of them.

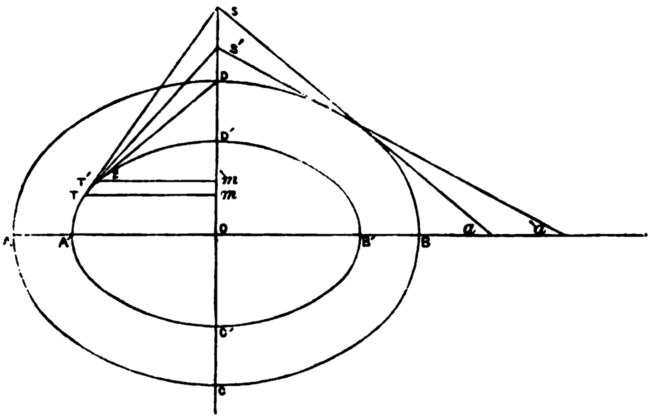
We have seen that if y be one of the limiting ordinates, $y = \frac{BKe' \cos \alpha}{\sqrt{(1 - e'^2 \cos^2 \alpha)}}$, or, substituting for e' and K,

$$y = \frac{B^2 \cos \alpha}{\sqrt{(c'^2 \sin^2 \alpha + b'^2 \cos^2 \alpha)}} = \frac{B^2}{p \sec \alpha},$$

where p is the perpendicular drawn from the centre of the ellipse whose axes are c', b', (*i. e.* the principal section of the reciprocal ellipsoid which contains its greatest and mean axis) on the tangent, which makes the angle α with the axis c'.

Similarly $$y' = \frac{B^2}{p'.\sec\alpha'}.$$

Hence we derive the following construction. Let $ACBD$ be the section of the reciprocal ellipsoid containing the greatest



and mean axis, $A'C'B'D'$ the focal ellipse of the same surface. Draw two tangents to the outer ellipse, making with the axis AB the angles α, α', and from the points s, s', where they cut the other axis, draw tangents to the inner ellipse. TT' is the arc which has become denoted by σ. For

$$Om = \frac{OD'^2}{OS} = \frac{B'^2}{p\sec\alpha}$$

equals one of the limiting ordinates. The other is of course Om'.

The same construction precisely applies to the focal hyperbola: if then t and t' be the points of contact of the similarly drawn tangents to it, the surface of the ellipsoidal belt will be

$$S = \pi c . \left(\frac{a}{b}\, TT' + \frac{b}{a}\, tt'\right).$$

For the semiellipsoid, $\alpha = \frac{\pi}{2}$, $\alpha' = 0$; therefore the point S will go to infinity, S' will coincide with D, and the arc TT' will become $A't$. Similar reductions hold for the hyperbolic arc tt'.

PROP. V. The superficial area of the hyperboloids may, to a certain extent, be expressed in the same manner.

In the hyperboloid of one sheet, if θ be measured from the imaginary axis, we shall find the preceding investigation strictly applicable for all values of θ between $\frac{\pi}{2}$ and $\tan^{-1}\frac{c}{b}$. For values less than this, the integrals I and I' become imaginary, which is explained by observing that the cones whose intersections with the surface give the bounding curves, will in this case fall partly outside the surface. In the hyperboloid of two sheets it is necessary to measure θ from the real axis, and in this I and I' continue real from

$$\theta = 0 \text{ to } \theta = \tan^{-1}\frac{c}{a};$$

after this they become imaginary. The explanation of this is the same as for the hyperboloid of one sheet.

PROP. VI. To find what expression is to be substituted for $a^2 b^2 c^2 \dfrac{\sin\theta\, d\theta\, d\phi}{p^4}$ in the case of either of the paraboloids.

Adopting the usual mode of deriving the properties of the paraboloid from those of the ellipsoid or hyperboloid, we shall put $a^2 = mc$, $b^2 = nc$, and then make c infinite. Performing these operations we shall find

$$\frac{a^2 b^2 c^2}{p^4} = \frac{mnc^4}{(mc\sin^2\theta\cos^2\phi + nc\sin^2\theta\sin^2\phi + c^2\cos^2\theta)^2}$$

$$= \frac{mn}{\left(\dfrac{m\sin^2\theta\cos^2\phi + n\sin^2\theta\sin^2\phi}{c} + \cos^2\theta\right)^2} = \frac{mn}{\cos^4\theta}.$$

Hence we have $dS = mn.\dfrac{\sin\theta d\theta\,.\,d\phi}{\cos^4\theta}$

PROP. VII. To integrate the above expression and construct the bounding curves.

$$S = mn\int_a^{a'}\frac{\sin\theta d\theta}{\cos^4\theta}\int_0^{2\pi}d\phi = \frac{2\pi mn}{3}(\sec^3 a' - \sec^3 a).$$

The equations of the bounding curves are as before, $\theta = a$, $\theta = a'$: and since the equation of the tangent plane to the paraboloid is

$$z - z' = \frac{x'}{m}(x - x') + \frac{y'}{n}(y - y'),$$

[redacted]

if x, y be two of the coordinates of a point on one of these curves (the axis of z being the axis of the paraboloid), we shall have $\frac{x^2}{m^2} + \frac{y^2}{n^2} = \tan^2 a$, and $\frac{x^2}{m^2} + \frac{y^2}{n^2} = \tan^2 a'$. Hence the curves may be constructed as follows. In any plane perpendicular to the axis of the paraboloid, describe two ellipses whose axes are in the planes of the principal sections of the paraboloid and proportional to their parameters, and on these ellipses as bases erect two cylinders whose generatrices are parallel to the axis of the paraboloid. These cylinders will cut the surface in the required curves.

Prop. viii. The paraboloidal belt intercepted between any two of the curves described in the foregoing proposition, is proportional to the difference between the radii of curvature of either of the principal sections at the points where they intersect the bounding curves.

It appears from the preceding proposition that

$$S = \frac{2\pi mn}{3} \cdot (\sec^3 a' - \sec^3 a),$$

a, a' being the angles made with the axis by the normal to the surface at any point on the bounding curves. Let R be the radius of curvature, and N the normal to the principal section whose semi-parameter is m at the point where it intersects the first of the bounding curves; then, since N is also normal to the surface, $\sec^3 a = \frac{N^3}{m^3} = \frac{R}{m}$, $\left(\text{since } R = \frac{N^3}{m^2}\right)$. Similarly $\sec^3 a' = \frac{R'}{m}$; therefore $S = \frac{2\pi n}{3}(R' - R)$. Q.E.D.

It is evident that if ρ, ρ' be the similar radii of curvature for the other principal section we shall have $S = \frac{2\pi m}{3}(\rho' - \rho)$. It appears also that if, with the same parameters and with the same principal planes, there be constructed two paraboloids, one elliptic, the other hyperbolic; the cylinders described in Prop. vii. will intercept on them portions whose superficial areas are the same.

Prop. ix. Let three curves be described on the surface of an ellipsoid along the first of which the perpendicular to the tangent plane makes with the axis of z the constant angle γ, along the second β with the axis of y, and along the third a with the axis of x, and let these angles be connected by the

equations $\frac{\tan\alpha}{a} = \frac{\tan\beta}{b} = \frac{\tan\gamma}{c}$;* then if A_3, A_2, A_1 be the included portions of the ellipsoidal surface, we shall have

$$\frac{A_3 - A_2}{a^2} + \frac{A_1 - A_3}{b^2} + \frac{A_2 - A_1}{c^2} = 0.$$

It appears from Prop. 3, that

$$dA_3 = \frac{\pi a^2 b^2 c^2 . \sin\theta d\theta}{\sqrt{\{(a^2\sin^2\theta + c^2\cos^2\theta).(b^2\sin^2\theta + c^2\cos^2\theta\}}} . \left\{\frac{1}{a^2\sin^2\theta + c^2\cos^2\theta} + \frac{1}{b^2\sin^2\theta + c^2\cos^2\theta}\right\}:$$

And in the same way, if we had supposed the angle θ to be measured from the axis of y, we should have had

$$dA_2 = \frac{\pi a^2 b^2 c^2 \sin\theta d\theta}{\sqrt{\{(a^2\sin^2\theta + b^2\cos^2\theta)(c^2\sin^2\theta + b^2\cos^2\theta)\}}} . \left(\frac{1}{a^2\sin^2\theta + b^2\cos\theta} + \frac{1}{c^2\sin^2\theta + b^2\cos^2\theta}\right).$$

and by measuring θ from the axis of x we should have a similar value for dA_1. Now if in the values of dA_1, dA_2, dA_3, respectively, for $\tan\theta$ we substitute ax, bx, cx, it is evident that the limits of integration with regard to x will be the same for all; and it is easy to see that the values of dA_1, dA_2, dA_3, may be put under the following forms,

$$dA_1 = \pi b^2 c^2 \{2 + (b^2 + c^2)x^2\}(1 + a^2x^2)^2 dL\ldots.(1),$$

$$dA_2 = \pi a^2 c^2 \{2 + (a^2 + c^2)x^2\}(1 + b^2x^2)^2 dL\ldots.(2),$$

$$dA_3 = \pi a^2 b^2 \{2 + (a^2 + b^2)x^2\}(1 + c^2x^2)^2 dL\ldots.(3),$$

where $\quad dL = \dfrac{xdx}{(1 + a^2x^2)^{\frac{3}{2}}.(1 + b^2x^2)^{\frac{3}{2}}(1 + c^2x^2)^{\frac{3}{2}}}$.

Multiply equation (1) by $(b^2 - c^2)a^2$, equation (2) by $(c^2 - a^2)b^2$, and equation (3) by $(a^2 - b^2)c^2$, and add them, and it is easy to see that the right-hand member of the new equation will vanish, hence

$$(b^2 - c^2)a^2 dA_1 + (c^2 - a^2).b^2 dA_2 + (a^2 - b^2)c^2 dA_3 = 0,$$

a result which may be put under the form

$$\frac{d(A_1 - A_2)}{a^2} + \frac{d(A_1 - A_3)}{b^2} + \frac{d(A_2 - A_1)}{c^2} = 0,$$

* A relation analogous to this subsists between the perpendiculars on the tangents at the extremities of the elliptical arcs used in Fagnani's theorem, for if α be the angle made with the axis of x by the perpendicular corresponding to the arc which terminates at the extremity of that axis (a) and β the similar angle for the axis b, we shall have $\frac{\tan\alpha}{a} = \frac{\tan\beta}{b}$, as is easily seen.

[redacted]

And since A_1, A_2, A_3, all begin together, the proposition is evident. If instead of supposing A_1, A_2, A_3, to be bounded, each by a single curve, we conceive each of these letters to denote the space included between two such curves, the same theorem holds, provided that the curves of the second series are connected by the same equations as those of the first.

ON THE POLAR EQUATION TO A CHORD OF A CONIC SECTION.

By the Rev. PERCIVAL FROST, M.A., St. John's College.

IN a previous number of the *Mathematical Journal* having noticed a form of the polar equation to the tangent to a conic section, I think that the corresponding equation to the chord, which appears nearly in the same form, may be thought worthy of notice by some of the readers of the Journal.

Let the equation to the conic section be

$$\frac{c}{r} = 1 + e \cos \theta,$$

$\alpha + \beta$, $\alpha - \beta$ the values of θ which correspond to the points of intersection of the chord and conic section, and

$$\frac{c}{r} = m \cos \theta + n \sin \theta$$

the equation to the chord.

At the points of intersection we obtain by equating the sides of the equations

$$(m - e) \cos (\alpha - \beta) + n \sin (\alpha - \beta) = 1,$$

$$(m - e) \cos (\alpha + \beta) + n \sin (\alpha + \beta) = 1.$$

Hence $\quad (m - e) \cos \alpha \cos \beta + n \sin \alpha \cos \beta = 1,$

and $\quad (m - e) \sin \alpha \sin \beta - n \cos \alpha \sin \beta = 0;$

then $\dfrac{m-e}{\cos \alpha} = \dfrac{n}{\sin \alpha} = \dfrac{(m-e) \cos \alpha + n \sin \alpha}{\cos^2 \alpha + \sin^2 \alpha} = \dfrac{\sec \beta}{1} = \sec \beta.$

Therefore the equation to the chord of the conic section is

$$\frac{c}{r} = (e + \sec \beta \cos \alpha) \cos \theta + \sec \beta \sin \alpha \sin \theta$$

$$= \sec \beta \cos (\theta - \alpha) + e \cos \theta.$$

COR. If $\beta = 0$, we obtain the equation to the tangent at the point $\theta = \alpha$,

$$\frac{c}{r} = \cos (\theta - \alpha) + e \cos \theta.$$

By means of this equation the problems proposed in vol. III. p. 87, may be readily solved. For, since the equation to the chord may be written

$$\frac{c \cos \beta}{r} = \cos(\theta - \alpha) + e \cos \beta \cos \theta,$$

this chord touches the conic section whose eccentricity and latus rectum are $e \cos \beta$ and $2c \cos \beta$, the point of contact being in the line bisecting the angle between the distance; if this angle be constant, the conic section is the envelope of the chords.

If the focal distance corresponding to the angle $\alpha - \beta$ be produced, α', β' the values of α, β, corresponding to the produced focal distance and the other

$$\alpha' - \beta' = \alpha + \beta,$$

$$\alpha' + \beta' = \alpha - \beta + \pi,$$

therefore $$2\beta' = \pi - 2\beta,$$

$$\beta' = \frac{\pi}{2} - \beta.$$

And the envelope to the corresponding chord has for its equation

$$\frac{c \sin \beta}{r} = \cos(\theta - \alpha') + e \sin \beta \cos \theta,$$

and $$c^2 = (c \sin \beta)^2 + (c \cos \beta)^2,$$

$$e^2 = (e \sin \beta)^2 + (e \cos \beta)^2,$$

which prove the propositions.

Several problems may be conveniently solved by means of this equation.

PROB. 1. If the angle between two focal distances be bisected by a third which remains fixed in position, the chords joining the extremities of the two focal distances, as they change their position, always pass through a fixed point whose locus is the directrix.

The equation to any chord is

$$\frac{c \cos \beta}{r} = \cos(\theta - \alpha) + e \cos \beta \cos \theta;$$

therefore at the point of intersection with any other, α being constant,

$$\cos(\theta - \alpha) = 0 \ldots\ldots\ldots\ldots\ldots (1),$$

and $$\frac{c}{r} = e \cos \theta \ldots\ldots\ldots\ldots\ldots (2);$$

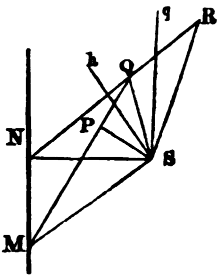
therefore, by (1), $$\theta = \alpha \mp \frac{\pi}{2} \ldots\ldots\ldots\ldots\ldots (3),$$

[redacted]

and by (2) the locus is the directrix (4). Hence the following geometrical construction may be easily performed.

PROB. 2. If three points P, Q, R be a conic section, and the focus be given, to construct the directrix.

Bisect the angles PSQ, QSR, by Sh and Sq. Draw SM and SN perpendicular to Sh, Sq. Produce QP and RQ to meet them in M and N. Join MN, which is the directrix, as appears from (3) and (4) of last problem.



PROB. 3. The locus of the intersection of chords drawn so that β in both is the same, and the difference between the α' constant, is a conic section.

ON THE REDUCTION OF $\dfrac{du}{\sqrt{U}}$, WHEN U IS A FUNCTION OF THE FOURTH ORDER.

By ARTHUR CAYLEY, M.A., Fellow of Trinity College.

IT is well known that the transformation of this differential expression into a similar one, in which the function in the denominator contains only even powers of the corresponding variable, is the first step in the process of reducing $\int\dfrac{du}{\sqrt{U}}$ to elliptic integrals. And, accordingly, the different modes of effecting this have been examined, more or less, by most of those who have written on the subject. The simplest supposition, that adopted by Legendre, and likewise discussed in some detail by Guderman, is that (u) is a fraction, the numerator and denominator of which are linear functions of the new variable. But the theory of this transformation admits of being developed further than it has yet been done, as regards the equation which determines the modulus of the elliptic function. This may be effected most easily as follows.

Suppose

$$U = a + 4bu + 6cu^2 + 4du^3 + eu^4,$$

$$P = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4.$$

Also let $P' = a'x'^4 + 4b'x'^3y' + 6c'x'^2y'^2 + 4d'x'y'^3 + e'y'^4$ be what P becomes after writing

$$x = \lambda x' + \mu y',$$

$$y = \lambda_{,}x' + \mu_{,}y':$$

and let $\quad U' = a' + 4b'u' + 6c'u'^2 + 4d'u'^3 + e'u'^4.$

Suppose, moreover,

$$\begin{cases} k = \lambda\mu_{,} - \lambda_{,}\mu, \\ I = ae - 4bd + 3c^2, \\ I' = a'e' - 4b'd' + 3c'^2, \\ J = ace - ad^2 - eb^2 - c^3 + 2bdc, \\ J' = a'c'e' - a'd'^2 - e'b'^2 - c'^3 + 2b'd'c'; \end{cases}$$

we have evidently

$$xdy - ydx = k.(x'dy' - y'dx'),$$

or

$$\frac{xdy - ydx}{P^{\frac{1}{2}}} = k.\frac{x'dy' - y'dx'}{P'^{\frac{1}{2}}}.$$

Or, writing $\quad u = \dfrac{y}{x}, \quad u' = \dfrac{y'}{x'};$

and therefore

$$\frac{xdy - ydx}{P^{\frac{1}{2}}} = \frac{du}{U^{\frac{1}{2}}}, \quad \frac{x'dy' - y'dx'}{P'^{\frac{1}{2}}} = \frac{du'}{U'^{\frac{1}{2}}}$$

$$\frac{du}{U^{\frac{1}{2}}} = k\,\frac{du'}{U'^{\frac{1}{2}}},$$

the equation between u and u' being

$$u = \frac{\lambda + \mu u'}{\lambda_{,} + \mu_{,}u'}.$$

Next, to determine the relations between the coefficients of U and U'. Since P, P' are obtained from each other by linear transformations (*Math. Journal*, vol. IV. p. 208), we have between the coefficients of these functions and of the transforming equations, the relations

$$I' = k^4.I,$$
$$J' = k^6.J;$$

whence also $\quad \dfrac{J'^2}{I'^3} = \dfrac{J^2}{I^3}.$

Suppose now $\quad U' = a'(1 + pu'^2)(1 + qu'^2),$

or $\quad b' = 0, \quad d' = 0, \quad 6c' = a'(p + q), \quad e' = a'pq;$

whence also $\quad I' = \dfrac{a'^2}{12}.(p^2 + q^2 + 14pq),$

$$J' = \frac{a'^3}{216}(p + q).(34pq - p^2 - q^2);$$

[redacted]

$$p^2 + q^2 + 14pq = 12 \,.\, \frac{k^4}{a'^2}\, I,$$

$$(p + q)\,.\,(34pq - p^2 - q^2) = 216 \,.\, \frac{k^6}{a'^3}\, J;$$

$$\therefore \frac{(p + q)^2 . (34pq - p^2 - q^2)^2}{(p^2 + q^2 + 14pq)^3}$$

$$= \frac{27J^2}{I^3}, \quad \text{whence also} \quad \frac{108pq\,(p - q)^4}{(p^2 + q^2 + 14pq)^3} = 1 - \frac{27J^2}{I^3},$$

which determines the relation between p and q. Also

$$\frac{k}{\sqrt{a'}} = \left(\frac{p^2 + q^2 + 14pq}{12I}\right)^{\frac{1}{4}},$$

so that $$\frac{du}{\sqrt{U}} = \left(\frac{p^2 + q^2 + 14pq}{12I}\right)^{\frac{1}{4}} \frac{du'}{\{(1 + pu'^2)\,(1 + qu'^2)\}^{\frac{1}{2}}}.$$

If in particular $p = -1$, writing also $-q$ for q,

$$\frac{du}{\sqrt{U}} = \left(\frac{q^2 + 14q + 1}{12I}\right)^{\frac{1}{4}} \frac{du'}{\{(1 - u'^2)\,(1 - qu'^2)\}^{\frac{1}{2}}},$$

where $$\frac{108q\,(1 - q)^4}{(q^2 + 14q + 1)^3} = 1 - \frac{27J^2}{I^3}.$$

Suppose, for shortness,

$$M = \frac{27}{4} \,.\, \frac{1}{\left(1 - \dfrac{27J^2}{I^3}\right)}, \quad \text{or} \quad \frac{1}{108}\left(1 - \frac{27J^2}{I^3}\right) = \frac{1}{16M},$$

$$(q^2 + 14q + 1)^3 - 16Mq\,(q - 1)^4 = 0, \quad i.e.$$

$$\left(q + \frac{1}{q} + 14\right)^3 - 16M\left(q^{\frac{1}{2}} - \frac{1}{q^{\frac{1}{2}}}\right)^4 = 0.$$

Let $$q^{\frac{1}{2}} - q^{-\frac{1}{2}} = \frac{4}{(\theta - 1)^{\frac{1}{2}}},$$

then $$\theta^3 - M(\theta - 1) = 0,$$

which determines θ. And then

$$q = \frac{7 + \theta + 4\,(3 + \theta)^{\frac{1}{2}}}{\theta - 1}.$$

Suppose $q = a$ is one of the values of q; the equation becomes

$$\frac{(q^2 + 14q + 1)^3}{q.(q - 1)^4} = \frac{(a^2 + 14a + 1)^3}{a\,(a - 1)^4}$$

$$= \frac{(\beta^8 + 14\beta^4 + 1)^3}{\beta^4.(\beta^4 - 1)^4}, \quad \text{if } a = \beta^4.$$

[redacted]

Now if $q = \left(\frac{1-\beta}{1+\beta}\right)^4$,

$$(q^2+14q+1) = \frac{16\,(\beta^8+14\beta^4+1)}{(1+\beta)^8}, \quad q-1 = -\frac{8\beta\,(1+\beta^2)}{(1+\beta)^4},$$

which satisfy the above equation: hence also, identically,

$$(q^2+14q+1)^3 - q\,(q-1)^4 . \frac{(\beta^8+14\beta^4+1)^3}{\beta^4 . (\beta^4-1)^4}$$

$$= (q-\beta^4)\left(q-\frac{1}{\beta^4}\right)\left\{q-\left(\frac{1-\beta}{1+\beta}\right)^4\right\}\left\{q-\left(\frac{1+\beta}{1-\beta}\right)^4\right\}$$

$$\left\{q-\left(\frac{1-\beta i}{1+\beta i}\right)^4\right\}\left\{q-\left(\frac{1+\beta i}{1-\beta i}\right)^4\right\};$$

or the values of q take the form

$$\beta^4, \ \frac{1}{\beta^4}, \ \left(\frac{1-\beta}{1+\beta}\right)^4, \ \left(\frac{1+\beta}{1-\beta}\right)^4, \ \left(\frac{1-\beta i}{1+\beta i}\right)^4, \ \left(\frac{1+\beta i}{1-\beta i}\right)^4.$$

(Comp. *Abel. Œuv.* tom. I. p. 310).

The equation $$\theta^3 - M\theta + M = 0$$

has its three roots real if $27 - 4M$ is negative, and only a single real root if $27-4M$ is positive. Writing the equation under the form

$$(\theta+3)^3 - 9\,(\theta+3)^2 + (27-M)\,(\theta+3) - (27-4M) = 0,$$

we see that in the former case θ has two values greater than -3, and a single value less than -3. Writing the equation under the form

$$(\theta-1)^3 + 3\,(\theta-1)^2 + (3-M)(\theta-1) + 1 = 0, \ (3-M \text{ is negative})$$

the positive roots are both greater than 1. Hence, in this case, q has four positive values and two imaginary ones. In the second case θ has a single real value, which is greater than -3 and less than 1. Hence q has two negative values and four imaginary ones. In the former case, $I^3 - 27J^2$ is positive, and the function U has either four imaginary factors or four real ones. In the second case, $I^3 - 27J^2$ is negative, or the function U has two real and two imaginary factors.

[redacted]

NOTE ON THE MAXIMA AND MINIMA OF FUNCTIONS OF THREE VARIABLES.

By Arthur Cayley, M.A., Fellow of Trinity College.

If A, B, C, F, G, H, be any real quantities, such that

$$BC + CA + AB - F^2 - G^2 - H^2,$$

and $(A + B + C)(ABC - AF^2 - BG^2 - CH^2 + 2FGH)$

are positive; the six quantities

$$BC - F^2,\ CA - G^2,\ AB - H^2,\ AK,\ BK,\ CK,$$

(where $K = ABC - AF^2 - BG^2 - CH^2 + 2FGH$)

are all of them positive. It is unnecessary to point out the connection of this property with the theory of maxima and minima.

To demonstrate this, writing as usual

$$\begin{array}{ll} BC - F^2 = A', & GH - AF = F', \\ CA - G^2 = B', & HF - BG = G', \\ AB - H^2 = C', & FG - CH = H', \end{array}$$

and K as above: then if $A'', B'', C'', F'', G'', H'', K'$ be formed from A', B', C', F', G', H', as these and K are from A, B, C, F, G, H, we have the well known formulæ

$$\begin{array}{lll} A'' = KA, & F'' = KF, & K' = K^2. \\ B'' = KB, & G'' = KG, & \\ C'' = KC, & H'' = KH, & \end{array}$$

It is required to show that if $A' + B' + C'$ and $A'' + B'' + C''$ are positive, $A', B', C', A'', B'', C''$ are so likewise.

Consider the cubic equation

$$(A' - k)(B' - k)(C' - k) - (A' - k)F'^2 - (B' - k)G'^2 - (C' - k)H'^2 + 2F'G'H' = 0,$$

the roots of which are all real. By the formulæ just given this may be written

$$k^3 - k^2(A' + B' + C') + k\,.\,(A'' + B'' + C'') - K^2 = 0;$$

and the terms of this equation are alternately positive and negative; *i.e.* the roots are all positive. Hence the roots of the limiting equation

$$(B' - k)(C' - k) - F'^2 = 0$$

are positive, *i.e.* $B' + C'$ and $B'C'$ are positive: but from the second condition B', C' are of the same sign. Consequently of the same sign with $B' + C'$ or positive. Also $A'' = B'C' - F'^2$

is positive. Similarly, considering the other limiting equations, A', B', C', A'', B'', C'' are all of them positive.

In connection with the above I may notice the following theorem. The roots of the equation

$$\begin{aligned}(A-ka)(B-kb)(C-ck)-(A-ka)(F-kf)^2\\ -(B-kb)(G-kg)^2-(C-kc)(H-kh)^2\\ +2(F-kf)(G-kg)(H-kh)=0,\end{aligned}$$

are all of them real, if either of the functions

$$Ax^2+By^2+Cz^2+2Fyz+2Gxz+2Hxy,$$
$$ax^2+by^2+cz^2+2fyz+2gxz+2hxy,$$

preserve constantly the same sign. The above form parts of a general system of properties of functions of the second order.

ON THE MATHEMATICAL THEORY OF ELECTRICITY IN EQUILIBRIUM.

By WILLIAM THOMSON, B.A., Fellow of St. Peter's College.

I. *On the Elementary Laws of Statical Electricity.**

1. The elementary laws which regulate the distribution of electricity on conducting bodies have been determined by means of direct experiments, by Coulomb, and in the form he has given them, which is independent of any hypothesis,† they have long been considered as rigorously established. The problem of the distribution of electricity in equilibrium on a conductor of any form was thus brought within the province of mathematical analysis; but the solution, even in the simplest cases, presented so much difficulty that Coulomb, after having investigated it experimentally for bodies of various forms, could only compare his measurements with the results of his theory by very rude processes of approximation. Without however giving rigorous solutions in particular cases, he examined the general problem with great care, and left nothing indefinite in the conditions to be satisfied, so that it was entirely by analytical difficulties that he was stopped As an example of the success of his theoretical investigations, we may refer to the well-known demonstration of the theorem (usually attributed to Laplace) relative to the

* This paper is a translation (with considerable additions) of one which appeared in Liouville's *Journal de Mathématiques*, vol. x. p. 209.

† See the first Note at the end of this paper.

[redacted]

repulsion exercised by a charged conductor on a point near its surface.*

The memoirs of Poisson, on the mathematical theory, contain the analytical determination of the distribution of electricity on two conducting spheres placed near one another, the solution being worked out in numbers in the case of two equal spheres in contact, which had been investigated experimentally by Coulomb (as well as in another case, not examined by Coulomb, which is given as a specimen of the numerical results that may be deduced from the formulæ). The calculated ratios of the intensities at different points of the surface he is therefore enabled to compare with Coulomb's measurements, and he finds an agreement which is quite as close as could be expected, when we consider the excessively difficult and precarious nature of quantitative experiments in electricity: but the most remarkable confirmation of the theory from these researches is the entire agreement of the principal features, even in some very singular phenomena, of the experimental results with the theoretical deductions. For a complete account of the experiments we must refer to Coulomb's fifth memoir (*Histoire de l' Académie*, 1787), and for the mathematical investigations to the first and second memoirs of Poisson (*Mémoires de l' Institut*, 1811), or to the treatise on Electricity in the *Encyclopædia Metropolitana*, where the substance of Poisson's first memoir is given.

The mathematical theory received by far the most complete development which it has hitherto obtained, in Green's *Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*,† in which a series of general theorems were demonstrated, and many interesting applications made to actual problems.‡

Of late years some distinguished experimentalists have begun to doubt the truth of the laws established by Coulomb, and have made extensive researches with a view to discover the laws of certain phenomena which they considered incompatible with his theory. The most remarkable works of this kind have been undertaken independently by Mr. Snow Harris and Mr. Faraday, and in their memoirs, published in the *Philosophical Transactions*, we find detailed accounts of their researches. All the experiments, however, which they have made, having direct reference to the distribution of electricity in equilibrium, are, I think, in full accordance with the laws of Coulomb, and must therefore, instead of objections to his theory, be considered as confirming it. As however many

* See Note II. † Nottingham, 1828. ‡ See Note III.

have believed Coulomb's theory to be overturned by these investigations, and as others have at least been led to entertain doubts as to its certainty or accuracy, the following attempt to explain the apparent difficulties is made the subject of the first of a series of papers in which various parts of the mathematical theory of electricity, and corresponding problems in the theories of magnetism and heat, will be considered.

2. We may commence by examining some experimental results published in Mr. Harris's first memoir *On the Elementary Laws of Electricity.** After describing the instruments employed in his researches, Mr. Harris gives the details of some experiments with reference to the attraction exercised by an insulated electrified body on an uninsulated conductor placed in its neighbourhood. The first result which he announces is that, when other circumstances remain the same, the attraction varies as the square of the quantity of electricity with which the insulated body is charged. It is readily seen, as was first remarked by Dr. Whewell in his *Report on the Theories of Electricity, &c.,*† that this is a rigorous deduction from the mathematical theory, following from the fact that the quantity of electricity induced upon the uninsulated body is proportional to the charge on the electrified body by which it is attracted.

The remaining results have reference to the force of attraction at different distances, and with bodies of different forms opposed. As these are generally very irregular (such as "plane circular areas backed by small cones"), we should not, according to Coulomb's theory, expect any very simple laws, such as Mr. Harris discovers, to be rigorously true. Accordingly, though they are announced by him without restriction, we must examine whether the experiments from which they have been deduced are of a sufficiently comprehensive character to lead to any general conclusions with respect to electrical action. Now in the first place, we find that in all of them the attraction is "independent of the form of the unopposed parts" of the bodies, which will be the case only when the intensity of the induced electricity on the unopposed parts of the uninsulated body is insensible. According to the mathematical theory, and according to Mr. Faraday's researches "on induction in curved lines," which will be referred to below, the intensity never absolutely vanishes at any point of the uninsulated body: but it is readily seen that in the case of

* Philosophical Transactions, 1834.
† British Association Report for 1837.

Mr. Harris's experiments, it will be so slight on the unopposed portions that it could not be perceived without experiments of a very refined nature, such as might be made by the proof plane of Coulomb, which is in fact, with a slight modification, the instrument employed by Mr. Faraday in the investigation. Now to the degree of approximation to which the intensity on the unopposed parts may be neglected, the laws observed by Mr. Harris when the opposed surfaces are plane may be readily deduced from the mathematical theory. Thus let v be the potential in the interior of the charged body, A, a quantity which will depend solely on the state of the interior coating of the battery with which in Mr. Harris's experiments A is connected, and will therefore be sensibly constant for different positions of A relative to the uninsulated opposed body, B. Let a be the distance between the plane opposed faces of A and B, and let S be the area of the opposed parts of these faces, which will in general be the area of the smaller, if they be unequal. When the distance a is so small that we may entirely neglect the intensity on all the unopposed parts of the bodies, it is readily shewn from the mathematical theory that (since the difference of the potentials at the surfaces of A and B is v) the intensity of the electricity produced by induction at any point of the portion of the surface of B which is opposed to A, is $\frac{v}{4\pi a}$, the intensity at any point which is not so situated being insensible. Hence the attraction on any small element ω, of the portion S of the surface of B, will be in a direction perpendicular to the plane and equal to $2\pi\left(\frac{v}{4\pi a}\right)^2$.* Hence the whole attraction on B is

$$\frac{v^2 S}{8\pi a^2}.$$

This formula expresses all the laws stated by Mr. Harris as results of his experiments in the case when the opposed surfaces are plane.

3. When the opposed surfaces are curved, for instance when A and B are equal spheres, we can make no approximation analogous to that which has led us to so simple an expression in the case of opposed planes; and we find accordingly that no such simple law for the attraction in this case has been announced by Mr. Harris. He has however found

* See *Mathematical Journal*, vol. III. p. 275.

that it is expressed with tolerable accuracy by the formula

$$F = \frac{k}{c\,(c - 2a)},$$

where c is the distance between the centres of the spheres, a the radius of each, k a constant, which will depend on a and on the charge of the battery with which A is in communication. Though however this formula may give results which do not differ very much from observation within a limited range of distances, it cannot, according to any theory, be considered as expressing the physical law of the phenomenon. For, according to it, when the balls are very distant, F ultimately varies as $\frac{1}{c^2}$. Now it is clear that the law of force must ultimately become the inverse cube of the distance, since the quantity of electricity induced upon B will be ultimately in the inverse ratio of the distance, and the attraction between the balls as the product of the quantities of electricity directly, and as the square of the distance inversely, and hence the formula given by Mr. Harris cannot express the law of force when the balls are very distant. In the experiments by which his formula is tested, the force of attraction is measured by means of an ordinary balance and weights: the only comparison of results which he publishes is transcribed in the following table.

Dist. of Centres.	Measured Force in Grains.	Values of $\frac{15c_1\,(c_1 - 2)}{c\,(c - 2)}$.
$c_1 = 2.3$	15	15
$c_2 = 2.5$	8.25 +	8.28
$c_3 = 2.8$	4.6 +	4.62
$c_4 = 3.0$	3.5 −	3.45

From this table we see that the formula is verified in three cases to the extent of accuracy of the experiments. Comparisons extended to a much wider range of distances would be required to establish it, and it would be necessary to take precautions to prevent the experimental results from being influenced by disturbing causes. In the experiments made by Mr. Harris we find that no precautions have been taken to avoid the disturbing influence of extraneous conductors, which, according to the descriptions and drawings he gives of his instruments, seem to exist very abundantly in the neighbourhood

of the bodies operated upon, being partly metal in connection with the insulated system with which the body A communicates, and partly uninsulated metal, in the fixed parts of the electrometer, and in the moveable parts by which B is supported. The general effect produced by the presence of such bodies in disturbing the observed law of force, must be to make it diminish less rapidly with the distance when A and B are separated by a considerable interval: and it is probably owing, at least in part, to such disturbing causes that Mr. Harris's results nearly agree, as far as they go, with a formula which would ultimately give for the law of force the inverse square of the distance between A and B, instead of the inverse cube.

4. The determination by the mathematical theory of the attraction or repulsion between two electrified conducting spheres has not hitherto, so far as I am aware, been attempted, and would present considerable difficulty by means of the formulæ ordinarily given for such problems. It may, however, very readily be effected by means of a general theorem on the attraction between electrified conductors, which will be given in a subsequent paper. Thus, if $F(c)$ be the force of attraction, corresponding to the distance c between the centres, in the particular case when the two spheres are equal (the radius of each being unity), and the potential in the interior of one of them is nothing (as will be the case when the body is uninsulated), the potential in the interior of the other being v, I have found the following formulæ which express $F(c)$ by a converging series.

$$\text{(A)} \qquad F(c) = v^2 c \left(\frac{P_1}{Q_1^2} + \frac{P_2}{Q_2^2} + \frac{P_3}{Q_3^2} + \&\text{c.} \right), \quad \text{where}$$

$$\text{(B)}, \qquad \begin{cases} Q_1 = c^2 - 1, \\ Q_3 = (c^2 - 2)\, Q_1 - 1, \\ Q_{n+2} = (c^2 - 2)\, Q_{n+1} - Q_n. \end{cases}$$

$$\text{(C)}, \qquad \begin{cases} P_1 = 1, \\ P_2 = 2c^2 - 3. \\ P_{n+2} = (c^2 - 2)\, P_{n+1} + (Q_{n+1} - P_n). \end{cases}$$

These formulæ enable us to calculate Q_1, Q_2, Q_3, Q_4, &c., and then P_1, P_2, P_3, P_4, &c., successively, by a simple and uniform arithmetical process, for any particular value of c. I have thus calculated the values of $\frac{F(c)}{v^2}$ in five cases, the

first four of which are those examined by Mr. Harris, and have obtained the following results, each of which is true to five places of decimals.

c	$v^{-2}F(c)$.
2·3	0·32926
2·5	0·17423
2·8	0·09168
3·0	0·06592
4·0	0·02075

To compare these with Mr. Harris's measurements we may calculate the value of the potential in his battery, during the observations, by means of his first result, and thence find the attraction for the other three cases by means of the calculated values of $v^{-2}F(c)$. Thus we have $v^{-2} \times 15 = 3293$, which gives

$$v^2 = 45·56,$$

and hence

$$F(2·5) = 7·94,$$
$$F(2·8) = 4·18,$$
$$F(3) = 3·00.$$

These numbers differ considerably from Mr. Harris's results, but in the direction indicated by the considerations mentioned above.

5. The most important part of the researches of Mr. Harris is that in which he investigates the insulating power of air of different densities. The result at which he arrives is, that the intensity necessary to produce a spark depends solely on the density of the air, and not otherwise on the pressure or temperature. He thus shews that the conducting power of flame, of heated bodies, and of a vacuum, are due solely to the rarefaction of the air in each case. He also shews that the intensities necessary to produce a spark, are in the simple ratios of the densities of the air.

6. In a subsequent memoir, by the same author,* we find additional experiments on the elementary principles of the theory of electricity. The first series which is described, was made for the purpose of testing the truth of Coulomb's law, that the repulsion of two similarly charged points is inversely as the square of the distance, and directly as the product of the masses. In experiments of this kind in which accurate quantitative results are aimed at, many precautions are ne-

* *Philosophical Transactions*, 1836.

[redacted]

cessary. Thus all conducting bodies except those operated upon, must be placed beyond the reach of influence, and the distance between the repelling bodies must be considerable with reference to their linear dimensions, so that the distribution of electricity on each may be uninfluenced by the presence of the other. Also the bodies should be spheres, so that the attraction may be the same as if the whole electricity of each were collected at its centre; and the distance to be measured will then be the distance between the centres. These conditions have been expressly mentioned by Coulomb, and they have been fulfilled, as far as possible, in his researches, as we see by the descriptions of the experiments made, which we find in his memoirs. He has thus arrived by direct measurement at the law, which we know by a mathematical demonstration,* founded upon independent experiments, to be the rigorous law of nature, for electrical action. None of these precautions however have been taken in the experiments described in Mr. Harris's memoir, and the results are accordingly unavailable for the accurate *quantitative* verification of any law, on account of the numerous unknown disturbing circumstances by which they are affected. The phenomena which he observes, however, afford *qualitative* illustrations of the mathematical theory of a very interesting nature, as may be seen from the following examples of his results.

(*a*) When the distance between the bodies is great with reference to their linear dimensions, the repulsion is inversely as the square of the distance, and directly as the product of the masses.

(*b*) When the distance is small, the action becomes apparently irregular. Thus if the quantities of electricity on the two bodies be equal, the force, which is always of repulsion, does not increase so rapidly when the bodies approach, as if it followed the law of the inverse square of the distance.

(*c*) If the charges be unequal; the repulsion ceases at a certain distance, and at all smaller distances there is attraction between the bodies.

These results are, with all their peculiarities, in full accordance with the theory of Coulomb, which indicates that, if the quantities of electricity be equal, and the bodies equal and similar, there will be repulsion in every position: but if there be any difference, however small, between the charges, the repulsion will necessarily cease, and attraction commence, before contact takes place, when one body is made to approach the other. Unless, however, the difference of the charges

* See Murphy's *Electricity*, p. 41, o[illegible] [illegible], Art. 154.

be sufficiently considerable, a spark may pass between the bodies, and render the charges equal, before attraction commences. In Mr. Harris's experiments, in which the bodies seem to have been nearly oblate spheroids, the attraction is generally sensible before the distance is small enough to allow a spark to pass, if the charge on one be double of that on the other.

Mr. Harris next proceeds to investigate the theory of the proof plane, and to examine whether it can be considered as indicating with certainty the intensity of electricity at any part of a charged body, and, principally from an experiment made on a charged non-conductor (a hollow sphere of glass), comes to a negative conclusion. It should be remembered, however, that, the proof plane having never been applied to determine the intensity at points of the surface of a charged non-conductor, such conclusions in no way interfere with adopted ideas. Since there can be no manner of doubt as to the theory of this valuable instrument, as we find it explained by M. Pouillet,* nor as to the experimental use of it made by Coulomb, it is unnecessary to enter more at length on the subject here.

7. Mr. Faraday's researches on electrostatical induction, which are published in a memoir forming the eleventh series of his Experimental Researches in Electricity, were undertaken with a view to test an idea which he had long possessed, that the forces of attraction and repulsion exercised by free electricity, are not the resultant of actions exercised at a distance, but are propagated by means of molecular action among the contiguous particles of the insulating medium surrounding the electrified bodies, which he therefore calls the *dielectric*. By this idea he has been led to some very remarkable views upon induction, or in fact upon electrical action in general. As it is impossible that the phenomena observed by Faraday can be incompatible with the results of experiment which constitute Coulomb's theory, it is to be expected that the difference of his ideas from those of Coulomb must arise solely from a different method of stating, and interpreting physically, the same laws: and farther, it may I think be shewn that either method of viewing the subject, when carried sufficiently far, may be made the foundation of a mathematical theory which would lead to the elementary principles of the other as consequences. This theory would accordingly be the expression of the ultimate law of the phenomena, inde-

* See Note IV.

pendently of any physical hypothesis we might, from other circumstances, be led to adopt. That there are necessarily two distinct elementary ways of viewing the theory of electricity, may be seen from the following considerations, founded on the principles developed in a previous paper in this Journal.*

Corresponding to every problem relative to the distribution of electricity on conductors, or to forces of attraction and repulsion exercised by electrified bodies, there is a problem in the uniform motion of heat which presents the same analytical conditions, and which therefore, considered mathematically, is the same problem. Thus, let a conductor A, charged with a given quantity of electricity, be insulated in a hollow conducting shell, B, which we may suppose to be uninsulated. According to the mathematical theory, an equal quantity of electricity of the contrary kind will be attracted to the interior surface of B, (or the surface of B, as we may call it to avoid circumlocution), and the distribution of this charge, and of the charge on A, will take place so that the resultant attraction at any point of each surface may be in the direction of the normal. This condition being satisfied, it will follow that there is no attraction on any point within A, or without the surface of B, that is, on any point within either of the conducting bodies. The most convenient mathematical expression for the condition of equilibrium, is that the potential at any point P† must have a constant value when P is on the surface of A, and the value nothing when P is on the surface of B; and it will follow from this that the potential will have the same constant value for any point within A, and will be equal to nothing for any point without the surface of B.

If A be subject to the influence of any uninsulated conductors, we must consider such bodies as belonging to the shell in which A is contained, and their surfaces as forming part of the surface of B: in such cases this surface will generally be the interior surface of the walls of the room in which A is contained, and of all uninsulated conductors in the room. If however we have to consider the case in which A is subject to no external influence, we must suppose every part of the surface of B to be very far from A. The most general problem we can contemplate in electricity

* On the Uniform Motion of Heat, and its Connection with the Mathematical Theory of Electricity, Vol. III. p. 73.

† The term used by Green for the sum of the quotients obtained by dividing the product of each element of the surfaces of A and B, and its electrical intensity, by its distance from P.

(exclusively of the case in which the insulating medium is heterogeneous, and exercises a special action, which will be alluded to below), is to determine the potential at any point when A, instead of being a single conductor, is a group of separate insulated conductors charged to different degrees, and when there are non-conductors electrified in a given manner, placed in the insulating medium, in the neighbourhood. The conditions of equilibrium will still be that the potential at each surface due to all the free electricity must be constant, and the theorems stated above will still be true: thus the attraction will be nothing in the interior of each portion of A, and without the surface of B; and the whole quantity of induced electricity on the latter surface will be the algebraic sum of the charges of all the interior bodies with its sign changed. When the potential due to such a system is determined for every point, the component of the resultant force at any point P, in any direction PL, may be found by differentiation, being the limit of the difference between the values of the potential at P, and at a point Q, in PL, divided by PQ, when P moves up towards and ultimately coincides with P, and the direction of the force, on a *negative* particle, being that in which the potential increases. By Coulomb's theorem, the intensity at any point in one of the conducting surfaces is equal to the attraction (on a negative unit) at that point, divided by 4π.

Now if we wish to consider the corresponding problem in the theory of heat, we must suppose the space between A and B, instead of being filled with a dielectric medium (that is a non-conductor for electricity), to be occupied by any homogeneous solid body, and sources of heat or cold to be so distributed over the terminating surfaces, or the interior surface of B and the surface of A, that the permanent temperature at the first surface may be zero, and at the second shall have a certain constant value, the same as that of the *potential* in the case of electricity. If A consist of different isolated portions, the temperature at the surface of each will have a constant value, which is not necessarily the same for the different portions. The problem of *distributing sources of heat, according to these conditions*, is mathematically identical with the problem of distributing *electricity in equilibrium* on the surfaces of A and B. In the case of heat, the *permanent temperature* at any point replaces the *potential* at the corresponding point in the electrical system, and consequently the *resultant flux of heat* replaces the *resultant attraction* of the electrified bodies, in direction and magnitude. The

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problem in each case is determinate, and we may therefore employ the elementary principles of one theory, as theorems, relative to the other. Thus, in the paper in which these considerations are developed, Coulomb's fundamental theorem relative to electricity is applied to the theory of heat; and self-evident propositions in the latter theory are made the foundation of Green's theorems in electricity.* Now the laws of motion for heat which Fourier lays down in his *Théorie Analytique de la Chaleur*, are of that simple elementary kind which constitute a mathematical theory properly so called; and therefore, when we find corresponding laws to be true for the phenomena presented by electrified bodies, we may make them the foundation of the mathematical theory of electricity: and this may be done if we consider them merely as actual truths, without adopting any physical hypothesis, although the idea they naturally suggest is that of the propagation of some effect by means of the mutual action of contiguous particles; just as Coulomb, although his laws naturally suggest the idea of material particles attracting or repelling one another at a distance, most carefully avoids making this a *physical hypothesis*, and confines himself to the consideration of the mechanical effects which he observes and their necessary consequences.†

All the views which Faraday has brought forward, and illustrated or demonstrated by experiment, lead to this method of establishing the mathematical theory, and, as far as the analysis is concerned, it would, in most *general* propositions, be even more simple, if possible, than that of Coulomb. (Of course the analysis of *particular* problems would be identical in the two methods). It is thus that Faraday arrives at a knowledge of some of the most important of the general theorems, which, from their nature, seemed destined never to be perceived except as mathematical truths. Thus, in his theory, the following proposition is an elementary principle. Let any portion a of the surface of A be projected on B, by means of lines (which will be in general curved) possessing the property that the resultant electrical force at any point of each of them is in the direction of the tangent: the quantity of electricity produced by induction on this projection is equal to the quantity of the opposite kind of elec-

* It was not until some time after that paper was published, that I was able to add the direct analytical demonstrations of the theorems, which are given in the papers on "General Propositions in the Theory of Attraction," vol. III. pp. 189, 201, and which I have since found are the same as those originally given by Green.

† See Note I.

tricity on a.* The lines thus defined are what Faraday calls the "curved lines of inductive action." For a detailed account of the experiments by which these phenomena are investigated, reference must be made to Mr. Faraday's own memoirs, published in the Philosophical Transactions, and in a separate form in his Experimental Researches.

8. The hypothesis adopted by Faraday, of the *propagation* of inductive action, naturally led him to the idea that its effects may be in some degree dependent upon the nature of the insulating medium or dielectric, by which, according to this view, it is transmitted. In the second part of his memoir he describes a series of researches instituted to put this to the test of experiment, and arrives at the following conclusions.

If the dielectric be air, the inductive action is quite independent of its density or temperature (which, as Mr. Faraday remarks, agrees perfectly with previous results obtained by Mr. Harris); and in general, if the dielectric be any gas or vapour capable of insulating a charge, the inductive action is invariable. Hence he concludes that "*all gases have the same power of*, or *capacity for*, sustaining induction through them, (which might have been expected when it was found that no variation of density or pressure produced any effect.)"

When the dielectric is solid, the induction is greater than through air, and varies according to the nature of the substance. Numbers which measure the "specific inductive capacities" of the dielectrics employed (sulphur, shell lac, glass, &c.), are deduced from the experiments.

To express these results in the language of the mathematical theory, let us recur to the supposition of a body, A, charged with a given quantity of electricity, and insulated in the interior of a closed conducting shell, B. The potential of the system at the interior surface of B, and at every point without this surface, will be nothing; at the surface and in the interior of A it will have a constant value, which will depend on the form, magnitude, and relative position of the surfaces A and B, on the quantity of electricity on A, and, according to Faraday's discovery, on the *dielectric power* of the insulating medium which fills the space between A and B. If this be gaseous, neither its nature, nor its state as to temperature, pressure, or density, will affect the value of the potential in A; but if it be a solid substance, such as sulphur or shell lac, the value of the potential will be less than when the space is occupied by air, and will vary with the nature of the insulating solid.

* See Note IV.

The result in the case of a gaseous dielectric is what would follow from Coulomb's theory, if we consider gases to be quite impermeable to electricity, and to be entirely unaffected by electrical influence. The phenomena observed with solid dielectrics, which agree with the circumstance observed by Nicholson, that the *dissimulating power* of a Leyden phial depends on the nature of the glass of which it is made, as well as on its thickness, have been by some attributed to a slight degree of conducting power, or of penetrability, possessed by solid insulators. This explanation, however, seems to be very insufficient; and besides, Faraday has estimated the nature of the effects of imperfect insulation, by independent experiments, and has established, in what seems to be a very satisfactory manner, the existence of a peculiar action in the interior of solid insulators when subjected to electrical influence. As far as can be gathered from the experiments which have yet been made, it seems probable that a dielectric, subjected to electrical influence, becomes excited in such a manner that every portion of it, however small, possesses *polarity* exactly analogous to the magnetic polarity induced in the substance of a piece of soft iron under the influence of a magnet. By means of a certain hypothesis regarding the nature of magnetic action,* Poisson has investigated the mathematical laws of the distribution of magnetism and of magnetic attractions and repulsions. These laws seem to represent in the most general manner the state of a body polarized by influence, and therefore, without adopting any particular mechanical hypothesis, we may make use of them to form a mathematical theory of electrical influence in dielectrics, the truth of which can only be established by a rigorous comparison of its results with experiment.

Let us therefore consider what would be the effect, according to this theory, which would be produced by the presence of a solid dielectric, C, placed in the space between A and B, the rest of which is occupied by air. The action of C, when

* Faraday adopts the corresponding hypothesis to explain the action of a solid dielectric, which he states thus:—"If the space round a charged globe were filled with a mixture of an insulating dielectric, as oil of turpentine or air, and small globular conductors, as shot, the latter being at a little distance from each other, so as to be insulated, then these in their condition and action exactly resemble what I consider to be the condition and action of the particles of the insulating dielectric itself. If the globe were charged, these little conductors would all be polar; if the globe were discharged, they would all return to their normal state, to be polarized again upon the recharging of the globe." (*Experimental Researches*, §. 1679.) The results of the mathematical analysis of such an action are given in the text. It may be added that the value of the coefficient k will differ sensibly from unity if the volume occupied by the small conducting balls bear a finite ratio to that occupied by the insulating medium.

excited by the influence of the electricities on A and B, may (as Poisson has shewn for magnetism) be represented, whether on points within or without C, by a certain distribution or positive electricity on one portion of the surface of C, and of an equal quantity of negative electricity on the remainder. The condition necessary and sufficient for determining this distribution may (as can be shewn from Poisson's analysis) be expressed as follows. Let R be the resultant force on a point P without C, and R' on a point P' within C, due to the electrified surfaces A and B, and to the imagined distribution on C. If P and P' be taken infinitely near one another, and consequently each infinitely near the surface of C, the component of R' in the direction of the normal must bear to the component of R in the same direction a constant ratio $\left(\frac{1}{k}\right)$ depending on the capacity for dielectric induction of the matter of C.* The components of R and R' in the tangent plane will of course be equal and in the same direction, and, if ρ be the intensity of the imagined distribution on the surface of C, in the neighbourhood of P and P', the difference of the normal components will be $4\pi\rho$, as is evident from Coulomb's theorem, referred to above.

Let us now suppose C to be a shell surrounding A, and let S and S', its interior and exterior surfaces, be *surfaces of equilibrium* in the system of forces due to the action of A and B, and of the polarity of C. It may be shewn that the same surfaces S, S', would necessarily be surfaces of equilibrium, if C were removed and the whole space were filled with air; and consequently, that the whole series of surfaces of equilibrium, commencing with A and ending with B, will be the same in the two cases. Hence the resultant force due to the excitation of the dielectric C, or to the imagined distributions of electricity on S and S' which produce it, on points within S or without S', must be such as not to alter the distributions on A and B when the quantity on A is given, and is therefore nothing. Accordingly, let Q be the total force on a point indefinitely near S, and within it; Q' the total force on a point without S', but indefinitely near it. Since the forces on points without S and within S' indefinitely near the former points are, according to the law stated above, $\frac{Q}{k}$ and $\frac{Q'}{k}$, it follows that the intensities of the imagined distributions on

* From this it follows that, in the case of heat, C must be replaced by a body whose conducting power is k times as great as that of the matter occupying the remainder of the space between A and B.

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S and S', in the neighbourhood of the points considered, are

$$-\frac{1}{4\pi}\left(Q - \frac{Q}{k}\right) \text{ and } \frac{1}{4\pi}\left(Q' - \frac{Q'}{k}\right).$$

Hence, if U, U' be the potentials at S, S', due to A and B alone, and v the potential at any point P, it follows* that the potential at P, due to the polarity of the dielectric, is

$$-\left(1 - \frac{1}{k}\right) U + \left(1 - \frac{1}{k}\right) U',$$

or
$$-\left(1 - \frac{1}{k}\right) v + \left(1 - \frac{1}{k}\right) U',$$

or
$$-\left(1 - \frac{1}{k}\right) v + \left(1 - \frac{1}{k}\right) v, \text{ that is, } 0,$$

according as P is within S, within S' and without S, or without S'. Hence the total potential will be, according to the position of P,

$$v - \left(1 - \frac{1}{k}\right)(U - U'),$$

or
$$\frac{v}{k} + \left(1 - \frac{1}{k}\right) U',$$

or
$$v.$$

Hence the sole effect of the dielectric C, on the state of A and B, is to diminish the potential in the interior of the former by the quantity

$$\left(1 - \frac{1}{k}\right)(U - U').$$

If the whole space between A and B be occupied by the solid dielectric, the surfaces S and A will coincide, as also, S' and B, and therefore $U = V$, $U' = 0$. Hence the potential in the interior of A will be

$$\frac{V}{k},$$

or the fraction $\frac{1}{k}$ of the potential, with the same charge on A, and with a gaseous dielectric. From this it follows that, when the dielectric is solid, it would require, to produce a given potential in the interior of A, k times the charge which would be necessary to produce the same potential when the dielectric is gaseous, and therefore the body A in a given state, defined by the potential in its interior, produces on the interior surface of B, by induction, through the solid dielectric, a quantity of electricity k times as great as through a gaseous dielectric. On this account Faraday calls the property of a dielectric measured by k, its "specific inductive capacity."

* See Green's *Essay*, Art. 12; or, *Math. Journal*, vol. III. p. 75.

In Faraday's experiments an apparatus (which is in fact a Leyden phial, in which any solid or fluid may be substituted for the glass dielectric of an ordinary Leyden phial) is used, corresponding to the case we have been considering, in which A is a conducting sphere (2.33 inches in diameter), and B a concentric spherical shell surrounding it (the distance between the surfaces of A and B being ·62 of an inch). In the shell B there is an aperture into which a shell lac stem is fixed; a wire, attached to A, passes through the centre of this stem to the outside of the shell, and supports a ball of metal, M, which is thus insulated and connected with A. It may be shewn that in such an apparatus the state of the ball A and of the shell B will approximately be not affected by the aperture in the latter, or by the wire supporting M, and that the distribution of electricity on M will be approximately the same as if the wire supporting it and the conductors A and B were removed. Hence the sole relation between A and M will be that the *potentials* in their interiors are the same; and therefore the latter, which is accessible, may be taken as an index of the state of the former.

To determine the specific inductive capacity of any dielectric, Faraday uses two apparatus of the kind just described, precisely equal and similar, in one of which the space between A and B is filled with air, and in the other with the dielectric to be examined. One of these apparatus is charged, and the intensity measured: the balls M, M' in the two are then made to touch and separated again, and the remaining intensity on the first (which is equal to the intensity imparted to the second) is measured. If this be found to differ from half the original intensity, it will follow that the specific inductive capacity of the substance examined differs from that of air, which is unity, and its value may be determined by means of a simple expression from the experimental data. To investigate this, let us first suppose each apparatus to be charged, and let it be required to find the intensity on the balls after they are made to touch, and then removed from mutual influence; and let the dielectrics be any two substances, whose inductive capacities are k, k'. Let ρ, ρ' be the intensities before, and σ the common intensity after contact. Then, denoting by Q, Q' the quantities of electricity constituting the charges before, and q, q' after contact, we shall have, by the principles already developed,

$$\frac{Q}{Q'} = \frac{k\rho}{k'\rho'}, \quad \frac{\sigma}{\rho} = \frac{q}{Q}, \quad \frac{\sigma}{\rho'} = \frac{q'}{Q'}.$$

Also $$Q + Q' = q + q'.$$

Hence we deduce $$\sigma = \frac{k\rho + k'\rho'}{k + k'}.$$

In the experiment described, one of the dielectrics is air. Hence, to obtain the required formula, we may put $k' = 1$, in this equation, and then resolve for k.

Thus we find $$k = \frac{\sigma - \rho'}{\rho - \sigma}.$$

If only one of the apparatus be originally charged, according as it is the first or the second, we shall have

$$k = \frac{\sigma}{\rho - \sigma},$$

or $$k = \frac{\rho' - \sigma}{\sigma}.$$

If the substance examined (the dielectric of the first apparatus) be any gas, or air in a different state as to pressure or temperature from the air of the second apparatus, Faraday always finds the intensity after contact to be half the original intensity, and hence for every gaseous body $k = 1$.

If the dielectric of the first apparatus be solid, the intensity after contact is found to be greater than half the original intensity when the first, and less than half when the second is the apparatus originally charged. Hence for a solid dielectric, $k > 1$. For sulphur Faraday finds the value to be rather more than 2·2; for shell-lac, about 2; and for flint-glass, greater than 1·76.

The commonly received ideas of attraction and repulsion exercised at a distance, independently of any intervening medium, are quite consistent with all the phenomena of electrical action which have been here adduced. Thus we may consider the particles of air in the neighbourhood of electrified bodies to be entirely uninfluenced, and therefore to produce no effect in the resultant action on any point: but the particles of a solid non-conductor must be considered as assuming a polarized state when under the influence of free electricity, so as to exercise attractions or repulsions on points at a distance, which, with the action due to the charged surfaces, produce the resultant force at any point. It is, no doubt, possible that such forces at a distance may be discovered to be produced entirely by the action of contiguous particles of some intervening medium, and we have an analogy for this in the case of heat, where certain effects which follow the same laws are undoubtedly propagated

from particle to particle. It might also be found that magnetic forces are propagated by means of a second medium, and the force of gravitation by means of a third. We know nothing however of the molecular action by which such effects could be produced, and in the present state of physical science it is necessary to admit the known facts in each theory as the foundation of the ultimate laws of action at a distance.

St. Peter's College, Nov. 22, 1845.

NOTES.

NOTE I.

Coulomb has expressed his theory in such a manner that it can only be attacked in the way of proving his experimental results to be inaccurate. This is shewn in the following remarkable passage in his sixth memoir, which follows a short discussion of some of the physical ideas then commonly held with reference to electricity. "*Je préviens pour mettre la théorie qui va suivre à l' abri de toute dispute systématique, que dans la supposition des deux fluides électriques, je n' ai d' autre intention que de présenter avec le moins d' élémens possible, les resultats du calcul et de l' expérience, et non d' indiquer les véritables causes de l' électricité. Je renverrai, à la fin de mon travail sur l' électricité, l' examen des principaux systèmes auxquels les phénomènes électriques ont donné naissance.*"—Histoire de l'Académie, 1788, p. 673.

NOTE II.

This theorem may be stated as follows. Let A be a closed surface of any form, and let matter, attracting inversely as the square of the distance, be so distributed over it that the resultant attraction on an interior point is nothing: the resultant attraction on an exterior point, indefinitely near any part of the surface, will be perpendicular to the surface and equal to $4\pi\rho$, if $\rho\omega$ be the quantity of matter on an element ω of the surface in the neighbourhood of the point. Coulomb's demonstration of this theorem may be found in a preceding paper in the *Mathematical Journal*, vol. III. p. 74. He gives it himself, in his sixth memoir on Electricity (*Histoire de l' Academie*, 1788, p. 677), in connection with an investigation of the theory of the proof plane in which, by an error that is readily rectified, he arrives at the result that a small insulated conducting disc, put in contact with an electrified conductor at any point, and then removed, carries with it as much electricity as lies on an element of the conductor at that point equal in area to the two faces of the disc; the quantity actually removed being only half of this. This result, however, does not at all affect the experimental use which he makes of the proof plane, which is merely to find the ratios of the intensities at different points of a charged conductor. As the complete theory of this valuable instrument has not, so far as I am aware, been given in any English work, I annex the following remarkably clear account of it, which is extracted from Pouillet's *Traité de Physique*:—"Quand le plan d' épreuve est tangent à une surface, il se confond avec l' élément qu' il touche, il prend en quelque sorte sa place relativement à l' électricité, ou plutôt il devient lui-même l' élément sur lequel la fluide se répand; ainsi, quand on retire

ce plan, on fait la même chose que si l'on avait découpé sur la surface un élément de même épaisseur et de même étendue que lui, et qu'on l'eût enlevé pour le porter dans la balance sans qu'il perdît rien de l'électricité qui le couvre; une fois séparé de la surface, cet élément n'aurait plus dans ses différents points qu'une épaisseur électrique moitié moindre, puisque la fluide devrait se répandre pour en couvrir les deux faces. Ce principe posé, l'expérience n'exige plus que de l'habitude et de la dextérité: après avoir touché un point de la surface avec le plan d'épreuve, on l'apporte dans la balance, où il partage son électricité avec le disque de l'aiguille qui lui est égale, et l'on observe la force de torsion à une distance connue. On répète la même experience en touchant un autre point, et le rapport des forces de torsion est le rapport des repulsions électriques; on en prend la racine carrée pour avoir le rapport des épaisseurs. Ainsi le génie de Coulomb a donné en même temps aux mathématiciens la loi fondamentale suivant laquelle la matière électrique s'attire et se repousse; et aux physiciens une balance nouvelle, et des principes d'éxpérience au moyen desquels ils peuvent en quelque sorte sonder l'épaisseur de l'électricité sur tous les corps, et determiner les pressions qu'elle exerce sur les obstacles qui l'arrêtent."

To this explanation it should be added that, when the proof plane is still very near the body to which it has been applied, the effect of mutual influence is such as to make the intensity be insensible at every point of the disc on the side next the conductor, and at each point of the conductor which is *under* the disc. It is only when the disc is removed to a considerable distance that the electricity spreads itself symmetrically on its two faces, and that the intensity at the point of the conductor to which it was applied, recovers its original value. It was the omission of this consideration that caused Coulomb to fall into the error alluded to above.

Note III.

This memoir of Green's has been unfortunately very little known, either in this country or on the continent. Some of the principal theorems in it have been re-discovered within the last few years, and published in the following works:—

Comptes Rendues for Feb. 11th, 1839, where part of the series of theorems is announced without demonstration, by Chasles.

Gauss's memoir on "General Theorems relating to Attractive and Repulsive Forces, varying inversely as the square of the distance," in the *Resultate aus den Beobachtungen des magnetischen Vereins im Jahre* 1839, *Leipsic*, 1840. (Translations of this paper have been published in *Taylor's Scientific Memoirs* for April 1842, and in the Numbers of *Liouville's Journal* for July and August 1842.)

Mathematical Journal, vol. III., Feb. 1842, in a paper "On the Uniform Motion of Heat, &c."

Additions to the Connaissance des Tems for 1845 (published June 1842), where Chasles supplies demonstrations of the theorems which he had previously announced.

I should add that it was not till the beginning of the present year (1845) that I succeeded in meeting with Green's Essay. The allusion made to his name with reference to the word "potential" (*Mathematical Journal*, vol. III. p. 190), was taken from a memoir of Murphy's, "On Definite Integrals with Physical Applications," in the *Cambridge Transactions*, where a mistaken definition of that term, as used by Green, is given.

NOTE IV.

This theorem may be proved as follows:—

Let S be any closed surface, containing no part of the electrified bodies within it, which we may conceive to be described between A and B; let P be the component in the direction of the normal, of the resultant force at any point of the surface S, and let ds be an element of the surface at the same point. Then it may be easily proved (see vol. III. p. 204), that

$$\iint P ds = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots \;(a),$$

the integrations being extended over the entire surface. Now let S be supposed to consist of three parts; the portion α, of the surface of A; its projection β, on the interior surface of B; and the surface generated by the curved lines of projection. The value of P at each point of the latter portion of S will be nothing, since the tangent at any point of a line of projection is the direction of the force. Hence, if $[\iint P ds]$, and $(\iint P ds)$ denote the values of $\iint P ds$, for the portions α and β of S, the equation (a) becomes

$$\left[\iint P ds\right] + \left(\iint P ds\right) = 0.$$

But if ρ be the intensity of the distribution on the surface A or B, at any point, we have, by Coulomb's theorem,

$$\rho = \frac{P}{4\pi}.$$

Hence $$\left[\iint \rho ds\right] + \left(\iint \rho ds\right) = 0,$$

which is the theorem quoted in the text.

MATHEMATICAL NOTES.

Solution of an Optical Problem proposed in the Senate-House Papers of 1844.

"If a polished plane have an indefinite number of very fine concentric circular grooves turned on its surface, and light be incident on it from a luminous point, the appearance presented to the eye of an observer will be that of a bright curve; find its equation."

The solution depends very simply on the principle that, when a ray of light is reflected at any surface, the length of the course of the ray, reckoned from any point in the incident ray to any point in the reflected, is a minimum. For, in the above case, let the polished plane be the plane of xy, and the centre of the circular grooves the origin, $x_1 y_1 z_1$ the coordinates of the luminous point, $e f g$ those of the eye, and $x\,y\,0$ those of the point of incidence on the plane corresponding to the groove whose radius is r; then the length of the path of the ray is

$$\sqrt{\{(x_1 - x)^2 + (y_1 - y)^2 + z_1^2\}} + \sqrt{\{(e - x)^2 + (f - y)^2 + g^2\}},$$

which is to be a minimum subject to the condition

$$x^2 + y^2 = r^2,$$

or
$$\frac{dy}{dx} = -\frac{x}{y}.$$

Hence the condition of minimum is
$$\frac{(x_1 - x)\, y - (y_1 - y)\, x}{\{(x_1 - x)^2 + (y_1 - y)^2 + z_1^2\}^{\frac{1}{2}}} + \frac{(e - x)\, y - (f - y)\, x}{(e - x)^2 + (f - y)^2 + g^2\}^{\frac{1}{2}}} = 0 \ldots (a),$$
or
$$\frac{x_1 y - y_1 x}{\{(x_1 - x)^2 + (y_1 - y)^2 + z_1^2\}^{\frac{1}{2}}} + \frac{ey - fx}{\{(e - x)^2 + (f - y)^2 + g^2\}^{\frac{1}{2}}} = 0 \ldots (\mathrm{A});$$
which, if x and y be taken as current coordinates, is the equation required. The radicals have been allowed to remain, because if they had been expelled by squaring, the result would have comprised the case in which the *difference* of the lengths of the incident and reflected ray is a minimum, and we should then have introduced a branch of the curve which is extraneous to the problem.

H. G.

[Equation (a) obviously expresses the condition, that straight lines drawn from any point P of the bright curve to the eye and to the luminous point, make equal angles with the tangent to the circle described from C as centre through P, and in the plane (x, y).

If light from a luminous point be incident upon a polished rod of any form, it would follow directly from the law of reflection, that a bright point will be seen on the rod in every position, such that lines drawn from it to the eye and to the luminous point, make equal angles with the tangent. From this we might immediately deduce the solution of the above problem as well as of the following.

A straight polished rod revolves rapidly in a plane about a fixed point; to find the bright curve which is seen by an eye in any position, when light is incident from a luminous point.

Taking, as in the preceding problem, $(x_1, y_1, z_1,)$ for the coordinates of the luminous point Q; $(e, f, g,)$ for those of the eye E, and $(x, y, 0,)$ for those of the image P of the luminous point seen in the rod at any instant, or, which is the same, of a point in the bright curve, we have, for the condition that QP and EP may be equally inclined to the rod OP,
$$\frac{(x_1 - x)\, x + (y_1 - y)\, y}{\{(x_1 - x)^2 + (y_1 - y)^2 + z_1^2\}^{\frac{1}{2}}} + \frac{(e - x)\, x + (f - y)\, y}{\{(e - x)^2 + (f - y)^2 + g^2\}^{\frac{1}{2}}} = 0,$$
which is therefore the equation of the bright curve.]

ON HOMOGENEOUS FUNCTIONS OF THE THIRD ORDER WITH THREE VARIABLES.

By Arthur Cayley.

The following problem corresponds to the geometrical question of determining the polar reciprocal of a plane curve of the third order: the solution of it is also important, with reference to the linear transformations of homogeneous functions of three variables of the third order; reasons for which it has appeared to me worth while to obtain the completely developed result.

Let
$$\begin{aligned}3U = {} & ax^3 + by^3 + cz^3 + 3iy^2z + 3jz^2x + 3kx^2y \\ & + 3i_1yz^2 + 3j_1zx^2 + 3k_1xy^2 + 6lxyz \quad \ldots\ldots (1).\end{aligned}$$

It is required to eliminate x, y, z, λ from the equations

$$U = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$\left.\begin{aligned}\frac{dU}{dx} + \lambda\xi = 0 \\ \frac{dU}{dy} + \lambda\eta = 0 \\ \frac{dU}{dz} + \lambda\zeta = 0\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (3).$$

From the equations (2), (3), we obtain immediately

$$\Theta = \xi x + \eta y + \zeta z = 0 \ldots\ldots\ldots\ldots (4);$$

and thence $\quad \Theta x = 0, \quad \Theta y = 0, \quad \Theta z = 0 \quad \ldots\ldots\ldots (5)$:

so that a single equation more, such as

$$\Phi = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

where Φ is homogeneous and of the second order in x, y, z, would, in conjunction with the equations (3) and (5), enable us to eliminate linearly the seven quantities $x^2, y^2, z^2, yz, zx, xy, \lambda$. Such an equation may be thus obtained.

Let L, M, N, R, S, T, be the second differential coefficients of U, each of them divided by two. The equations (3) may be written

$$\begin{aligned}Lx + Ty + Sz + \lambda\xi &= 0, \quad \ldots\ldots\ldots\ldots (7), \\ Tx + My + Rz + \lambda\eta &= 0, \\ Sx + Ry + Nz + \lambda\zeta &= 0.\end{aligned}$$

And joining to these the equation (4),

$$\xi x + \eta y + \zeta z = 0,$$

[redacted]

we have, by the elimination of x, y, z, in so far as they explicitly appear, and λ, an equation $\Phi = 0$ of the required form. Hence we may write

$$\Phi = -\begin{vmatrix} L, & T, & S, & \xi \\ T, & M, & R, & \eta \\ S, & R, & N, & \zeta \\ \xi, & \eta, & \zeta, & \end{vmatrix} \quad \ldots\ldots\ldots\ldots (8);$$

or substituting for L, M, N, R, S, T, and expanding,

$$\Phi = Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy \ldots (9);$$

where

$$\left.\begin{aligned}
A &= (k_1 j - l^2)\,\xi^2 + (ja - j_1^2)\,\eta^2 + (ak_1 - k^2)\,\zeta^2 + 2\,(j_1 k - al)\,\eta\zeta \\
&\qquad + 2\,(kl - k_1 j_1)\,\zeta\xi + 2\,(lj_1 - jk)\,\eta\xi, \\
B &= (bi_1 - i^2)\,\xi^2 + (i_1 k - l^2)\,\eta^2 + (bk - k_1^2)\,\zeta^2 + 2\,(lk_1 - ik)\,\eta\zeta \\
&\qquad + 2\,(k_1 i - bl)\,\zeta\xi + 2\,(il - i_1 k_1)\,\eta\xi, \\
C &= (ci - i_1^2)\,\xi^2 + (cj_1 - j^2)\,\eta^2 + (j_1 i - l^2)\,\zeta^2 + 2\,(jl - j_1 i_1)\,\eta\zeta \\
&\qquad + 2\,(li_1 - ij)\,\zeta\xi + 2\,(i_1 j - cl)\,\eta\xi, \\
2F &= (bc - ii_1)\,\xi^2 + (i_1 j_1 + ck - 2lj)\,\eta^2 + (ki + bj_1 - 2lk_1)\,\zeta^2 \\
&\qquad + (l^2 + k_1 j - ki_1 - j_1 i)\,\eta\zeta + (k_1 i_1 - bj)\,\zeta\xi + (ij - ck_1)\,\eta\xi, \\
2G &= (ij + ck_1 - 2li_1)\,\xi^2 + (ca - jj_1)\,\eta^2 + (j_1 k_1 + ai - 2lk)\,\zeta^2 \\
&\qquad + (jk - ai_1)\,\eta\zeta + (l^2 + i_1 k - ij_1 - k_1 j)\,\zeta\xi + (i_1 j_1 - ck)\,\eta\xi, \\
2H &= (k_1 i_1 + bj - 2li)\,\xi^2 + (jk + ai_1 - 2lj_1)\,\eta^2 + (ab - kk_1)\,\zeta^2 \\
&\qquad + (j_1 k_1 - ai)\,\eta\zeta + (ki - bj_1)\,\zeta\xi + (l^2 + j_1 i - jk_1 - i_1 k)\,\eta\xi,
\end{aligned}\right\} \quad \ldots\ldots (10).$$

Performing the elimination indicated, the result may be represented by

$$\mathsf{F}U = \begin{vmatrix} a, & k_1, & j, & l, & j_1, & k, & \xi \\ k, & b, & i_1, & i, & l, & k_1, & \eta \\ j_1, & i, & c, & i_1, & j, & l, & \zeta \\ 2\xi, & . & . & . & \zeta, & \eta & . \\ . & 2\eta & . & \zeta & . & \xi & . \\ . & . & 2\zeta & \eta & \xi & . & . \\ A & B & C & F & G & H & . \end{vmatrix} = 0 \quad \ldots\ldots (11).$$

Partially expanding,

$$\mathsf{F}U = A\mathrm{a} + B\mathrm{b} + C\mathrm{c} + 2F\mathrm{f} + 2G\mathrm{g} + 2H\mathrm{h} \ldots (12).$$

[redacted]

The values of the coefficients a, b, c, f, g, h, may be useful on other occasions: they are as follows.

$$\left.\begin{aligned}
\mathrm{a} = {} & 0\xi^4 + 2(cj_1 - j^2)\eta^4 + 2(bk - k_1^2)\zeta^4 \\
& + 2(4jl - 3i_1j_1 - ck)\eta^3\zeta + 4(k_1 i - lb)\zeta^3\xi + 0\xi^3\eta \\
& + 2(4k_1l - 3ik - j_1b)\eta\zeta^3 + 0\zeta\xi^3 + 4(i_1j - lc)\xi\eta^3 \\
& + 2(3i_1k + 3j_1i - 2jk_1 - 4l^2)\eta^2\zeta^2 + 2(bi_1 - i^2)\zeta^2\xi^2 \\
& \qquad + 2(ci - i_1^2)\xi^2\eta^2 \\
& + 2(ii_1 - bc)\xi^2\eta\zeta + 4(ck_1 + i_1l - 2ij)\xi\eta^2\zeta \\
& \qquad + 4(bj + il - 2i_1k_1)\xi\eta\zeta^2. \\
\mathrm{b} = {} & 2(ci - i_1^2)\xi^4 + 0\eta^4 + 2(ak_1 - k^2)\zeta^4 \\
& + 0\eta^3\zeta + 2(4kl - 3j_1k_1 - ai)\zeta^3\xi + 4(i_1j - lc)\xi^3\eta \\
& + 4(j_1k - al)\eta\zeta^3 + 2(4i_1l - 3ji - k_1c)\zeta\xi^3 + 0\xi\eta^3 \\
& + 2(aj - j_1^2)\eta^2\zeta^2 + 2(3j_1i + 3k_1j - 2ki_1 - 4l^2)\zeta^2\xi^2 \\
& \qquad + 2(cj_1 - j^2)\xi^2\eta^2 \\
& + 4(ck + jl - 2j_1i_1)\xi^2\eta\zeta + 2(jj_1 - ca)\xi\eta^2\zeta \\
& \qquad + 4(ai_1 + j_1l - 2jk)\xi\eta\zeta^2. \\
\mathrm{c} = {} & 2(bi_1 - i^2)\xi^4 + 2(aj - j_1^2)\eta^4 + 0\zeta^4 \\
& + 4(j_1k - al)\eta^3\zeta + 0\zeta^3\xi + 2(4il - 3k_1i_1 - bj)\xi^3\eta \\
& + 0\eta\zeta^3 + 4(k_1i - bl)\zeta\xi^3 + 2(4j_1l - 3kj - i_1a)\xi\eta^3 \\
& + 2(ak_1 - k^2)\eta^2\zeta^2 + 2(bk - k_1^2)\zeta^2\xi^2 \\
& \qquad + 2(3k_1j + 3i_1k - 2ij_1 - 4l^2)\xi^2\eta^2 \\
& + 4(bj_1 + k_1l - 2ki)\xi^2\eta\zeta + 4(ai + kl - 2k_1j_1)\xi\eta^2\zeta \\
& \qquad + 2(kk_1 - ab)\xi\eta\zeta^2. \\
\mathrm{f} = {} & (ii_1 - bc)\xi^4 + 0\eta^4 + 0\zeta^4 \\
& + 2(j_1^2 - aj)\eta^3\zeta + (ab - kk_1)\zeta^3\xi + (3ck_1 - 2i_1l - ij)\xi^3\eta \\
& + 2(k^2 - ak_1)\eta\zeta^3 + (3bj - 2il - i_1k_1)\zeta\xi^3 + (ca - jj_1)\xi\eta^3 \\
& + 4(al - j_1k)\eta^2\zeta^2 + (ki + 2k_1l - 3bj_1)\zeta^2\xi^2 \\
& \qquad + (i_1j_1 + 2lj - 3ck)\xi^2\eta^2 \\
& + (4l^2 + 2i_1k + 2ij_1 - 8jk_1)\xi^2\eta\zeta + (7kj - 6j_1l - ai_1)\xi\eta^2\zeta \\
& \qquad + (7k_1j_1 - 6kl - ai)\xi\eta\zeta^2. \\
\mathrm{g} = {} & 0\xi^4 + (jj_1 - ca)\eta^4 + 0\zeta^4 \\
& + (3ai_1 - 2j_1l - jk)\eta^3\zeta + 2(k_1^2 - bk)\zeta^3\xi + (bc - ii_1)\xi^3\eta \\
& + (ab - kk_1)\eta\zeta^3 + 2(i^2 - bi_1)\zeta\xi^3 + (3ck - 2jl - j_1i_1)\xi\eta^3 \\
& + (j_1k_1 + 2lk - 3ai)\eta^2\zeta^2 + 4(bl - k_1i)\zeta^2\xi^2 \\
& \qquad + (ij + 2i_1l - 3ck_1)\eta^2\xi^2 \\
& + (7i_1k_1 - 6il - bj)\xi^2\eta\zeta + (4l^2 + 2j_1i + 2jk_1 - 8ki_1)\xi\eta^2\zeta \\
& \qquad + (7ik - 6k_1l - bj_1)\xi\eta\zeta^2.
\end{aligned}\right\}$$

...... (13).

[redacted]

$$\begin{aligned}
\mathrm{h} = 0\xi^4 + 0\eta^4 + (kk_1 - ab)\,\zeta^4 & \\
+ (ca - jj_1)\,\eta^3\zeta + (3bj_1 - 2k_1l - ki)\,\zeta^3\xi + 2\,(i_1^2 - ci)\,\xi^3\eta & \\
+ (3ai - 2kl - k_1j_1)\,\eta\zeta^3 + (bc - ii_1)\,\zeta\xi^3 + 2\,(j^2 - cj_1)\,\xi\eta^3 & \\
+ (jk + 2j_1l - 3ai_1)\,\eta^2\zeta^2 + (k_1i_1 + 2li - 3bj)\,\zeta^2\xi^2 + 4\,(cl - i_1j)\,\xi^2\eta^2 & \\
+ (7ji - 6i_1l - ck_1)\,\xi^2\eta\zeta + (7j_1i_1 - 6jl - ck)\,\xi\eta^2\zeta & \\
+ (4l^2 + 2k_1j + 2ki_1 - 8ij_1)\,\xi\eta\zeta^2. &
\end{aligned} \qquad \ldots\ldots (13).$$

Substituting these values, the result after all reductions becomes

$$0 = \mathsf{F}U = \ldots\ldots\ldots\ldots\ldots \quad (14),$$

$$\begin{aligned}
& \xi^6\,(6bcii_1 - 4i^3c - 4i_1^3b + 3i^2i_1^2 - b^2c^2) \\
& + \eta^6\,(6cajj_1 - 4j^3a - 4j_1^3c + 3j^2j_1^2 - c^2a^2) \\
& + \zeta^6\,(6abkk_1 - 4k^3b - 4k_1^3a + 3k^2k_1^2 - a^2b^2) \\
& + \eta^5\zeta\,(5ca^2i_1 - 17jj_1ai_1 + 24aj^2l + 12j_1^3i_1 - 5cajk \\
& \qquad + 12cj_1^2k - 7j^2j_1k - 12caj_1l - 12j_1^2jl) \\
& + \zeta^5\xi\,(5ab^2j_1 - 17kk_1bj_1 + 24bk^2l + 12k_1^3j_1 - 5abki \\
& \qquad + 12ak_1^2i - 7k^2k_1i - 12abk_1l - 12k_1^2kl) \\
& + \xi^5\eta\,(5bc^2k_1 - 17ii_1ck_1 + 24ci^2l + 12i_1^3k_1 - 5bcij \\
& \qquad + 12bi_1^2j - 7i^2i_1j - 12bci_1l - 12i_1^2il) \\
& + \eta\zeta^5\,(5a^2bi - 17kk_1ai + 24ak_1^2l + 12k^3i - 5baj_1k_1 \\
& \qquad + 12bk^2j_1 - 7kk_1^2j_1 - 12bakl - 12k^2k_1l) \\
& + \zeta\xi^5\,(5b^2cj - 17ii_1bj + 24bi_1^2l + 12i^3j - 5cbk_1i_1 \\
& \qquad + 12ci^2k_1 - 7ii_1^2k_1 - 12cbil - 12i^2i_1l) \\
& + \xi\eta^5\,(5c^2ak - 17jj_1ck + 24cj_1^2l + 12j^3k - 5aci_1j_1 \\
& \qquad + 12aj^2i_1 - 7jj_1^2i_1 - 12acjl - 12j^2j_1l) \\
& + \eta^4\zeta^2\,(-5ca^2i + 15jj_1ai - 46ajl^2 - 10j_1^3i + 12cakl - 12cj_1k^2 + 4j^2k^2 \\
& \qquad + 5caj_1k_1 + 5jj_1^2k_1 - 10aj^2k_1 - 34j_1^2ki_1 + 26jj_1kl + 34aj_1i_1l \\
& \qquad - 6a^2i_1^2 + 10j_1^2l^2 + 12ajki_1) \\
& + \zeta^4\xi^2\,(-5ab^2j + 15kk_1bj - 46bkl^2 - 10k_1^3j + 12abil - 12ak_1i^2 \\
& \qquad + 4k^2i^2 + 5abk_1i_1 + 5kk_1^2i_1 - 10bk^2i_1 - 34k_1^2ij_1 + 26kk_1il \\
& \qquad + 34bk_1j_1l - 6b^2j_1^2 + 10k_1^2l^2 + 12bkij_1) \\
& + \xi^4\eta^2\,(-5bc^2k + 15ii_1ck - 46cil^2 - 10i_1^3k + 12bcjl - 12bi_1j^2 + 4i^2j^2 \\
& \qquad + 5bci_1j_1 + 5ii_1^2j_1 - 10ci^2j_1 - 34i_1^2jk_1 + 26ii_1jl + 34ci_1k_1l \\
& \qquad - 6c^2k_1^2 + 10i_1^2l^2 + 12cijk_1) \\
& + \eta^2\zeta^4\,(-5ba^2i_1 + 15kk_1ai_1 - 46ak_1l^2 - 10k^3i_1 + 12baj_1l - 12bkj_1^2 \\
& \qquad + 4k_1^2j_1^2 + 5bakj + 5k_1k^2i - 10ak_1^2j - 34k^2j_1i + 26kk_1j_1l \\
& \qquad + 34akil - 6a^2i^2 + 10k^2l^2 + 12ak_1j_1i)
\end{aligned}$$

$$
\begin{aligned}
&+ \zeta^2\xi^4(-5cb^2j_1 + 15ii_1bj_1 - 46bi_1l^2 - 10i^3j_1 + 12cbk_1l - 12cik_1^2 \\
&\qquad + 4i_1^2k_1^2 + 5cbik + 5i_1i^2j - 10bi_1^2k - 34i^2k_1j + 26ii_1k_1l \\
&\qquad + 34bijl - 6b^2j^2 + 10i^2l^2 + 12bi_1k_1j) \\
&+ \xi^2\eta^4(-5ac^2k_1 + 15jj_1ck_1 - 46cj_1l^2 - 10j^3k_1 + 12aci_1l - 12aji_1^2 \\
&\qquad + 4j_1^2i_1^2 + 5acji + 5j_1j^2k - 10cj_1^2i - 34j^2i_1k + 26jj_1i_1l \\
&\qquad + 34cjkl - 6c^2k^2 + 10j^2l^2 + 12cj_1i_1k) \\
&+ \eta^3\zeta^3(32j_1i_1k^2 + 32j_1^2ki - 20j_1kl^2 - 10jj_1kk_1 - 30ali_1k - 30alij_1 \\
&\qquad + 28al^3 + 44aljk_1 - 14aijk - 14ai_1j_1k_1 - 6bajj_1 - 6cakk_1 \\
&\qquad + 2a^2bc + 4bj_1^3 + 4ck^3 - 14jk^2l - 14j_1^2k_1l + 12a^2ii_1) \\
&+ \zeta^3\xi^3(32k_1j_1i^2 + 32k_1^2ij - 20k_1il^2 - 10kk_1ii_1 - 30blj_1i - 30bljk_1 \\
&\qquad + 28bl^3 + 44blki_1 - 14bijk - 14bi_1j_1k_1 - 6cbkk_1 - 6abii_1 \\
&\qquad + 2ab^2c + 4ck_1^3 + 4ai^3 - 14ki^2l - 14k_1^2i_1l + 12b^2jj_1) \\
&+ \xi^3\eta^3(32i_1k_1j^2 + 32i_1^2jk - 20i_1jl^2 - 10ii_1jj_1 - 30clk_1j - 30clki_1 \\
&\qquad + 28cl^3 + 44clij_1 - 14cijk - 14ci_1j_1k_1 - 6acii_1 - 6bcjj_1 \\
&\qquad + 2abc^2 + 4ai_1^3 + 4bj^3 - 14ij^2l - 14i_1^2j_1l + 12c^2kk_1 \\
&+ \xi^4\eta\zeta(65ii_1k_1j + 49ii_1l^2 + 11ii_1^2k + 11i^2i_1j_1 - 21cbjk_1 + 23bcl^2 \\
&\qquad + 9bcki_1 + 9bcij_1 - 14cik_1l - 14bi_1jl + 14bij^2 + 14ci_1k_1^2 \\
&\qquad - 58i^2jl - 58i_1^2k_1l - 20bi_1^2j_1 - 20ci^2k) \\
&+ \eta^4\zeta\xi(65jj_1i_1k + 49jj_1l^2 + 11jj_1^2i + 11j^2j_1k_1 - 21acki_1 + 23cal^2 \\
&\qquad + 9caij_1 + 9cajk_1 - 14aji_1l - 14cj_1kl + 14cjk^2 + 14cj_1i^2 \\
&\qquad - 58j^2kl - 58j_1^2i_1l - 20cj_1^2k_1 - 20aj^2i) \\
&+ \zeta^4\eta\xi(65kk_1j_1i + 49kk_1l^2 + 11kk_1^2j + 11k^2k_1i_1 - 21baij_1 + 23abl^2 \\
&\qquad + 9abjk_1 + 9abki_1 - 14bkj_1l - 14ak_1il + 14aki^2 + 14ak_1j^2 \\
&\qquad - 58k^2il - 58k_1^2j_1l - 20ak_1^2i_1 - 20bk^2j) \\
&+ \xi^3\eta^2\zeta(19bcjk - 32bcj_1l - 5abci_1 - 70ii_1jk - 60ii_1j_1l - 5aii_1^2 \\
&\qquad + 50cikl - 7cij_1k_1 + 10aci^2 + 38i_1^2kl + 56i_1^2k_1j_1 + 21ck_1^2j \\
&\qquad - 19ck_1l^2 + 8i_1k_1jl - 42i_1l^3 - 43ij^2k_1 + 85ijl^2 - 37ci_1kk_1 \\
&\qquad + 28bi_1jj_1 - 10bj^2l + 15i^2jj_1) \\
&+ \eta^3\zeta^2\xi(19caki - 32cak_1l - 5abcj_1 - 70jj_1ki - 60jj_1k_1l - 5bjj_1^2 \\
&\qquad + 50ajil - 7ajk_1i_1 + 10baj^2 + 38j_1^2il + 56j_1^2i_1k_1 + 21ai_1^2k \\
&\qquad - 19ai_1l^2 + 8j_1i_1kl - 42j_1l^3 - 43jk^2i_1 + 85jkl^2 - 37aj_1ii_1 \\
&\qquad + 28cj_1kk_1 - 10ck^2l + 15j^2kk_1) \\
&+ \zeta^3\xi^2\eta(19abij - 32abi_1l - 5abck_1 - 70kk_1ij - 60kk_1i_1l - 5ckk_1^2 \\
&\qquad + 50bkjl - 7bki_1j_1 + 10cbk^2 + 38k_1^2jl + 56k_1^2j_1i_1 + 21bj_1^2i \\
&\qquad - 19bj_1l^2 + 8k_1j_1il - 42k_1l^3 - 43ki^2j_1 + 85kil^2 - 37bk_1jj_1 \\
&\qquad + 28ak_1ii_1 - 10ai^2l + 15k^2ii_1)
\end{aligned}
$$

[redacted]

multiplied by a numerical factor. If U is of the form

$$U = PQR \ldots\ldots\ldots\ldots\ldots (25),$$

then

$$\nabla U = \rho PQR = \rho U \ldots\ldots\ldots\ldots (26).$$

And this equation is consequently the condition of the function U being resolvable into linear factors. The equation in question resolves itself into

$$\frac{A}{a} = \frac{B}{b} = \frac{C}{c} = \frac{I}{i} = \frac{J}{j} = \frac{K}{k} = \frac{I_1}{i_1} = \frac{J_1}{j_1} = \frac{K_1}{k_1} = \frac{\Lambda}{l} \ldots (27);$$

a system which must contain three independent equations only. It would be interesting to verify this *à posteriori*.

ON LINEAR TRANSFORMATIONS.*

By ARTHUR CAYLEY.

[*Continued from Vol. IV. p.* 209.]

IN continuing my researches on the present subject, I have been led to a new manner of considering the question, which, at the same time that it is much more general, has the advantage of applying directly to the only case which one can possibly hope to develope with any degree of completeness, that of functions of two variables. In fact the question may be proposed, "To find all the derivatives of any number of functions, which have the property of preserving their form unaltered after any linear transformations of the variables." By Derivative I understand a function deduced in any manner whatever from the given functions, and I give the name of Hyperdeterminant Derivative, or simply of Hyperdeterminant, to those derivatives which have the property just enunciated. These derivatives may easily be expressed explicitly, by means of the known method of the separation of symbols. We thus obtain the most general expression of a hyperdeterminant. But there remains a question to be resolved, which appears to present very great difficulties, that of determining the *independent* derivatives, and the relation between these and the remaining ones. I have only succeeded in treating a very particular case of this

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* The present paper was originally written for the *Mathematical Journal;* but it has already been published, together with the one to which it forms a sequel, in *Crelle's Journal,* tom. xxx.

as will be shown in a subsequent paper "On Points of Inflection."

And from the six equations (15), (18), the six quantities $x^2, y^2, z^2, yz, zx, xy$, may be linearly eliminated, we have

$$\nabla U = Ax^3 + By^3 + Cz^3 + 3Iy^2z + 3Jz^2x + 3Kx^2y + 3I_1yz^2 + 3J_1zx^2 + 3K_1xy^2 + 6\Lambda xyz \quad \ldots\ldots (19),$$

where

$$\left.\begin{aligned}
A &= ak_1j + 2kj_1l - al^2 - jk^2 - j_1^2k_1, \\
B &= bi_1k + 2ik_1l - bl^2 - ki^2 - k_1^2i_1, \\
C &= cj_1i + 2ji_1l - cl^2 - ij^2 - i_1^2j_1, \\
3I &= bck + bi_1j_1 - ck_1^2 + 2jik_1 - 2bjl + il^2 - i^2j_1 - ii_1k, \\
3J &= cai + cj_1k_1 - ai_1^2 + 2kji_1 - 2ckl + jl^2 - j^2k_1 - jj_1i, \\
3K &= abj + ak_1i_1 - bj_1^2 + 2ikj_1 - 2ail + kl^2 - k^2i_1 - kk_1j, \\
3I_1 &= bcj_1 + cik - bj^2 + 2k_1i_1j - 2ck_1l + i_1l^2 - i_1^2k - ii_1j_1, \\
3J_1 &= cak_1 + aji - ck^2 + 2i_1j_1k - 2ai_1l + j_1l^2 - j_1^2i - jj_1k_1, \\
3K_1 &= abi_1 + bkj - ai^2 + 2j_1k_1i - 2bi_1l + k_1l^2 - k_1^2j - kk_1i_1, \\
6\Lambda &= abc + 3ijk + 3i_1j_1k_1 + 2l^3 - aii_1 - bjj_1 - ckk_1 - 2ljk_1 \\
&\quad - 2lki_1 - 2lij_1.
\end{aligned}\right\} \ldots (20),$$

And the result of the elimination is $\quad \ldots\ldots (21)$

$$K(U) = \left|\begin{matrix}
a, & k_1, & j, & l, & j_1, & k, \\
k, & b, & i_1, & i, & l, & k_1, \\
j_1, & i, & c, & i_1, & j, & l, \\
A, & K_1, & J, & L, & J_1, & K, \\
K, & B, & I_1, & I, & L, & K_1, \\
J_1, & I, & C, & I_1, & J, & L,
\end{matrix}\right| = 0.$$

[$K(U)$ is consequently, as is well known, a function of the twelfth order in $a, b, c, i, j, k, i_1, j_1, k_1, l$].

The equation
$$\nabla U = 0,$$
combined with that of the curve, determine, as Dr. Hesse has demonstrated in the paper quoted, the points of inflection of the curve. It may be inferred from this, that if U reduce itself to the form

$$U = (\alpha x^2 + \beta y^2 + \gamma z^2 + 2\iota yz + 2\kappa xz + 2\lambda xy)\,P = VP \ldots (22),$$

P a linear function of x, y, z: then ∇U takes the form

$$\nabla U = P(\rho V + \sigma P^2) \ldots\ldots\ldots\ldots (23),$$

where ρ is of the second order in the coefficients of P, and also in the coefficients $\alpha, \beta, \gamma, \iota, \kappa, \lambda$: and σ is equal to the determinant

$$\left|\begin{matrix}
\alpha, & \lambda, & \kappa \\
\lambda, & \beta, & \iota \\
\kappa, & \iota, & \gamma
\end{matrix}\right| \ldots\ldots\ldots\ldots (24),$$

i.e. the terms on the one side are respectively equal to the terms on the other. Hence if

$$\square = F(\|\Omega\|^{f}, \ \|\Omega'\|^{f'}, ..)$$

i.e. $\square$ a rational and integral function, homogeneous of the order f in the quantities of the series $\|\Omega\|$, homogeneous of the order f' in the quantities of the series $\|\Omega'\|$, &c., we have immediately

$$\dot{\square} = E^{f} E'^{f'} .. \square.$$

Or if U be any function whatever of the variables $x, y....$ which is transformed by the linear substitutions above into $\dot{U}$, then

$$\dot{\square}\dot{U} = E^{f} E'^{f'} .. \square U.$$

Or the function $\square U$

is by the above definition a hyperdeterminant derivative. The symbol $\square$ may be called "symbol of hyperdeterminant derivation," or simply " hyperdeterminant symbol."

Let $A, B..$ represent the different quantities of the series $\|\Omega\|$,—$A', B'..$ those of the series' $\|\Omega'\|$, &c.. . Then $\square$ may be reduced to a single term, and we may write

$$\square = A^{\alpha}.B^{\beta}..A'^{\alpha'}.B'^{\beta'}..$$

Also U may be supposed of the form

$$U = \Theta.\Phi..$$

where Θ, Φ are functions of the variables of one of the sets $x, y..$ of one of the sets $x', y'..$ &c. Thus Θ is of the form

$$F(x_1, y_1.. x_1', y_1 ..),$$

and so on. The functions Θ, Φ.... may be the same or different. It may be supposed after the differentiations that several of the sets $x, y...$ or of the sets $x', y'....$ become identical. In such cases it will always be assumed that the functions Θ,.. into which these sets of variables enter, are similar; so that they become absolutely identical, when the variables they contain are made so. Thus the general expression of a hyperdeterminant is

$$\square U = A^{\alpha}B^{\beta}..A'^{\alpha'}B'^{\beta'}..\Theta\Phi..$$

in which, after the differentiations, any number of the sets of variables are made equal. For instance, if all the sets $x, y..$ and all the sets $x', y'..$ are made equal, the hyperdeterminant refers to a single function $F(x, y.. x', y'..)$. In any other case it refers not to a single function but to several.

What precedes, is the general theory: it might perhaps have been made clearer b ... to a particular case:

and by doing this from the beginning, it will be seen that it presents no real difficulties. Passing at present to some developments, to do this, I neglect entirely the sets x', y'.. and I assume that the number m of variables in each of the sets x, y.. reduces itself to two; so that I consider functions of two variables x, y only. The functions Θ, Φ, &c. reduce themselves to functions $V_1, V_2 .. V_p$ of the variables x_1, y_1, or $x_2, y_2 \ldots$ or x_p, y_p. Writing also

$$\xi_1\eta_2 - \xi_2\eta_1 = \overline{12}, \text{ \&c.}$$

the symbols A, B.. reduce themselves to $\overline{12}, \overline{13}$.. Hence for functions of two variables, there results the following still tolerably general form

$$\Box U = \overline{12}^{\alpha}\,\overline{13}^{\beta}\,\overline{14}^{\gamma} .. \overline{23}^{\beta'}\,\overline{24}^{\gamma'} .. \overline{34}^{\gamma''} .. V_1V_2V_3V_4 ..$$

The functions V_1, V_2.. may be the same or different: but they will be supposed the same whenever the corresponding variables are made equal. This equality will be denoted by writing, for instance, $\Box VV'VV$..

to represent the value assumed by

$$\Box V_1V_2V_3V_4 ..$$

when after the differentiations

$$\begin{aligned} &x_1, y_1 = x_3, y_3 = x_4, y_4 = x, y \\ &x_2, y_2 = x', y'. \\ &\text{\&c.} \end{aligned}$$

It is easy to determine the general term of $\Box U$. To do this, writing for shortness

$$\begin{aligned} &\alpha + \beta + \gamma + .. = f_1 \\ &\alpha + \beta' + \gamma' .. \quad = f_2 \\ &\beta + \beta' + \gamma'' \quad = f_3 \\ &\text{\&c.} \end{aligned}$$

$$N = (-)^{r+s+t..+r'+s'..+t'..}\,\frac{[\alpha]^r}{[r]^r}\,\frac{[\beta]^s}{[s]^s}\,\frac{[\gamma]^t}{[t]^t} .. \frac{[\beta']^{s'}}{[s']^{s'}}\,\frac{[\gamma']^{t'}}{[t']^{t'}} .. \frac{[\gamma'']^{t''}}{[t'']^{t''}} ..$$

$$\xi^{f-r}\eta^r V \text{ or } \delta_x^{f-r}\delta_y^{\,r} V = \overset{f}{V}{}_{,r} \text{ or } V_{,r}.$$

The general term is

$$N\,\overset{f_1}{V_1}{}_{,r+s+t..}\,\overset{f_2}{V_2}{}_{,\alpha-r+s'+t'..}\,\overset{f_3}{V_3}{}_{,\beta-s+\beta'-s'+t''..}$$

where $r, s, t, .. s', t' .. t'', ..$ extend from 0 to $\alpha, \beta, \gamma .. \beta', \gamma' .. \gamma''$.. respectively. It would be easy to change this general term in a way similar to that which will be employe… sently for the particular case of $\Box V_1V_2V_3$.

If several of the functions become identical, and for these some of the letters f are equivalent, it is clear that the derivative $\square\, U$ refers to a certain number of functions $V_1, V_2 \ldots$ the same or different, of the variables $x, y;\ x', y' \ldots$ and besides that this derivative is homogeneous, of the degrees $\theta_1, \theta_1' \ldots$ with respect to the differential coefficients of the orders $f_1, f_1' \ldots$ &c. of V_1, (consequently homogeneous of the order $\theta_1 + \theta_1' + \ldots$ with respect to these differential coefficients collectively), homogeneous and of the degrees $\theta_2, \theta_2' \ldots$ with respect to the differential coefficients of the orders $f_2, f_2' \ldots$ of V_2, (consequently of the order $\theta_2 + \theta_2' \ldots$ with respect to these collectively), and so on. The degree with respect to all the functions is of course $\theta_1 + \theta_1' \ldots + \theta_2 + \theta_2' + \ldots = p$ suppose. In general, only a single function will be considered, and it will be assumed that $\square\, U$ only contains the differential coefficients of the f^{th} order. In this case, the derivative is said to be of the degree p and of the order f. The most convenient classification is by degrees, rather than by orders.

Commencing with the simplest case, that of functions of the second order (and writing V, W instead of V_1, V_2), we have

$$\square\, VW = \overline{12}^{a}\, VW.$$

(where ξ_1, η_1 apply to V and ξ_2, η_2 to W). This will be constantly represented in the sequel by the notation

$$\overline{12}^{a}\, VW = B_a(V, W).$$

Hence, writing $\delta_x^{\,a} V = V^{,0}\quad \delta_x^{\,a-1}\delta_y V = V^{,1} \ldots$

we have $B_a(V, W) = V^{,0} W^{,a} - \dfrac{[a]'}{[1]'}\, V^{,1} W^{,a-1} + \ldots$

And in particular, according as a is odd or even,

$$B_a(V, V) = 0$$

$$\tfrac{1}{2} B_a(V, V) = V^{,0} V^{,a} - \frac{a}{1}\, V^{,1} V^{,a-1} + \ldots$$

continued to the term which contains $V^{,\frac{1}{2}a} V^{,\frac{1}{2}a}$, the coefficient of this last term, divided by two.

Thus, for the functions $\frac{1}{2}(ax^2 + 2bxy + cy^2)$, $\frac{1}{24}(ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4)$, &c., if a be made equal to 2, 4, &c. respectively, we have the *constant* derivatives

$$\begin{aligned}
&ac - b^2\\
&ae - 4bd + 3c^2\\
&ag - 6bf + 15ce - 10d^2\\
&ai - 8bh + 28cg - 56df + 35e^2.\\
&\ \vdots
\end{aligned}$$

which have all of them the property of remaining unaltered,

à un facteur près, when the variables are transformed by means of $x = \lambda\dot{x} + \mu\dot{y}$, $y = \lambda'\dot{x} + \mu'\dot{y}$. Thus, for instance, if these equations give

$$ax^2 + 2bxy + cy^2 = \dot{a}\dot{x}^2 + 2\dot{b}\dot{x}\dot{y} + \dot{c}\dot{y}^2,$$

then $$\dot{a}\dot{c} - \dot{b}^2 = (\lambda\mu' - \lambda'\mu)^2.(ac - b^2),$$

and so on. This is the general property, which we call to mind for the case of these constant derivatives.

The above functions may be transformed by means of the identical equation

$$B_a(V, W) = \overline{12}^{a-k} B_k(V, W),$$

to make use of which, it is only necessary to remark the general formula

$$\xi_1^\lambda\, \eta_1^\mu\, \xi_2^\rho\, \eta_2^\sigma\, B_k(V, W) = B_k(\xi^\lambda\, \eta^\mu V, \xi^\rho\, \eta^\sigma W).$$

Thus, if $k = 1$, we obtain for the above series, the new forms

$$ac - b^2$$
$$(ae - bd) - 3(bd - c^2)$$
$$(ag - bf) - 5(bf - ce) + 10(ce - d^2)$$
$$(ai - bh) - 7(bh - cg) + 21(cg - df) - 35(df - e^2)$$
&c.,

the law of which is evident. This shows also that these functions may be linearly expressed by means of the series of determinants

$$\left\|\begin{matrix} a, b \\ b, c \end{matrix}\right\| \quad \left\|\begin{matrix} a, b, c \\ b, c, d \end{matrix}\right\| \quad \text{\&c.}$$

We may also immediately deduce from them the derivatives B which relate to two functions. For example, for functions of the sixth order this is

$$ag' + a'g - 6(bf' + b'f) + 15(ce' + c'e) - 20dd',$$

which has an obvious connection with

$$ag - 6bf + 15ce - 10d^2;$$

and the same is the case for functions of any order.

The following theorem is easily verified; but I am unacquainted with the general theory to which it belongs.

"If U, V are any functions of the second order, and $W = \lambda U + \mu V$; then

$$B_2'[B_2(W, W), B_2(W, W)] = 0$$

(where B_2' relates to λ, μ) is the same that would be obtained by the elimination of x, y between $U = 0$, $V = 0$." (See Note.*)

* Not given with the present paper.

[redacted]

In fact this becomes

$$4(ac-b^2)(a'c'-b'^2)-(ac'+a'c-2bb')^2=0,$$

which is one of the forms under which the result of the elimination of the variables from two quadratic equations may be written. This is a result for which I am indebted to Mr. Boole.

Passing to the third degree, we may consider in particular the derivatives

$$\square\, UVW = \overline{23}^a\, \overline{31}^a\, \overline{12}^a\, UVW = C_a(U, V, W)$$

Writing for shortness

$$A_r = \frac{[a]^r}{[r]^r}, \quad \delta_x^{2a-r}\delta_y^{\,r}U = U^{,r},$$

we have the general term

$$C_a(U, V, W) = \Sigma\left\{(-)^{r+s+t}A_rA_sA_t\,U^{,a+t-s}\,V^{,a+r-t}\,W^{,a+s-r}\right\},$$

where r, s, t extend from 0 to a. By changing the suffixes r, s the following more convenient formula

$$C_a(U,V,W)=\Sigma\Sigma\left\{(-)^{\sigma+\rho}\,U^{,\rho}\,V^{,\sigma}\,W^{,3a-\rho-\sigma}\,\Sigma\left[(-)^tA_{\rho-t}A_{\sigma+t-a}A_t\right]\right\},$$

where t extends from 0 to $2a$: ρ, σ, and $3a-\rho-\sigma$ must be positive and not greater than $2a$.

In particular, according as a is odd or even,

$$C_a(U,U,U)=0,$$

$$C_a(U,U,U)=6\Sigma\Sigma\left\{(-)^{\rho+\sigma}\,U^{,\rho}\,U^{,\sigma}\,U^{,3a-\rho-\sigma}\,\Sigma\left[(-)^tA_{\rho-t}A_{\sigma+t-a}A_t\right]\right\},$$

in omitting such values of ρ, σ for which $\rho>\sigma$ or $\sigma>3a-\rho-\sigma$, and dividing by two the terms in which $\rho=\sigma$ or $\sigma=3a-\rho-\sigma$, and by six the term for which $\rho=\sigma=3a-\rho-\sigma=a$.

In particular, for functions of the fourth or eighth orders we have the constant derivatives

$$ace-ad^2-b^2e-c^3+2bcd$$

$$aei-4ibd-4afh+3ag^2+3ic^2+12beh-8chd-8bgf-22ceg$$
$$+24cf^2+24d^2g-36def+15e^3.$$

The first of which is a simple determinant. Thus we have been led to the functions $ae-4bd+3c^2$ and $ace-ad^2-eb^2-c^3+2bcd$, which occur in my "Note sur quelques formules, &c." (*Crelle*, tom. XXIV.), and in the forms which M. Eisenstein has given for the solutions of equations of the four first degrees.

Let U be a function of the order $4a$: the derivative C may be expressed by means of the derivatives B.

For, consider the function

$$B_{4a}[U, B_{2a}(V, W)],$$

[redacted]

Paying attention to the signification of B, this may be written

$$\overline{1\theta}^{4\alpha}\,\overline{23}^{2\alpha}\,UVW,$$

where the symbols ξ_θ, η_θ refer to the two systems $x_2, y_2 : x_3, y_3$. Thus it is easily seen that we may write

$$\xi_\theta = \xi_2 + \xi_3, \quad \eta_\theta = \eta_2 + \eta_3, \quad \text{or} \quad \overline{1\theta} = \overline{12} + \overline{13} = \overline{12} - \overline{31},$$

whence the function becomes

$$(\overline{12} - \overline{31})^{4\alpha}\,\overline{23}^{2\alpha}\,UVW.$$

Of which all the terms vanish except

$$\frac{[4\alpha]^{2\alpha}}{[2\alpha]^{2\alpha}}\,\overline{12}^{2\alpha}\,\overline{23}^{2\alpha}\,\overline{31}^{2\alpha}\,UVW.$$

Or putting $\quad K = \dfrac{[4\alpha]^{2\alpha}}{[2\alpha]^{2\alpha}} = \dfrac{2^{4\alpha}\,1.3\,..(4\alpha - 1)}{2.4\,..\,4\alpha}$,

we have $\quad B_{4\alpha}[U, B_{2\alpha}(V, W)] = KC_\alpha(U, V, W).$

Or in particular

$$B_{4\alpha}[U, B_{2\alpha}(U, U)] = KC_\alpha(U, U, U).$$

Thus for example, neglecting a numerical factor,

$$(ax^2 + 2bxy + cy^2)(cx^2 + 2dxy + ey^2) - (bx^2 + 2cxy + dy^2)^2$$
$$= (ac - b^2)\,x^4 + 2\,(ad - bc)\,x^3y + (ae + 2bd - 3c^2)\,x^2y^2$$
$$+ 2\,(be - cd)\,xy^3 + (ce - d^2)\,y^4.$$

And then

$$e(ac - b^2) - 4d\,\tfrac{2}{4}(ad - bc) + 6c\,\tfrac{1}{6}(ae + 2bd - 3c^2)$$
$$- 4b\,\tfrac{2}{4}(be - cd) + a(ce - d^2)$$
$$= 3\,(ace - ad^2 - be^2 - c^3 + 2bcd).$$

We have likewise the singular equation

$$B_{2\alpha}(V, W) = K\left(x^{4\alpha}\frac{d}{da_{4\alpha}} - x^{4\alpha-1}y\frac{d}{da_{4\alpha-1}}\,..+ y^{4\alpha}\frac{d}{da_0}\right)C_\alpha(U, V, W)$$

where $U = \dfrac{1}{[4\alpha]^{4\alpha}}\left(a_0 x^{4\alpha} - \dfrac{[4\alpha]^1}{1}\,a_1 x^{4\alpha-1}y\,..+ a_{4\alpha}y^{4\alpha}\right)$, &c.

If however $U = V = W$, we must write

$$B_{2\alpha}(U, U) = \tfrac{1}{3}K\left(x^{4\alpha}\frac{d}{da_{4\alpha}} - x^{4\alpha-1}y\frac{d}{da_{4\alpha-1}}\,..+ y^{4\alpha}\frac{d}{da_0}\right)C_\alpha(U, U, U),$$

the reason of which is easily seen. This subject will be resumed in the sequel.

[redacted]

The functions C may be transformed in the same way as the functions B have been. In fact

$$C_a(U, V, W) = \overline{12}^{a-k}\,\overline{23}^{a-k}\,\overline{31}^{a-k}\, C_k(U, V, W),$$

if in particular $k = 1$.

$$C_1(U, V, W) = \begin{vmatrix} U^{,0} & U^{,1} & U^{,2} \\ V^{,0} & V^{,1} & V^{,2} \\ W^{,0} & W^{,1} & W^{,2} \end{vmatrix} \quad U^{,0} \text{ for } \overset{2}{U}{}^{,0} \ldots$$

But in general

$\xi_1{}^{\rho'}\eta_1{}^{\rho}\xi_2{}^{\sigma'}\eta_2{}^{\sigma}\xi_3{}^{\tau'}\eta_3{}^{\tau}\, C_1(U, V, W)$, where $\rho+\rho'=\sigma+\sigma'=\tau+\tau'=2a-2$,

$$= C_1(\overset{2a-2}{U^{,\rho}}\,\overset{2a-2}{V^{,\sigma}}\,\overset{2a-2}{W^{,\tau}}) = \begin{vmatrix} U^{,\rho} & U^{,\rho+1} & U^{,\rho+2} \\ V^{,\sigma} & V^{,\sigma+1} & V^{,\rho+2} \\ W^{,\tau} & W^{,\tau+1} & W^{,\rho+2} \end{vmatrix} \quad U^{,\rho} \text{ for } \overset{2a}{U}{}^{,\rho}, \text{ \&c.}$$

whence $C_a(U, V, W)$

$$= \Sigma\Sigma \left\{(-)^{\rho+\sigma} \begin{vmatrix} U^{,\rho} & V^{,\sigma-1} & W^{,3a-\rho-\sigma-2} \\ U^{,\rho+1} & V^{,\sigma} & W^{,3a-\rho-\sigma-1} \\ U^{,\rho+2} & V^{,\sigma+1} & W^{,3a-\rho-\sigma} \end{vmatrix} \Sigma[(-)^t A_t' A'_{\rho-t} A'_{\sigma-a+t}]\right\},$$

where $A_t' = \dfrac{[a-1]^t}{[t]^t}$; t extends from 0 to $\overline{a-1}$; ρ, $\sigma - 1$, and $3a - \rho - \sigma - 2$ may have any positive values not greater than $2a - 2$.

In particular $C_a(U, U, U)$

$$= 6\Sigma\Sigma \left\{(-)^{\rho+\sigma} \begin{vmatrix} U^{,\rho} & U^{,\sigma-1} & U^{,3a-\rho-\sigma-2} \\ U^{,\rho+1} & U^{,\sigma} & U^{,3a-\rho-\sigma-1} \\ U^{,\rho+2} & U^{,\sigma+1} & U^{,3a-\rho-\sigma} \end{vmatrix} \Sigma[(-)^t A'_t A'_{\rho-t} A'_{\sigma-a+t}]\right\},$$

where ρ, σ need only have such values that $\rho < \sigma - 1$, $\sigma - 1 < 3a - \rho - \sigma - 2$.

In particular the derivative $aei - \ldots + 15e^3$ may be transformed into

$$\begin{vmatrix} a, d, g \\ b, e, h \\ c, f, i \end{vmatrix} - 3 \begin{vmatrix} a, e, f \\ b, f, g \\ c, g, h \end{vmatrix} - 3 \begin{vmatrix} b, c, g \\ c, d, h \\ d, e, i \end{vmatrix} + 6 \begin{vmatrix} b, d, f \\ c, e, g \\ d, f, h \end{vmatrix} - 15 \begin{vmatrix} c, d, e \\ d, e, f \\ e, f, g \end{vmatrix}$$

in which form it is obviously a linear function of the determinants

$$\left\| \begin{matrix} a, b, c, d, e, f, g \\ b, c, d, e, f, g, h \\ c, d, e, f, g, h, i \end{matrix} \right\|$$

which is true generally.

Omitting for the present the theory of derivatives of the form

$$\square\, UVW = \overline{23}^{\alpha}\, \overline{31}^{\beta}\, \overline{12}^{\gamma}\, UVW,$$

we pass on to the derivatives of the fourth degree, considering those forms in which all the differential coefficients are of the same order. We may write

$$\begin{aligned}\square\, UVWX &= (\overline{12}.\overline{34})^{\alpha}\, (\overline{13}.\overline{42})^{\beta}\, (\overline{14}.\overline{23})^{\gamma}\; UVWX \\ &= D_{\alpha,\beta,\gamma}(U,\, V,\, W,\, X) = D_{\alpha,\,\beta,\,\gamma}\,;\end{aligned}$$

or if, for shortness,

$$\overline{12}\,.\,\overline{34} = \mathfrak{A},\quad \overline{13}\,.\,\overline{42} = \mathfrak{B},\quad \overline{14}\,.\,\overline{23} = \mathfrak{C},$$

we have $\qquad D_{\alpha,\,\beta,\,\gamma} = \mathfrak{A}^{\alpha}\,\mathfrak{B}^{\beta}\,\mathfrak{C}^{\gamma}.\,UVWX.$

Suppose $U = V = W = X$, and consider the derivatives which correspond to the same value f of $\alpha + \beta + \gamma$. The question is to determine how many of these are independent, and to express the remaining ones in terms of these. Since the functions become equal after the differentiations, we are at liberty before the differentiations to interchange the symbolic number 1, 2, 3, 4 in any manner whatever. We have thus

$$D_{\alpha,\beta,\gamma} = D_{\beta,\gamma,\alpha} = D_{\gamma,\alpha,\beta} = (-)^f D_{\alpha,\gamma,\beta} = (-)^f D_{\gamma,\beta,\alpha} = (-)^f D_{\beta,\alpha,\gamma}.$$

But the identical equation

$$\mathfrak{A} + \mathfrak{B} + \mathfrak{C} = 0:$$

multiplied this by $\mathfrak{A}^a\,\mathfrak{B}^b\,\mathfrak{C}^c$ and applied to each term to the product $UVWX$, gives

$$D_{a+1,\,b,\,c} + D_{a,\,b+1,\,c} + D_{a,\,b,\,c+1} = 0\,;$$

whence if $a + b + c = f - 1$, we have a set of equations between the derivatives $D_{\alpha,\,\beta,\,\gamma}$ for which $\alpha + \beta + \gamma = f$. Reducing these by the conditions first found, suppose Θf is the number of divisions of an integer f into three parts, zero admissible, but permutations of the same three parts rejected. The number of derivatives is Θf, and the number of relations between them is $\Theta(f-1)$. Hence $\Theta f - \Theta(f-1)$ of these derivatives are independent: only when f is even, one of these is $D_{f,0,0}$, *i.e.* $\overline{12}^f\,\overline{34}^f.UVWX$, *i.e.* $\overline{12}^f UV.\,\overline{34}^f WX$, or $B_f(U,\,V)\,B_f(X,\,W)$, *i.e.* $[B_f(U,\,U)]^2$. Rejecting this, the number of independent derivatives when f is even, is $\Theta f - \Theta(f-1) - 1$. Let $E\left(\dfrac{a}{b}\right)$ be the greatest integer contained in the fraction $\dfrac{a}{b}$; the number required may be shown to be

$$E\frac{f}{6} \text{ or } E\frac{f+3}{6},$$

[redacted]

according as f is even or odd. Giving to f the six forms

$$6g,\quad 6g+1,\quad 6g+2,\quad 6g+3,\quad 6g+4,\quad 6g+5,$$

the corresponding numbers of the independent derivatives are

$$g,\quad g,\quad g,\quad g+1,\quad g,\quad g+1.$$

Thus there is a single derivative for the orders 3, 5, 6, 7, 8, 10 .... two for the orders 9, 11, 12, 13, 14, 16....&c.

When f is even, the terms $D_{f-3,3,0}$, $D_{f-6,6,0}$ .. and when f is odd, the terms $D_{f-1,1,0}$, $D_{f-4,4,0}$, $D_{f-7,7,0}$, &c. may be taken for independent derivatives: by stopping immediately before that in which the second suffix exceeds the first, the right number of terms is always obtained. Thus, when $f=9$ the independent derivatives are $D_{8\,1\,0}$, $D_{5\,4\,0}$, and we have the system of equations

$$D_{900}+D_{810}+D_{801}=0,\qquad D_{621}+D_{531}+D_{522}=0,$$
$$D_{810}+D_{720}+D_{711}=0,\qquad D_{540}+D_{450}+D_{441}=0,$$
$$D_{720}+D_{630}+D_{621}=0,\qquad D_{531}+D_{441}+D_{432}=0,$$
$$D_{711}+D_{621}+D_{612}=0,\qquad D_{522}+D_{432}+D_{423}=0,$$
$$D_{630}+D_{531}+D_{540}=0,\qquad D_{432}+D_{342}+D_{333}=0,$$

which are to be reduced by

$$D_{900}=-D_{900}=0,\qquad D_{801}=-D_{810},\ \&c.$$

It is easy to form the table

$D_{200}=B_2^2$	$D_{500}=0$	$D_{700}=0$
$D_{110}=-\frac{1}{2}B_2^2$	D_{410},	D_{610}
	$D_{320}=-D_{410}$	$D_{520}=-D_{610}$
$D_{300}=0$	$D_{311}=0$	$D_{511}=0$
D_{210}	$D_{221}=0$	$D_{430}=D_{610}$
$D_{111}=0$		$D_{421}=0$
	$D_{600}=B_6^2$	$D_{331}=0$
$D_{400}=B_4^2$	$D_{510}=-\frac{1}{2}B_6^2$	$D_{222}=0$
$D_{310}=-\frac{1}{2}B_4^2$	$D_{420}=-\frac{2}{3}D_{330}+\frac{1}{6}B_6^2$	
$D_{220}=\frac{1}{2}B_4^2$	$D_{411}=\frac{2}{3}D_{330}+\frac{1}{3}B_6^2$	
$D_{211}=0$	D_{330},	
	$D_{321}=-\frac{1}{3}D_{330}-\frac{1}{6}B_6^2$	
	$D_{222}=\frac{2}{3}D_{330}+\frac{1}{3}B_6^2$	

[redacted]

$$\begin{array}{ll}
D_{800} = B_8^2 & D_{900} = 0 \\
D_{710} = -\tfrac{1}{2} B_8^2 & D_{810}\,, \\
D_{620} = -\tfrac{2}{3} D_{530} + \tfrac{1}{6} B_8^2 & D_{720} = -\,D_{810} \\
D_{611} = \tfrac{2}{3} D_{530} + \tfrac{1}{3} B_8^2 & D_{711} = 0 \\
D_{530}\,, & D_{630} = \tfrac{1}{2} D_{810} - \tfrac{1}{2} D_{540} \\
D_{521} = -\tfrac{1}{3} D_{530} - \tfrac{1}{12} B_8^2 & D_{621} = \tfrac{1}{2} D_{810} + \tfrac{1}{2} D_{540} \\
D_{440} = -\tfrac{16}{15} D_{530} - \tfrac{1}{30} B_8^2 & D_{540}\,, \\
D_{431} = \tfrac{1}{15} D_{530} - \tfrac{1}{30} B_8^2 & D_{531} = -\tfrac{1}{2} D_{810} - \tfrac{1}{2} D_{540} \\
D_{422} = \tfrac{4}{15} D_{530} + \tfrac{2}{15} B_8^2 & D_{522} = 0 \\
D_{332} = -\tfrac{2}{15} D_{530} - \tfrac{1}{15} B_8^2 & D_{441} = 0 \\
 & D_{432} = \tfrac{1}{2} D_{810} + \tfrac{1}{2} D_{540} \\
 & D_{333} = 0.
\end{array}$$

Whatever be the value, all the tables except the three first commence thus, according as f is even or odd,

$$\begin{array}{lll}
D_{f\cdot 0\cdot 0} = B_f^2 & \text{or} & D_{f,0,0} = 0 \\
D_{f-1,1,0} = -\tfrac{1}{2} B_f^2 & & D_{f-1,1,0} \\
D_{f-2,2,0} = -\tfrac{2}{3} D_{f-3,3,0} + \tfrac{1}{6} B_f^2 & & D_{f-2,2,0} = -\,D_{f-1,1,0} \\
D_{f-2,1,1} = \tfrac{2}{3} D_{f-3,3,0} + \tfrac{1}{3} B_f^2 & & D_{f-2,1,1} = 0 \\
D_{f-3,3,0} & & \vdots \\
\vdots & &
\end{array}$$

but beyond this I am not acquainted with the law.

To give some formulæ for the transformation of these derivatives; we have, for example,

$$D_{f-1,1,0} = (\overline{12}\,.\,\overline{34})^{f-1}\,\overline{13}\,.\,\overline{42}\ UUUU$$
$$= \overline{13}\,.\,\overline{42}\ B_{f-1}(U,\,U)\,B_{f-1}(U,\,U).$$

But $\quad \overline{12}\,.\,\overline{42} = \xi_1\eta_2\eta_3\xi_4 - \xi_1\xi_2\eta_3\eta_4 - \eta_1\eta_2\xi_3\xi_4 + \eta_1\xi_2\xi_3\eta_4,$

and $\quad \xi_1\eta_2\eta_3\xi_4\ B_{f-1}(U,U)\,B_{f-1}(U,U)$

$$= B_{f-1}(\xi U,\,\eta U)\,B_{f-1}(\eta U,\,\xi U)$$
$$= B_{f-1}(U^{,0}U^{,1})\,B_{f-1}(U^{,1}U^{,0}),\ \&c.$$

(where $U^{,0}$, $U^{,1}$ stand for $\overset{1}{U}{}^{,0}\overset{1}{U}{}^{,1}$), &c.; or

$$D_{f-1,1,0} = -\,2\,\{B_{f-1}(U^{,0}U^{,0})\,B_{f-1}(U^{,1}U^{,1})$$
$$-\,B_{f-1}(U^{,0}U^{,1})\,B_{f-1}(U^{,1}U^{,0})\},$$

which reduces itself to

$$D_{f-1,1,0} = -\,2\,\{B_{f-1}(U^{,0}U^{,1})\}^2,$$
$$D_{f-1,1,0} = -\,2\,\{B_{f-1}(U^{,0}U^{,0})\,B_{f-1}(U^{,1}U^{,1}) - [B_{f-1}(U^{,0}U^{,1})]^2\},$$

according as f is even or odd.

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For example, for the orders 3, 5, 7, 9, we have

$$D_{210} = -2\left\{4(ac-b^2)(bd-c^2)-(ad-bc)^2\right\},$$

$$D_{410} = -2\left\{4(ae-4bd+3c^2)(bf-4ce+3d^2)-(af-3be+2cd)^2\right\}$$

$$D_{610} = -2\left\{4(ag-6bf+15ce-10d^2)(bh-6cg+15df-10e^2)\right.$$
$$\left.-(ah-5bg+9cf-5de)^2\right\}.$$

$$D_{810} = -2\left\{4(ai-8bh+28cg-56df+35e^2)(bj-8ci\right.$$
$$\left.+28dh-56eg+35f^2)-(aj-7bi+20ch-28dg+14ef)^2\right\}.$$

The derivatives D will be presently calculated in a completely expanded form up to the ninth order. We have, therefore, still to find the derivatives of the sixth and eighth orders and a second derivative of the ninth order. For the sixth order, the simplest method is to make use of D_{222}, which is easily seen to be equal to

$$24\begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix}$$

For the two others we have the general formulæ

$$D_{f-2,2,0}$$
$$= 2\left\{B_{f-2}(U^{,0}U^{,0})\,B_{f-2}(U^{,2}U^{,2}) - 4B_{f-2}(U^{,0}U^{,1})\,B_{f-2}(U^{,2}U^{,1})\right.$$
$$\left.+ B_{f-2}(U^{,0}U^{,2})\,B_{f-2}(U^{,2}U^{,0}) + 2\left[B_{f-2}(U^{,1}U^{,1})\right]^2\right\},$$

where $U^{,0}$, $U^{,1}$, $U^{,2}$ have been written for $\overset{2}{U}{}^{,0}$, $\overset{2}{U}{}^{,1}$, $\overset{2}{U}{}^{,2}$; a formula which is demonstrated in precisely the same way as that for $D_{f-1,1,0}$.

$$D_{f-3,3,0}$$
$$= -2\left\{B_{f-3}(U^{,0}U^{,3})\,B_{f-3}(U^{,3}U^{,0}) - 6B_{f-3}(U^{,0}U^{,1})\,B_{f-3}(U^{,3}U^{,2})\right.$$
$$+ 6B_{f-3}(U^{,0}U^{,2})\,B_{f-3}(U^{,3}U^{,1}) + 9B_{f-3}(U^{,1}U^{,1})\,B_{f-3}(U^{,2}U^{,2})$$
$$\left.- 9B_{f-3}(U^{,1}U^{,2})\,B_{f-3}(U^{,2}U^{,1}) - B_{f-3}(U^{,0}U^{,3})\,B_{f-3}(U^{,3}U^{,0})\right\},$$

(in which $U^{,0}$, &c. stand for $\overset{0}{U}{}^{,0}$, &c.). In particular

$$D_{620} = 2\left\{4(ag-6bf+15ce-10d^2)(ci-6dh+15eg-10f^2)\right.$$
$$-4(ah-5bg+9cf-5de)(bi+5ch+9dg-5ef)$$
$$+(ai-6bh+16cg-26df+15e^2)^2+8(bh-6cg$$
$$\left.+15df-10e^2)^2\right\},$$

$$D_{630} = -2\{4(ag - 6bf + 15ce - 10d^2)(dj - 6ei + 15fh - 10g^2)$$
$$- 6(ah - 5bg + 9cf - 5de)(cj - 5di + 9eh - 5fg)$$
$$+ 6(ai - 6bh + 16cg - 26df + 15e^2)(bj - 6ci + 16dh$$
$$- 26eg + 15f^2) + 36(bh - 6cg + 15df - 10e^2)(ci - 6dh$$
$$+ 15eg - 10f^2) - 9(bi - 5ch + 9dg - 5ef)^2 - (aj - 6bi$$
$$+ 15ch - 19dg + 9ef)^2\}.$$

Hence we have all the elements necessary for the calculation of the following table of the independent constant derivatives of the fourth degree, up to the ninth order.

$$D_{210} = -2(6abcd - 4ac^3 - 4b^3d + 3b^2c^2 - a^2d^2),$$

$$D_{410} = -2(10aebf - 16ae^2c - 16b^2df + 12aed^2 + 12c^2bf - 48c^3e$$
$$- 48d^3b + 76bcde + 32c^2d - a^2f^2 - 4acdf - 9b^2e^2),$$

$$^{*}D_{222} = 24(aceg + 2adef + 2gdbc - agd^2 - ae^3 - gc^3 - acf^2 - geb^2$$
$$- 2bd^2f - 2bcef + bde^2 + fdc^2 + b^2f^2 + e^2c^2 - 3ecd^2$$
$$+ bde^4 + fdc^2 + d^4),$$

$$D_{610} = -2(14agbh + 234bgcf + 990cedf - 375d^2e^2 - a^2h^2$$
$$- 25b^2g^2 - 81c^2f^2 - 18ahcf + 10ahde - 50bgde - 24acg$$
$$- 24b^2fh + 60agdf + 60cebh - 40age^2 - 40d^2bh$$
$$- 360bdf^2 - 36c^2eg + 240bfe^2 + 240d^2cg - 600ce^3$$
$$- 600d^2bf),$$

$$^{\dagger}D_{620} = 2(36agci + 696bfdh + 2340cge^2 + 2876d^2f^2 + 40b^2h^2$$
$$+ 544c^2g^2 + 1025e^4 - 388bgch - 340bhe^2 - 2596cgdf$$
$$- 4180dfe^2 - 16ahbi + a^2i^2 - 52aidf + 30aie^2 - 60agdh$$
$$- 60bfci + 60aeg^2 + 60iec^2 - 360befg - 360cdeh - 40agf^2$$
$$- 40d^2ci + 240bf^3 + 240hd^3 - 420cef^2 - 420egd^2$$
$$+ 20ach^2 + 20gib^2 + 20ahef + 20debi + 180bdg^2 + 180fhc^2$$
$$- 100bgef - 100dech),$$

$$D_{810} = -2(18aibj + 536bhci + 4256cgdh + 13328defg + 4704e^2f^2$$
$$- a^2j^2 - 49b^2i^2 - 400c^2h^2 - 784d^2g^2 - 40ajch + 56ajdg$$
$$- 28ajef - 392bidg + 196bief - 560chef - 32aci^2$$
$$- 32b^2hj + 112aidh + 112cgbj - 224aieg - 224dfbj$$
$$+ 140aif^2 + 140e^2bj - 896bdh^2 - 896c^2gi + 1792bheg$$
$$+ 1792dfci - 1120bhf^2 - 1120e^2ci - 6272ceg^2 - 6272d^2fh$$
$$+ 3920cgf^2 + 3920e^2dh - 7840df^3 - 7840ge^3),$$

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* $D_{330} = \frac{3}{4} D_{222} + \frac{1}{2} B^2_{6}$. † $D_{530} = -\frac{3}{2} D_{620} + \frac{1}{4} B^2_{8}$.

Equations which determine D_{330} and D_{530}, the quantities by means of which the remaining derivatives of the sixth and eighth orders have been expressed.

$$*\,D_{621} = -4\,(7bhci + 22cgdh + 39defg + 30e^2f^2 - 2b^2i^2 + 25c^2h^2$$
$$- 47d^2g^2 - 2ajch + 7ajdg - 5ajef + 74bgdi - 73befi$$
$$- 127chef + 2aci^2 + 2b^2hj - 7aidh - 7cgbj - 22aieg$$
$$- 22dfbj + 25aif^2 + 25bje^2 - 52bdh^2 - 52c^2gi + 23bgeh$$
$$+ 23cfdi + 70bhf^2 + 70ce^2i + 32ceg^2 + 32d^2fh + 25cgf^2$$
$$+ 25dhe^2 - 50df^3 - 50ge^3 - 45agfh - 45cedj - 45bfg^2$$
$$- 45eid^2 + 27aeh^2 + 27c^2fj - 20ag^3 - 20jd^3).$$

We may now proceed to demonstrate an important property of the derivatives of the fourth degree, analogous to the one which exists for the third degree. Let U, V, W, X be functions of any order f: then, investigating the value of the expression

$$B_{2f-2a}\,[B_a(U, V),\ B_a(W, X)].$$

This reduces itself in the first place to

$$\overline{\theta\phi}^{2f-2a}\ \overline{12}^a\ \overline{34}^a\ UVWX,$$

where ξ_θ, η_θ refer to U and V, and ξ_ϕ, η_ϕ to W and X: this comes to writing $\xi_\theta = \xi_1 + \xi_2$, $\eta_\theta = \eta_1 + \eta_2$, and $\xi_\phi = \xi_3 + \xi_4$, $\eta_\phi = \eta_3 + \eta_4$; whence

$$\overline{\theta\phi} = \overline{13} + \overline{14} + \overline{23} + \overline{24},$$

or the function in question is

$$(13 + 14 + 23 + 24)^{2f-2a}\ \overline{12}^a\ \overline{34}^a\ UVWX.$$

But all the terms of this where the sum of the indices of ξ_1, η_1 or ξ_2, η_2 or ξ_3, η_3 or ξ_4, η_4, exceed f, vanish: whence it is only necessary to consider those of the form

$$K_r\,(\overline{13}.\overline{42})^r\ (\overline{14}.\overline{23})^{f-a-r}\ (\overline{12}.\overline{34})^a\ UVWX,$$

where K_r denotes the numerical coefficient

$$\frac{(-)^r\,.\,[2f-2a]^{2f-2a}}{[r]^r\,[r]^r\,[f-a-r]^{f-a-r}\,[f-a-r]^{f-a-r}},$$

or $B_{2f-2a}\,[B_a(U, V),\ B_a(W, X)]$

$$= \Sigma\left\{K_r\,D_{a,r,f-a-r}(U, V, W, X)\right\}.$$

In particular, if $U = V = W = X$, writing also B_a for $B_a(U, U)$,

$$B_{2f-2a}(B_a, B_a) = \Sigma\,(K_r\,D_{a,r,f-a-r}).$$

$*\ D_{540} = 2D_{621} + D_{810}$.

Equation to determinate D_{540}.

If a is odd, this becomes

$$0 = \Sigma (K_r\ D_{a, r, f-a-r}),$$

an equation which must be satisfied identically by the relations that exist between the quantities D. If, on the contrary, a is even, we see that there are as many independent functions of the form

$$B_{2f-2a}(B_a, B_a)$$

as there are of the form D; and that these two systems may be linearly expressed, either by means of the other. Thus, for the orders 3, 5, 7, the derivatives D are respectively equal, neglecting a numerical factor, to

$$B_6(U^2, U^2),\quad B_{10}(U^2, U^2),\quad B_{14}(U^2, U^2).$$

For the sixth order they may be linearly expressed by means of
$$B_{12}(U^2, U^2),\quad B_6^2,$$
and so on. All that remains to complete the theory of the fourth degree is to find the general solution of this system of equations, as also of the system connecting the derivatives D.

Passing on to a more general property. Let $U_1, U_2 \dots U_p$ be functions of the orders $f_1, f_2 \dots f_p$; and suppose

$$\Theta(U_2 \dots U_p) = \square\, U_2 \dots U_p$$

a function of the degree f_1 in the variables: suppose that $\Theta(U_2 \dots U_p)$ contains the differential coefficients of the order r_2 for U_2, r_3 for U_3, &c., so that $f_1 = (f_2 - r_2) + \dots (f_p - r_p)$. Consider the expression

$$B_{f_1}\{U_1,\ \Theta(U_2 \dots U_p)\},$$

which reduces itself in the first place to

$$(\overline{12} + \overline{13} \dots + \overline{1p})^{f_1}\ \square\ U_1 U_2 \dots U_p,$$

then to $\quad K(\overline{12}^{f_2-r_2}\ \overline{13}^{f_3-r_3}\ \vdots\ \overline{1p}^{f_p-r_p}\ \square\ U_1 U_2 \dots U_p;$

where for shortness

$$K = \frac{[f_1]^{f_1}}{[f_2 - r_2]^{f_2-r_2} \dots [f_p - r_p]^{f_p-r_p}}.$$

For if one of the indices were smaller another would be greater, for instance that of $\overline{12}$: and the symbols ξ_2, η_2 in $\overline{12}^{f_2-r_2-\lambda}\ \square$ would rise to an order higher than f_2, or the term would vanish. Hence, writing

$$\square' = \overline{12}^{f_2-r_2}\ \overline{13}^{f_3-r_3} \dots \overline{1p}^{f_p-r_p}$$

and $\quad \Theta'(U_1, U_2 \dots, U_p) = \square' U_1 U_2 \dots U_p),$

we have $B_{f_1}\{U_1.\ \Theta(U_2\ldots U_p)\} = K\Theta'(U_1,\ U_2\ldots\ U_p)$;
i.e. the first side is a constant derivative of $U_1,\ U_2\ \ldots U_p$.

Suppose $$U_1 = \frac{1}{[f_1]^{f_1}}(a_0 x^{f_1} + \ldots),$$

$$\Theta(U_2\ldots U_p) = \frac{1}{[f_1]^{f_1}}(A_0 x^{f_1} + \ldots),$$

then $$K\Theta'(U_1\ldots U_p) = a_0 A_{f_1} - \frac{f_1}{1}\,a_1 A_{f_1-1} + \ldots;$$

i.e. $A_{f_1} = K\dfrac{d}{da_0}\,\Theta'(U_1\ldots U_p),\ \dfrac{f_1}{1}A_{f_1-1} = K\dfrac{d}{da_1}\,\Theta'(U_1,\ U_2\ldots U_p)\ldots$
or finally,

$$\Theta(U_2\ldots U_p) = \frac{K}{[f_1]^{f_1}}\left(x^{f_1}\frac{d}{da_{f_1}} - x^{f_1-1}y\frac{d}{da_{f_1-1}} + \ldots\right)\Theta'(U_1\ldots U_p),$$

an equation which holds good; changing, however, the numerical factor, when several of the functions $U_1\ldots U_p$ become identical. Hence the theorem: if U be a function given by

$$U = \frac{1}{[f]^f}(a_0 x^f + a_1 x^{f-1}y + \ldots),$$

and Θ be any constant derivative whatever of U, then

$$\left(x^f\frac{d}{da_f} - x^{f-1}y\frac{d}{da_{f-1}} + \ldots\right)\Theta$$

is a derivative of U, and its value, neglecting a numerical factor, may be found by omitting in the symbol $\square$, which corresponds to the derivative Θ, the factors which contain any one, no matter which, of the symbolic numbers. (See Note.*)

If, for example,

$$-\tfrac{1}{2}D_{210} = \Theta = 6abcd - 4ac^3 - 4bd^3 + 3b^2c^2 - a^2d^2,$$

or $$\square = \overline{12}^2.\overline{34}^2.\overline{13}.\overline{42};$$

then $$\left(x^3\frac{d}{da} - x^2y\frac{d}{dc} + xy^2\frac{d}{db} - y^3\frac{d}{da}\right)\Theta$$

reduces itself, omitting a numerical factor, to

$$\overline{12}^2\,\overline{13}\,UUU = -\tfrac{1}{2}B_1\{U,\ B_1(U.U)\}.$$

This may be compared with some formulæ of M. Eisenstein's, (*Crelle*, XXVII.); adopting his notation, we have

* Not given with the present paper.

[redacted]

$$\Phi = ax^3 + 3bx^2y + 3cxy^2 + dy^3,$$

$$F = \tfrac{1}{36} B_2(\Phi, \Phi) = (ac - b^2)\, x^2 + (ad - bc)\, xy + (bd - c^2)\, y^2,$$

$$\Phi = -\tfrac{1}{2}\left(x^3 \frac{d}{da} - x^2y \frac{d}{dc} + xy^2 \frac{d}{db} - y^3 \frac{d}{da}\right) D,$$

where D is the same as Θ. Hence to the system of formulæ which he has given, we may add the two following:

$$\Phi_1 = \tfrac{1}{3}\left(\frac{d\Phi}{dx}\frac{dF}{dy} - \frac{d\Phi}{dy}\frac{dF}{dx}\right),$$

$$\Phi_1 = -\tfrac{1}{216}\left\{\frac{d^3\Phi}{dx^3}\frac{d^2\Phi}{dx^2}\frac{d\Phi}{dy} - \frac{d^3\Phi}{dx^2\,dy}\left(2\frac{d^2\Phi}{dx\,dy}\frac{d\Phi}{dy} + \frac{d^2\Phi}{dy^2}\frac{d\Phi}{dx}\right)\right.$$
$$\left. + \frac{d^3\Phi}{dx\,dy^2}\left(2\frac{d^2\Phi}{dx\,dy}\frac{d\Phi}{dx} + \frac{d^2\Phi}{dx^2}\frac{d\Phi}{dy}\right) - \frac{d^3\Phi}{dy^3}\frac{d^2\Phi}{dx^2}\frac{d\Phi}{dx}\right\},$$

the first of which explains most simply the origin of the function Φ_1.

It will be sufficient to indicate the reductions which may be applied to derivatives of the form

$$C_{\alpha,\beta,\gamma}(U, V, W) = \overline{23}^{\alpha}\,.\,\overline{31}^{\beta}\,.\,\overline{12}^{\gamma}\; UVW,$$

where U, V, W are *homogeneous* functions. In fact, if

$$\xi x + \eta y = \Xi,$$

the above becomes, neglecting a numerical factor,

$$(\Xi_1\,.\,\overline{23})^{\alpha}\,(\Xi_2\,.\,\overline{31})^{\beta}\,(\Xi_3\,.\,\overline{12})^{\gamma}\; UVW,$$

where the symbols ξ, η are supposed not to affect the x, y which enter into the expressions Ξ. But we have identically

$$\Xi_1\overline{23} + \Xi_2\overline{31} + \Xi_3\overline{12} = 0,$$

an equation which gives rise to reductions similar to those which have been found for the derivatives $D_{\alpha,\beta,\gamma}$, but which require to be performed with care, in order to avoid inaccuracies with respect to the numerical factors. It may, however, be at once inferred, that the number of independent derivatives $D_{\alpha,\beta,\gamma}$ is the same with that of the independent derivatives $D_{\alpha,\beta,\gamma}$ for the same value of α, β, γ.

From similar reasonings to those by which $B\{U, B(U,U)\}$ has been found, the following general theorem may be inferred.

"The derivative of any number of the derivatives of one or more functions, or even of any number of functions of these derivatives, is itself a derivative of the original functions."

[redacted]

For the complete reduction of these double derivatives, it would be sufficient, theoretically, to be able to reduce to the smallest number possible, the derivatives of any given degree whatever. This has been done for the derivatives of the third degree $C_{\alpha,\beta,\gamma}$, and for those of the fourth degree, in which all the differentiations rise to the same order $(D_{\alpha,\beta,\gamma})$: it seems, however, very difficult to extend these methods even to the next simplest cases,—extensive researches in the theory of the division of numbers would probably be necessary. Important results might be obtained by connecting the theory of hyperdeterminants with that of elimination, but I have not yet arrived at anything satisfactory upon this subject. I shall conclude with the remark, that it is very easy to find a series, or rather a series of series's of hyperdeterminants of all degrees, viz. the determinants

$$\begin{vmatrix} a, & b \\ b, & c \end{vmatrix}, \quad \begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix}, \quad \begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix} \;\&c.$$

$$\begin{vmatrix} \cdot & a, & b, & c \\ \cdot & b, & c, & d \\ a, & b, & c & \cdot \\ b, & c, & d & \cdot \end{vmatrix}, \quad \begin{vmatrix} \cdot & a, & b, & c, & d, & e \\ \cdot & b, & c, & d, & e, & f \\ \cdot & c, & d, & e, & f, & g \\ a, & b, & c, & d, & e & \cdot \\ b, & c, & d, & e, & f & \cdot \\ c, & d, & e, & f, & g & \cdot \end{vmatrix} \;\&c. \quad \begin{vmatrix} \cdot & \cdot & a, & b, & c, & d \\ \cdot & \cdot & b, & c, & d, & e \\ \cdot & a, & b, & c, & d & \cdot \\ \cdot & b, & c, & d, & e & \cdot \\ a, & b, & c, & d & \cdot & \cdot \\ b, & c, & d, & e & \cdot & \cdot \end{vmatrix} \;\&c.$$

However, these functions are not all independent; *e.g.* the last may be linearly expressed by the square of the second and the cube of $(ae - 4bd + 3c^2)$: nor do I know the symbolical form of these hyperdeterminant determinants.

INVESTIGATION OF PROPERTIES OF THE HYPERBOLA.

By Edmond R. Turner, Caius College.

The following is a mode of treating the hyperbola, in a manner similar to that given by Mr. O'Brien in a former number of the Journal. It does not require the use of an imaginary angle or quantity, as the method does which is given in his treatise on Analytical Geometry.

It is evident that we may put the equation to the hyperbola in the form of two equations, by means of a subsidiary angle ϕ, by assuming

$$x = a \sec \phi, \quad y = b \tan \phi;$$

and if ϕ and ϕ' be the angles corresponding to P and D, the extremities of two conjugate diameters (ϕ' being a subsidiary angle, for expressing the equation to the conjugate hyperbola)

$$\phi = \phi'.$$

Since the equation to the conjugate hyperbola is

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1,$$

we must for this curve assume

$$y = b \sec \phi', \quad x = a \tan \phi'.$$

Let xy, $x'y'$ be the points P and D respectively, then we have

$$\left.\begin{aligned} x &= a \sec \phi, & y &= b \tan \phi \\ x' &= a \tan \phi', & y' &= b \sec \phi' \end{aligned}\right\} \ldots\ldots\ldots (1);$$

but,

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 0;$$

therefore, substituting from (1),

$$\sin \phi - \sin \phi' = 0,$$

or

$$\phi = \phi'.$$

If $r\theta$, $r'\theta'$ be the polar coordinates of P and D,

$$r^2 = x^2 + y^2 = a^2 \sec^2 \phi + b^2 \tan^2 \phi,$$

$$r'^2 = x'^2 + y'^2 = a^2 \tan^2 \phi + b^2 \sec^2 \phi;$$

therefore

$$r^2 - r'^2 = a^2 - b^2.$$

Also if A be the area of the parallelogram completed upon CP and CD,

$$\begin{aligned} A &= rr' \sin (\theta' - \theta) \\ &= xy' - x'y \\ &= ab \sec^2 \phi - ab \tan^2 \phi \\ &= ab. \end{aligned}$$

If in the expressions $x = a \sec \phi$, $y = b \tan \phi$, we put ϕ' for ϕ, and change x and y into x' and y', therefore

$$x' = a \tan \phi' = a \tan \phi = \frac{a}{b} y,$$

$$y' = b \sec \phi' = b \sec \phi = \frac{b}{a} x.$$

[redacted]

To find the evolute to the hyperbola.
The equation to the normal, being

$$y - y_1 = -\frac{dx_1}{dy_1}(x - x_1),$$

may be written

$$y - b\tan\phi = -\frac{a}{b}\sin\phi\,(x - a\sec\phi),$$

or $$y = -x\frac{a}{b}\sin\phi + \frac{a^2+b^2}{b}\tan\phi;$$

therefore, differentiating with regard to ϕ,

$$0 = -x\frac{a}{b}\cos\phi + \frac{a^2+b^2}{b}\sec^2\phi:$$

therefore $$\sec^3\phi = \frac{x}{\alpha}; \quad \text{where } \alpha = \frac{a^2+b^2}{a}.$$

Changing $\sec\phi, x, y, a, b$ into $\tan\phi, y, x, b, a$, respectively,

$$\tan^3\phi = \frac{y}{\beta}, \quad \text{where } \beta = \frac{a^2+b^2}{b};$$

therefore $$\left(\frac{x}{\alpha}\right)^{\frac{2}{3}} + \left(\frac{y}{\beta}\right)^{\frac{2}{3}} = 1.$$

It is evident that, if a circle be described on the axis major, and a tangent be drawn to it from the foot of the ordinate at any point; the radius passing through the point of contact will be inclined at the angle ϕ to the axis of abscissas.

NOTE ON THE RINGS AND BRUSHES IN THE SPECTRA PRODUCED BY BIAXAL CRYSTALS.

By William Thomson, B.A.

It has been shown in this Journal (vol. III. p. 286) that if any system of isothermal plane curves be given, the orthogonal system, which is proved to be necessarily isothermal also, may in every case be determined. Thus if $v = a$ be the equation to the first system, v being a function of x and y which satisfies the equation

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = 0,$$

we shall have for the equation to the orthogonal system

$$u = \int\left(\frac{dv}{dy}\,dx - \frac{dv}{dx}\,dy\right) = \beta,$$

the expression under the sign of integration being in this case a complete differential, and the equation

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0$$

will be satisfied; results which may be readily verified.

To take an example, let Q, Q', &c. be any number of fixed points determined by the coordinates (a, b), (a', b'), &c., and let r, r', &c. be the distances of the point $P(xy)$ from those points. We may take for the equation of an isothermal system of curves

$$v = m \log r + m' \log r' + \&c. = \alpha \;\ldots\ldots\; (1),$$

where $r^2 = (x - a)^2 + (y - b)^2$, &c., and m, m', &c. are constants.

In this case we have

$$u = m \tan^{-1}\frac{x - a}{y - b} + m' \tan^{-1}\frac{x - a'}{y - b'} + \&c. = \beta \;\ldots\; (2)$$

for the orthogonal system, which, as may be readily verified, is also isothermal.

To take a simple case, let there be only two fixed points, Q, Q', and let $m = m' = 1$. The equation of the first system becomes

$$rr' = \epsilon^{\alpha} = c \;\ldots\ldots\ldots\ldots\ldots\; (3);$$

and, if we take the origin as the point of bisection of QQ', and make this line the axis of x, the equation of the second system becomes

$$\tan^{-1}\frac{x - a}{y} + \tan^{-1}\frac{x + a}{y} = \beta \;\ldots\ldots\ldots\; (4),$$

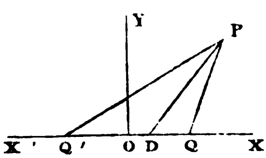
or

$$\frac{2xy}{y^2 - x^2 + a^2} = \tan\beta,$$

which may be put under the form

$$x^2 + 2kxy - y^2 = a^2 \;\ldots\ldots\ldots\; (5).$$

The equation (3) represents the series of *lemniscates* which Herschel has shown to be the forms of the rings in a biaxal crystal. Also (4) is the equation of a brush, since, if we draw PD bisecting the angle QPQ' and meeting QQ' in D, we have



$$PDQ = PQX - DPQ = PQ'X + Q'PD$$

$$= \tfrac{1}{2}(PQX + PQ'X) = \tfrac{1}{2}\left(\tan^{-1}\frac{x - a}{y} + \tan^{-1}\frac{x + a}{y}\right).$$

Hence (4) represents the locus of the points P, when the angle PDQ is constant, which is the characteristic property of a brush.

Thus we see that the rings in a biaxal crystal form a system of isothermal plane curves, and the brushes the conjugate orthogonal system.

Some curious properties of the second system, which is a series of hyperbolas, may be deduced from equation (5). Let $\frac{a}{h_1^{\frac{1}{2}}}$ and $\frac{a}{h_2^{\frac{1}{2}}}$ be the semiaxes, real and imaginary, and θ the angle which the former makes with OX. To determine h_1, h_2, and θ, we have

$$(h - 1)(h + 1) = k^2,$$

$$\tan^2\theta = \frac{h_1 - 1}{h_1 + 1},$$

from which we deduce $h_1 = (k^2 + 1)^{\frac{1}{2}}$,

$$h_2 = -(k^2 + 1)^{\frac{1}{2}},$$

$$\tan^2\theta = \frac{(k^2 + 1)^{\frac{1}{2}} - 1}{(k^2 + 1)^{\frac{1}{2}} + 1}.$$

Hence
$$(k^2 + 1)^{\frac{1}{2}} = \frac{1 + \tan^2\theta}{1 - \tan^2\theta} = \frac{1}{\cos 2\theta},$$

and
$$\left(\frac{a}{h_1^{\frac{1}{2}}}\right)^2 = a^2 \cos 2\theta.$$

Thus the second system is a series of rectangular hyperbolas whose vertices lie on the lemniscate of which the equation is

$$\rho^2 = a^2 \cos 2\theta.$$

By putting $y = 0$ in (5), we have, for the two values of x, $\pm a$, and therefore each hyperbola passes through the points Q and Q'. The series is determined by this and the preceding property.

In addition it may be remarked that, by putting $\epsilon^2 = a^2$ in (3), we find
$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2),$$

or, in polar coordinates,

$$\rho^2 = 2a^2 \cos 2\theta,$$

for the equation of one of the curves of the first system. This curve is a lemniscate similar to that which is the locus of the vertices of the second system, and similarly situated, but of different magnitude.

ON THE PRINCIPAL AXES OF A RIGID BODY.

By William Thomson.

§ 1. *Definition of principal axes of a solid body.*—2...5. *Centrifugal forces generated by revolution round a fixed axis.* 6, 7. *Determination of the principal axes through any point.* 8. *Demonstration that they are perpendicular to one another.* 9. *Determination of principal moments.*—10, 11. *Examination of cases in which there are principal axes in any of the coordinate planes.*—12. *Conditions that the axes of coordinates may be principal axes.*

In the following paper no new results are brought forward, but the method of investigation is in some respects simpler than that which is usually given in treatises on Dynamics, and I am not aware that the definition which I have adopted has been employed by other writers.

1. The definition of principal axes usually given in treatises on Dynamics involves the consideration of three at right angles to one another, and it is not easy from it to deduce a simple definition of a single principal axis through any point. The dynamical property which is proved, that the principal axes through a point are the only "permanent axes of rotation" of the body when held by the point, affords a very simple definition of a principal axis, which may be stated as follows:—

Def. *A principal axis through any point of a solid body is a line such that, if the body revolve round it, the centrifugal forces generated are either in equilibrium with one another, or have a single resultant passing through the point.*

2. Let OM be any axis through a point O of a rigid body, determined relatively to three rectangular axes OX, OY, OZ, by the direction cosines l, m, n, and let the body revolve round OM with an angular velocity ω. Let P be any point (x, y, z) and let PM be drawn perpendicular to OM. The centrifugal force on an element $\delta\mu$ of the body at P will be $\omega^2 PM\delta\mu$, and its component in any direction will be found by substituting for PM in this expression its corresponding projection. Now the coordinates of P, relatively to M, are

$$x - l(lx + my + nz),\ y - m(lx + my + nz),\ z - n(lx + my + nz).$$

Hence the components of the centrifugal force on $\delta\mu$, parallel to OX, OY, OZ, are

$$\left.\begin{array}{l} \omega^2\delta\mu\,\{x - l(lx + my + nz)\} \\ \omega^2\delta\mu\,\{y - m(lx + my + nz)\} \\ \omega^2\delta\mu\,\{z - n(lx + my + nz)\} \end{array}\right\} \ldots\ldots\ldots\ldots (1).$$

The components of the couple obtained by transferring the force on $\delta\mu$ to the origin are consequently

$$\left.\begin{array}{l} \omega^2\delta\mu\,(mz - ny)(lx + my + nz) \\ \omega^2\delta\mu\,(nx - lz)(lx + my + nz) \\ \omega^2\delta\mu\,(ly - mx)(lx + my + nz) \end{array}\right\} \ldots\ldots\ldots\ldots (2).$$

To get the components (X, Y, Z) of the resultant force at the origin, and the components (L, M, N) of the resultant couple of the centrifugal forces, we must take the sum of the preceding expressions for every element of the body. Thus, if μ be the mass of the body and $\bar{x}, \bar{y}, \bar{z}$ the coordinates of the centre of gravity: and if

$$U = \Sigma\delta\mu\, x\,(lx + my + nz),$$
$$V = \Sigma\delta\mu\, y\,(lx + my + nz),$$
$$W = \Sigma\delta\mu\, z\,(lx + my + nz),$$

we have

$$\left.\begin{array}{l} X = \omega^2\mu\,\{\bar{x} - l\,(l\bar{x} + m\bar{y} + n\bar{z})\} \\ Y = \omega^2\mu\,\{\bar{y} - m\,(l\bar{x} + m\bar{y} + n\bar{z})\} \\ Z = \omega^2\mu\,\{\bar{z} - n\,(l\bar{x} + m\bar{y} + n\bar{z})\} \end{array}\right\} \ldots\ldots (3),$$

$$\left.\begin{array}{l} L = \omega^2\,(mW - nV) \\ M = \omega^2\,(nU - lW) \\ N = \omega^2\,(lV - mU) \end{array}\right\} \ldots\ldots\ldots\ldots (4).$$

If we denote the sums

	$\Sigma\delta\mu x^2$,	$\Sigma\delta\mu y^2$,	$\Sigma\delta\mu z^2$,	$\Sigma\delta\mu yz$,	$\Sigma\delta\mu zx$,	$\Sigma\delta\mu xy$
by	F	G	H	A'	B'	C'

we shall have the following expressions for U, V, W,

$$\left.\begin{array}{l} U = Fl + C'm + B'n \\ V = C'l + Gm + A'n \\ W = B'l + A'm + Hn \end{array}\right\} \ldots\ldots\ldots\ldots (5).$$

If we denote the moments of inertia of the body round the axes of coordinates by A, B, C, so that

$$A = \Sigma\delta\mu\,(y^2 + z^2) = G + H, \quad B = H + F, \quad C = F + G,$$

we have for L, M, N, the modified expressions

$$\left.\begin{array}{ll} & L = \omega^2\,(nv - mw) \\ & M = \omega^2\,(lw - nu) \\ & N = \omega^2\,(mu - lv) \\ \text{where} & u = Al - C'm - B'n \\ & v = -C'l + Bm - A'n \\ & w = -B'l - A'm + Cn \end{array}\right\} \ldots\ldots\ldots\ldots (6).$$

3. Comparing the values of X, Y, Z, given by equations (3), with the expressions (1), we conclude that the absolute amount of the centrifugal force is the same as if the whole mass of the body were collected at its centre of gravity; but equations (4) shew that its moments are in general not the same as they would be in this case, and that there is not generally a single resultant of the centrifugal forces.

4. From equations (3) and (4) we deduce the relations

$$\left.\begin{array}{l} lX + mY + nZ = 0 \\ lL + mM + nN = 0 \end{array}\right\} \dots\dots\dots\dots (7),$$

and hence the resultant force and the axis of the resultant couple are each perpendicular to the axis of rotation. These conclusions might have been anticipated from the circumstance that each component of the centrifugal force is in a line perpendicular to the axis of rotation, and passing through it.

5. If O and $\overline{M}$ be fixed points in the axis, and $O\overline{M}$ be given, the equations (3) and (4) enable us to find the pressure which the body, when revolving uniformly, exerts upon them. If we take $l = 0$, $m = 0$, $n = 1$, so that M may be a point in OZ, we obtain the ordinary expressions for the pressure on a fixed axis.

If the axes of coordinates be such that

$$A' = 0, \quad B' = 0, \quad C' = 0,$$

(principal axes according to the ordinary definition), the expressions for the components of the centrifugal couple become much simplified. Thus if we take the equations (6) we have, in this case,

$$L = \omega^2 (B - C)\, mn,$$
$$M = \omega^2 (C - A)\, nl,$$
$$N = \omega^2 (A - B)\, lm,$$

which are the expressions indicated by Poinsot in his memoir on Rotatory Motion.*

6. If the centrifugal forces are either in equilibrium or have a single resultant passing through the origin, the couples L, M, N must vanish. Hence the conditions that OM may be a principal axis, according to the definition stated above, are

* For analytical demonstrations of the various theorems indicated by Poinsot in this Memoir, reference may be made to an elegant paper in Liouville's Journal, entitled "*Thèse sur le mouvement d' un corps solide autour d' un point fixe, par M. Briot, Professeur au Collège Royale d' Orléans.*"

[redacted]

$$\left.\begin{array}{l} m\,(B'l + A'm + Hn) - n\,(C'l + Gm + A'n) = 0 \\ n\,(Fl + C'm + B'n) - l\,(B'l + A'm + Hn) = 0 \\ l\,(C'l + Gm + A'n) - m\,(Fl + C'm + B'n) = 0 \end{array}\right\} \ldots\ldots (8),$$

or

$$\left.\begin{array}{l} n\,(-C'l + Bm - A'n) - m(-B'l - A'm + Cn) = 0 \\ l\,(-B'l - A'm + Cn) - n\,(Al - C'm - B'n) = 0 \\ m\,(Al - C'm - B'n) - l\,(-C'l + Bm - A'n) = 0 \end{array}\right\} \ldots (9);$$

the first set being obtained from equations (4), and the second, which differ only in form, from (6). In the investigation of principal axes commonly given, the quantities F, G, H are made use of instead of the moments of inertia, A, B, C; but as by using the latter quantities a geometrical representation of the analysis, by means of Poinsot's "momental ellipsoid", may be introduced, (a different "ellipsoid of construction" being made use of in the ordinary method,) in the present article equations (9) will be employed.

If none of the quantities l, m, n vanish, (the cases in which the equations are satisfied when one or more of these quantities vanish will be examined below,) the three equations (9), on account of the peculiar form of their first members, which satisfy the relation $Ll + Mm + Nn = 0$, will be equivalent to two, which may be written thus:

$$\frac{Al - C'm - B'n}{l} = \frac{-C'l + Bm - A'n}{m} = \frac{-B'l - A'm + Cn}{n} \ldots (10).$$

Hence we infer that the diameters of the surface

$$Ax^2 + By^2 + Cz^2 - 2A'yz - 2B'zx - 2C'xy = D \ldots. (11),$$

which cut their diametral planes at right angles, are principal axes of the solid.*

7. The two equations (10) are sufficient for determining the ratios $l : m : n$; but it is convenient to assume a third unknown quantity P, to represent each member of the equations. We thus obtain the three equations

$$\left.\begin{array}{l} Pl = Al - C'm - B'n \\ Pm = -C'l + Bm - A'n \\ Pn = -B'l - A'm + Cn \end{array}\right\} \ldots\ldots\ldots (12).$$

Eliminating $l : m : n$ in the usual manner, we find

$$(A - P)(B - P)(C - P)$$
$$- A'^2(A - P) - B'^2(B - P) - C'^2(C - P) - 2A'B'C' = 0 \ldots (13),$$

* The same property may be proved of the surface

$$Fx^2 + Gy^2 + Hz^2 + 2A'yz + 2B'zx + 2C'xy = D,$$

by using equations (8) instead of (9).

a cubic equation to determine P, which may be shewn to have three real roots. Substituting for P in (12) any one of these values, we have three linear equations, from any two of which the same values of the ratios $l : m : n$ may be obtained. Hence there are three principal axes through O.

8. We may shew that these are at right angles to one another in the following manner.*

Let l_1, m_1, n_1, and l_2, m_2, n_2, be the direction-cosines of any two principal axes, and P_1, P_2 the corresponding roots of the cubic. Then, substituting in (12) the values l_1, m_1, n_1, P_1, for l, m, n, P, multiplying the first equation by l_2, the second by m_2, and the third by n_2, and adding, we find

$$P_1(l_1l_2 + m_1m_2 + n_1n_2) = Q,$$

where $Q = Al_1l_2 + Bm_1m_2 + Cn_1n_2$
$$- A'(m_1n_2 + n_1m_2) - B'(n_1l_2 + l_1n_2) - C'(l_1m_2 + m_1l_2).$$

Commencing with the system l_2, m_2, n_2, P_2, and following a similar process, we should have found

$$P_2(l_1l_2 + m_1m_2 + n_1n_2) = Q,$$

on account of the symmetry of Q. Hence, by subtraction,

$$(P_1 - P_2)(l_1l_2 + m_1m_2 + n_1n_2) = 0;$$

and therefore, unless P_1 be equal to P_2,

$$l_1l_2 + m_1m_2 + n_1n_2 = 0;$$

which shews that any two principal axes, corresponding to different roots of the discriminating cubic, are at right angles.

9. The moment of inertia of the body round any axis may, as is well known, be expressed in terms of the sums or integrals A, B, C, A', B', C', and the quantities which determine the position of the axis, and therefore may be found without farther summation or integration. Thus, if P be the moment of inertia round an axis (l, m, n) through O, we have

$$\begin{aligned} P &= \Sigma\delta\mu\,\{(mz - ny)^2 + (nx - lz)^2 + (lx - my)^2\} \\ &= l^2\,\Sigma\delta\mu\,(y^2 + z^2) + m^2\,\Sigma\delta\mu\,(z^2 + x^2) + n^2\,\Sigma\delta\mu\,(x^2 + y^2) \\ &\qquad - 2mn\,\Sigma\delta\mu\,yz - 2nl\,\Sigma\delta\mu\,zx - 2lm\,\Sigma\delta\mu\,xy \end{aligned}$$

or $$P = Al^2 + Bm^2 + Cn^2 - 2A'mn - 2B'nl - 2C'lm \ldots (14).$$

* This demonstration was first given by Cauchy.

Hence, the moment of inertia of the body about any axis OQ, cutting the surface (11) in Q, is $\dfrac{D}{OQ^2}$; whence the surface is called *the momental ellipsoid.*

Each member of equations (10) is found, by multiplying numerators and denominators by l, m, n, and adding respectively, to be equal to

$$Al^2 + Bm^2 + Cn^2 - 2A'mn - 2B'nl - 2C'lm.$$

Hence, if (l, m, n) be a principal axis, the moment of inertia, given by (14), is the quantity P, of which the values are determined by the discriminating cubic, and therefore the three principal moments of inertia, relative to the point O, are the roots of the cubic equation (13).

10. Let us now consider the case in which equations (9) can be satisfied when one of the quantities l, m, n vanishes. Thus, if $n = 0$, the equations become

$$m(lB' + mA') = 0,$$
$$l(lB' + mA') = 0,$$
$$m(lA - mC') - l(-lC' + mB) = 0 \;\ldots\ldots (a).$$

Since l and m cannot both vanish when $n = 0$, the first two equations give

$$lB' + mA' = 0 \;\ldots\ldots\ldots\ldots\ldots (b).$$

Eliminating $l : m$ from (a) by means of this, we obtain

$$B'(AA' + B'C') - A'(BB' + C'A') = 0 \;\ldots\ldots (c),$$

which is therefore the condition that there may be a principal axis in the plane (xy). Its position in the plane will be given by equation (b), unless both A' and B' vanish; in which case, the quadratic (a) giving two values of $\dfrac{l}{m}$, and (b) being identically true, there would be two principal axes in the plane (xy).

11. Let us next suppose both m and n to vanish, and investigate under what conditions equations (9) can be satisfied. In this case the first will be identically true, and the second and third will give

$$B' = 0, \quad C' = 0 \;\ldots\ldots\ldots\ldots (a);$$

which are therefore the conditions that OX may be a principal axis. The remaining principal axes, which will lie in the plane (yz), will be found by making B', C', and l

vanish in equations (9). The second and third will thus be satisfied identically, and the first will become

$$n(Bm - A'n) - m(-A'm + Cn) = 0;$$

which will determine two axes at right angles, being in fact the equation for determining the principal axes of the ellipse in which the surface (11) cuts the plane of yz.

12. From equations (a) (§ 11) it follows, that if OX, OY, OZ be principal axes, we must have $A' = 0$, $B' = 0$, $C' = 0$, or

$$\Sigma\,\delta\mu\,yz = 0, \quad \Sigma\,\delta\mu\,zx = 0, \quad \Sigma\,\delta\mu\,xy = 0.$$

These equations are usually given as the definition of a system of principal axes.

In the next Number another method of treating the equations of condition for principal axes, corresponding to the method followed in a paper "*On the Reduction of the General Equation of the Second Degree*," (1st Series, vol. IV. p. 227); and application will be made to the investigation of the relations which exist between the principal axes through different points of a solid.

Glasgow, Jan. 6, 1846.

ON CIRCULAR SECTIONS OF THE LOCUS OF THE GENERAL EQUATION OF THE SECOND ORDER.

By the Rev. PERCIVAL FROST, M.A.

LET the equation to the surface be

$$u = ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy$$
$$+ 2a''x + 2b''y + 2c''z + d = 0 \ldots .(a).$$

Let l, m, n be the direction-cosines of a plane cutting the surface in a circle, f, g, h the coordinates of its centre, and $l', m'\ n'$ the direction cosines of any diameter of this circle, of which the equations will consequently be

$$\frac{x-f}{l'} = \frac{y-g}{m'} = \frac{z-h}{n'} = r \ldots\ldots\ldots (b).$$

Therefore at the points of intersection with the surface we have, by combining these with equation (a),

$$Mr^2 + 2Nr + P = 0,$$

the roots of which must be equal and of opposite signs: hence $N = 0$; and, since P is independent of l', m', n', M

must have the same value for all values of l', m', n' consistent with the equations

$$\left.\begin{array}{l} ll' + mm' + nn' = 0 \\ l'^2 + m'^2 + n'^2 = 1 \end{array}\right\} \dots\dots\dots\dots (1).$$

and

If u' be the value of u when f, g, h are written for x, y, z, we have

$$N = l'\frac{du'}{df} + m'\frac{du'}{dg} + n'\frac{du'}{dh} =$$

and therefore, since l', m', n' are indeterminate,

$$\frac{du'}{ldf} = \frac{du'}{mdg} = \frac{du'}{ndh} \dots\dots\dots\dots (2).$$

Also $\quad M = al'^2 + bm'^2 + cn'^2 + 2a'm'n' + 2b'n'l' + 2c'l'm'$.

Let $\quad a - M = \alpha,\quad b - M = \beta,\quad c - M = \gamma$.

Then, since $M = M(l'^2 + m'^2 + n'^2)$, the preceding equation may be put under the form

$$\alpha l'^2 + \beta m'^2 + \gamma n'^2 + 2a'm'n' + 2b'n'l' + 2c'l'm' = 0 \,..(c).$$

Multiplying by n^2 and substituting from the first of equations (1), we have

$$(\alpha l'^2 + \beta m'^2)\, n^2 + \gamma\,(ll' + mm')^2 + 2c'l'm'n^2$$
$$- 2n\,(a'm' + b'l')\,(ll' + mm') = 0 \dots (d).$$

Since (c) must be satisfied for all values of the ratio $l' : m' : n'$ consistent with the first of equations (1), in the latter equation (d) we may equate the coefficients of l'^2, $l'm'$, and n'^2 to zero. Hence

$$\alpha n^2 - 2b'ln + \gamma l^2 = 0 \dots\dots\dots\dots (3),$$
$$\gamma m^2 - 2a'mn + \beta n^2 = 0 \dots\dots\dots\dots (4),$$

and

$$a'ln + b'mn - \gamma lm - c'n^2 = 0 \dots\dots\dots\dots (5).$$

Substituting in (5) for a' and b' their values by (3) and (4), we obtain

$$\beta l^2 - 2c'lm + \alpha m^2 = 0 \dots\dots\dots\dots (6).$$

To eliminate the ratios $l : m : n$ from the equations (3), (4), and (6), we have, by (3) and (4),

$$(\alpha a'^2 + \beta b'^2)\, n^2 + \gamma\,(a'^2l^2 + b'^2m^2) = 2a'b'n\,(a'l + b'm)$$
$$= 2a'b'\,(\gamma lm + c'n^2) \text{ by } (5).$$

Hence $\quad (\alpha a'^2 + \beta b'^2 - 2a'b'c')\, n^2 + \gamma\,(a'l - b'm)^2 = 0 \dots (e)$.

By (3) and (4),

$$(\alpha m^2 - \beta l^2)\, n^2 + 2lmn\,(a'l - b'm) = 0,$$

and by (6), $\quad (\alpha m^2 + \beta l^2)^2 = 4c'^2l^2m^2$,

which gives $\quad (\alpha m^2 - \beta l^2)^2 = 4\,(c'^2 - \alpha\beta)\, l^2m^2$.

Therefore $\quad (a'l - b'm)^2 = (c'^2 - \alpha\beta)\, n^2$.

Hence, by comparison with (e),

$$(\alpha a'^2 + \beta b'^2 + \gamma c'^2 - 2a'b'c' - \alpha\beta\gamma)\, n^2 = 0,$$

and therefore

$$\alpha a'^2 + \beta b'^2 + \gamma c'^2 - 2a'b'c' - \alpha\beta\gamma = 0 \;\ldots\ldots (7).$$

Substituting for α, β, γ their values, we have

$$(M-a)(M-b)(M-c) - a'^2(M-a) - b'^2(M-b) - c'^2(M-c) - 2a'b'c' = 0 \ldots (8),$$

a cubic equation to determine M, which has necessarily three real roots. Substituting any one of them for M in (3), (4), and (6), we obtain three equations, any two of which would give the same values of $l:m:n$, thus fixing the positions of the planes of circular section.* Equations (2) will give the loci of their centres. If two of the equations be used, we should obtain four systems of values of $l:m:n$; of these the two only which are compatible with the third equation must be taken.

Cor. 1. If $a' = b' = c' = 0$, equation (8) becomes

$$(M-a)(M-b)(M-c) = 0.$$

If we take $M = b$, (4) and (5) will become

$$\gamma m^2 = 0, \quad \text{and} \quad \alpha m^2 = 0:$$

therefore

$$m = 0,$$

* [The elimination of l, m, n between the equations (3), (4), (6) has been given by Mr. Sylvester (first series, vol. II. p. 283), as an example of the dialytic method according to which, in this case, he forms equation (5) and the two which correspond in symmetry, and eliminates $l^2, m^2, n^2, mn, nl, lm$ from the six equations, in which they enter linearly.

Equations (7) may also be found by expressing the condition that the cone

$$\alpha x^2 + \beta y^2 + \gamma z^2 + 2a'yz + 2b'zx + 2c'xz = 0 \;\ldots\ldots\ldots\ldots (f),$$

may have a plane for one sheet. Since the surface is of the second order, it follows that it must consist of two planes, and therefore one root of its discriminating cubic must vanish, which is expressed by (7). Either of these planes would be parallel to (real or imaginary) circular sections of the given surface, and consequently for each root of the cubic in M there are two, andonly two (both real or both imaginary), systems of circular sections. The intersection of the pair of planes corresponding to one value of M will be the principal axis of the surface (f) which corresponds to the root zero of the discriminating cubic. Hence, by the formulæ in a paper "On the Reduction of the General equation of the Second Degree," (first series vol. IV. p. 227), if l, m, n be the direction cosines of either plane, we have

$$\frac{l}{\alpha a' - b'c'} + \frac{m}{\beta b' - c'a'} + \frac{n}{\gamma c' - a'b'} = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots (g),$$

and it may be verified that this equation, and one of the equations (3), (4), (6), would, by elimination, lead to the two others, if the relation (7) be attended to. Equation (g) along with any one of the three equations in the text gives the two required systems of values of $l:m:n$, without ambiguity.]

and $$\frac{l^2}{a-b} = \frac{n^2}{b-c},$$

and the locus of the centres is given by equations

$$bg + b' = 0,$$

and $$\frac{af + a'}{(a-b)^{\frac{1}{2}}} = \frac{ch + c'}{(b-c)^{\frac{1}{2}}}.$$

The values of l and n are impossible unless b be intermediate between a and c, and therefore the planes which cut the surface in circles are perpendicular to the plane of greatest and least axes.

Cor. 2. If $a' = c' = 0$,
the cubic becomes

$$(M - b)\{(M - a)(M - c) - b'^2\} = 0.$$

I. Let $M = b$,

$$\gamma m^2 = 0, \quad m = 0,$$

$$(b - c)l^2 + 2b'ln + (b - a)n^2 = 0;$$

therefore $(b - c)(b - a) < b'^2$ if the values of $\dfrac{l}{n}$ be real, and different.

II. $$(M - a)(M - c) = b'^2$$

$$\frac{l^2}{a} = -\frac{m^2}{\beta} = \frac{n^2}{\gamma},$$

hence $\dfrac{\beta}{a}$ must be negative, and therefore the quadratic in M must have a root between b and a, which requires that

$$(b - a)(b - c) > b'^2.$$

When this is satisfied there will be one root between b and that one of the quantities a, c which differs least from b, and the others will lie between a and c. The former root will make both $\dfrac{\beta}{a}$ and $\dfrac{\beta}{\gamma}$ negative, and the latter will make one positive and the other negative. Hence the principal sections corresponding to the former are real, and to the latter, imaginary.

Cor. 3. In the case of a surface of revolution the roots of equations (3), (4), (6) are equal; therefore

$$a\gamma = b'^2,$$

$$\beta\gamma = a'^2$$

$$a\beta = c'^2.$$

Hence
$$\gamma = \frac{a'b'}{c'},$$
and
$$M = c - \frac{a'b'}{c'} = b - \frac{c'a'}{b'} = a - \frac{b'c'}{a'} \quad \ldots\ldots (a).$$

The axis of revolution will be the locus of the centres and its equations will be found as follows:

Since
$$\frac{b'c'}{a'} m^2 - 2c'lm + \frac{a'c'}{b'} l^2 = 0,$$
we have
$$a'l = b'm = c'n;$$
and therefore
$$a'(af + c'g + b'h + a'') = b'(c'f + bg + a'h + b'')$$
$$= c'(b'f + a'g + ch + c''),$$
which are the required equations. These may be modified as follows:
$$(aa' - b'c')\left(f + \frac{a'a''}{aa' - b'c'}\right) = (bb' - a'c')\left(g + \frac{b'b''}{bb' - a'c'}\right).$$
Hence, by (a),
$$a'\left(f + \frac{a'a''}{aa' - b'c'}\right) = b'\left(g + \frac{b'b''}{bb' - c'a'}\right) = c'\left(h + \frac{c'c''}{cc' - a'b'}\right),$$
which are the required equations in their simplest forms.

Cambridge, Jan. 28, 1846.

ON SYMBOLICAL GEOMETRY.

By Professor Sir William Rowan Hamilton, LL.D.

(*Continued from* p. 57.)

Determinateness of the first Four Operations on Geometrical Fractions (or Quotients).

8. Meanwhile the principles and definitions which have been already laid down, are sufficient to conduct to clear and determinate interpretations of all operations of combining geometrical quotients among themselves, by any number of additions, subtractions, multiplications, and divisions: each *quotient* of the kind here mentioned being regarded, by what has been already shown, as the *mark of a certain complex relation between two straight lines in space*, depending not only on their *relative lengths*, but also on their *relative*

directions. If we denote now by a symbol of fractional form, such as $\frac{b}{a}$, the quotient thus obtained by dividing one line b by another line a, when directions as well as lengths are attended to, the definitional equations (26), (27), (28), (29), will take these somewhat shorter forms:*

$$\frac{c}{a} + \frac{b}{a} = \frac{c+b}{a}; \qquad \frac{c}{a} - \frac{b}{a} = \frac{c-b}{a}; \ldots. \ (46),$$

$$\frac{c}{a} \times \frac{a}{b} = \frac{c}{b}; \qquad \frac{c}{a} \div \frac{b}{a} = \frac{c}{b}; \ \ldots\ldots \ (47),$$

which agree in all respects with the corresponding formulæ of ordinary algebra, and serve to fix, in the present system, the meanings of the operations +, −, ×, ÷, on what may be called *geometrical fractions*. These FRACTIONS being only other forms for what we have called *geometrical quotients* in earlier articles of this paper, we may now write the identity,

$$\frac{b}{a} = b \div a \ldots\ldots\ldots\ldots\ldots\ldots \ (48).$$

* On the principles alluded to in former notes, the formulæ for the addition, subtraction, multiplication, and division, of any two geometrical fractions, might be thus written:

$$\frac{D-C}{B-A} + \frac{C-A}{B-A} = \frac{D-A}{B-A},$$

$$\frac{D-A}{B-A} - \frac{C-A}{B-A} = \frac{D-C}{B-A},$$

$$\frac{D-A}{C-A} \times \frac{C-A}{B-A} = \frac{D-A}{B-A},$$

$$\frac{D-A}{B-A} \div \frac{C-A}{B-A} = \frac{D-A}{C-A};$$

A, B, C, D being symbols of any four points of space, and B − A being a symbol of the straight line drawn to B from A. If we denote this line by the biliteral symbol BA, we obtain the following somewhat shorter forms, which do not however all agree so closely with the forms of ordinary algebra:

$$\frac{DC}{BA} + \frac{CA}{BA} = \frac{DA}{BA},$$

$$\frac{DA}{BA} - \frac{CA}{BA} = \frac{DC}{BA},$$

$$\frac{DA}{CB} \times \frac{CA}{BA} = \frac{DA}{BA},$$

$$\frac{DA}{BA} \div \frac{CA}{BA} = \frac{DA}{CA}.$$

For the same reason, an *equation between any two such fractions*, for example the following,

$$\frac{\mathrm{f}}{\mathrm{e}} = \frac{\mathrm{b}}{\mathrm{a}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (49),$$

is to be understood as signifying, 1st, that the *length* of the one *numerator* line f is to the length of its own *denominator* line e *in the same ratio* as the length of the other numerator line b to the length of the other denominator line a; 2nd, that these four lines are *co-planar*, that is to say, in or parallel to one common plane; and 3rd, that the *same amount and direction of rotation*, round an axis perpendicular to this common plane, which would bring the line a into the direction originally occupied by b, would also bring the line e into the original direction of f. The same complex relation between the same four lines may also (by what has been already seen) be expressed by the *inverse* equation

$$\frac{\mathrm{e}}{\mathrm{f}} = \frac{\mathrm{a}}{\mathrm{b}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (50),$$

or by the *alternate* form $\qquad \dfrac{\mathrm{f}}{\mathrm{b}} = \dfrac{\mathrm{e}}{\mathrm{a}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (51).$

Two fractions which are, in this sense, *equal* to the same third fraction, are also equal to each other; and the *value* of such a fraction is not altered by altering the lengths of its numerator and denominator in any common ratio; nor by causing both to turn together through any common amount of rotation, in a common direction, round an axis perpendicular to both; nor by transporting either or both, without rotation, to any other positions in space. When the lengths and directions of any three co-planar lines, a, b, e, are given, it is always possible to determine the length and direction of a fourth line f, which shall be co-planar with them, and shall satisfy an equation between fractions, of the form (49). It is therefore possible to *reduce any two geometrical fractions to a common denominator;* or to satisfy not only the equation (49), but also this other equation,

$$\frac{\mathrm{h}}{\mathrm{g}} = \frac{\mathrm{c}}{\mathrm{a}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (52),$$

by a suitable choice of the three lines a, b, c, when the four lines e, f, g, h, are given; since, whatever may be the given directions of these four lines, it is always possible to find (or to conceive as found) a fifth line a, which shall be at once co-planar with the pair e, f, and also with the pair g

For a similar reason it is always possible to transform two given geometrical fractions into two others equivalent to them, in such a manner, that the new denominator of one shall be equal to the new numerator of the other; or to satisfy the two equations

$$\frac{h}{g} = \frac{c'}{a'}, \quad \frac{f}{e} = \frac{a'}{b'} \ldots\ldots\ldots\ldots (53),$$

by a suitable choice of the three lines a′, b′, c′, whatever the four given lines e, f, g, h, may be. Making then for abridgment

$$c + b = d, \quad c - b = d' \ldots\ldots\ldots\ldots (54),$$

and interpreting a sum or difference of lines as has been done in former articles, we see that it is always possible to choose eight lines a, b, c, d, a′, b′, c′, d′, so as to satisfy the conditions (49), (52), (53), (54); and thus, by (46) and (47), to interpret the sum, the difference, the product, and the quotient of *any two* given geometrical fractions, $\frac{f}{e}$ and $\frac{h}{g}$, as being each equal to *another given fraction* of the same sort, as follows:

$$\frac{h}{g} + \frac{f}{e} = \frac{d}{a}, \quad \frac{h}{g} - \frac{f}{e} = \frac{d'}{a} \ldots\ldots\ldots (55),$$

$$\frac{h}{g} \times \frac{f}{e} = \frac{c'}{b'}, \quad \frac{h}{g} \div \frac{f}{e} = \frac{c}{b} \ldots\ldots\ldots (56),$$

any variations in the new numerators and denominators, which are consistent with the foregoing conditions, being easily seen to make no changes in the values of the fractions which result. The *interpretations* of those four symbolic combinations, which are the first members of the four equations (55) and (56), are thus entirely *fixed:* and we are *no longer at liberty, in the present system,* to introduce arbitrarily any *new meanings* for those symbolic forms, or to subject them to any *new laws* of combination among themselves, without examining whether such meanings or such laws are consistent with the principles and definitions which it has been thought right to establish already, as appearing to be more simple and primitive, and more intimately connected with the application of symbolical language to geometry, or at least with the plan on which it is here attempted to make that application, than any of those other laws or meanings. If, for example, it shall be found that, in virtue of the foregoing principles, the *successive addition* of any number of geometrical fractions gives a result which is independent of

their order, this consequence will be, for us, a *theorem*, and not a definition. And if, on the contrary, the same principles shall lead us to regard the *multiplication* of geometrical fractions as being in general a *non-commutative* operation, or as giving a result which is *not* independent of the order of the factors, we shall be obliged to accept this conclusion also, that we may preserve consistency of system.

Separation of the Scalar and Vector parts of Sums and Differences of Geometrical Fractions.

9. To develope the geometrical meaning of the first equation (46), we may conceive each of the two numerator lines b, c, and also their sum d, to be orthogonally projected, first on the common denominator line a itself, and secondly on a plane perpendicular to that denominator. The former projections may be called b_1, c_1, d_1; the latter, b_2, c_2, d_2; and thus we shall have the nine relations,

$$\left.\begin{array}{lll} b_2 + b_1 = b, & b_1 \parallel a, & b_2 \perp a, \\ c_2 + c_1 = c, & c_1 \parallel a, & c_2 \perp a, \\ d_2 + d_1 = d, & d_1 \parallel a, & d_2 \perp a, \end{array}\right\} \ldots\ldots (57),$$

together with the three equations

$$c + b = d, \qquad c_1 + b_1 = d_1, \qquad c_2 + b_2 = d_2 \ldots (58);$$

of which the two last are deducible from the first, by the geometrical properties of projections. We have, therefore, by (46),

$$\frac{c}{a} + \frac{b}{a} = \frac{d}{a} = \frac{d_2}{a} + \frac{d_1}{a} \ldots\ldots\ldots\ldots (59),$$

$$\frac{d_1}{a} = \frac{c_1}{a} + \frac{b_1}{a}, \qquad \frac{d_2}{a} = \frac{c_2}{a} + \frac{b_2}{a} \ldots\ldots (60).$$

Since the three projections b_1, c_1, d_1, are parallel to a (in that sense of the word *parallel* which does not exclude coincidence), the three quotients in the first equation (60) are what we have already named *scalars;* that is, they are what are commonly called real numbers, positive, negative, or zero: they are also the scalar parts of the three quotients in the first equation (59), so that we may write

$$\frac{b_1}{a} = S\,\frac{b}{a}, \qquad \frac{c_1}{a} = S\,\frac{c}{a}, \qquad \frac{d_1}{a} = S\,\frac{d}{a} \ldots\ldots (61),$$

using the letter S here, as in a former article, for the characteristic of the operation of *taking the scalar part* of any geometrical quotient, or fraction. (If any confusion should

[redacted]

be apprehended, on other occasions, from this use of the letter S, and if the abridged word Scal. should be thought too long, the sign $\underset{e}{\mathrm{S}}$ might be employed.) Eliminating the four symbols b_1, c_1, d_1, d, between the first equation (59), the first equation (60), and the three equations (61), we obtain the result

$$\mathrm{S}\left(\frac{c}{a}+\frac{b}{a}\right)=\mathrm{S}\,\frac{c}{a}+\mathrm{S}\,\frac{b}{a}\ \ldots\ldots\ldots\ (62);$$

in which, by the foregoing article, $\frac{b}{a}$ and $\frac{c}{a}$ may represent any two geometrical fractions: so that we may write generally

$$\mathrm{S}\left(\frac{h}{g}+\frac{f}{e}\right)=\mathrm{S}\,\frac{h}{g}+\mathrm{S}\,\frac{f}{e}\ \ldots\ldots\ldots\ (63),$$

and may enunciate in words the same result by saying, that the *scalar of the sum* of any two such fractions is equal to the *sum of the scalars*. In like manner, the three other projections b_2, c_2, d_2, being each perpendicular to a, the three other partial quotients, which enter into the second equation (60), are what we have already called *vectors* in this paper, or more fully they are the vector parts of the three quotients in the first equation (59); so that we may write

$$\frac{b_2}{a}=\mathrm{V}\,\frac{b}{a},\quad \frac{c_2}{a}=\mathrm{V}\,\frac{c}{a},\quad \frac{d_2}{a}=\mathrm{V}\,\frac{d}{a}\ \ldots\ldots\ (64),$$

V being here used, as in a former article, for the characteristic of the operation of *taking the vector part;* we have, therefore,

$$\mathrm{V}\left(\frac{c}{a}+\frac{b}{a}\right)=\mathrm{V}\,\frac{c}{a}+\mathrm{V}\,\frac{b}{a}\ \ldots\ldots\ldots\ (65),$$

$$\mathrm{V}\left(\frac{h}{g}+\frac{f}{e}\right)=\mathrm{V}\,\frac{h}{g}+\mathrm{V}\,\frac{f}{e}\ \ldots\ldots\ldots\ (66),$$

and may assert that the *vector of the sum* of any two geometrical fractions is equal to the *sum of the vectors*. These formulæ (63) and (66) are important in the present system; they are however, as we see, only symbolical expressions of those very simple geometrical principles from which they have been derived, through the medium of the equations (58); namely, the principles that, *whether on a line or on a plane*, the *projection of a sum* of lines is equal to the *sum of the projections*, if the word *sum* be suitably interpreted. The analogous interpretation of a *difference* of lines, combined with similar considerations, gives in like manner the formulæ

$$\mathrm{S}\left(\frac{\mathrm{h}}{\mathrm{g}} - \frac{\mathrm{f}}{\mathrm{e}}\right) = \mathrm{S}\,\frac{\mathrm{h}}{\mathrm{g}} - \mathrm{S}\,\frac{\mathrm{f}}{\mathrm{e}} \ldots\ldots\ldots\ldots (67),$$

$$\mathrm{V}\left(\frac{\mathrm{h}}{\mathrm{g}} - \frac{\mathrm{f}}{\mathrm{e}}\right) = \mathrm{V}\,\frac{\mathrm{h}}{\mathrm{g}} - \mathrm{V}\,\frac{\mathrm{f}}{\mathrm{e}} \ldots\ldots\ldots\ldots (68);$$

that is to say, the *scalar and vector of the difference* of any two geometrical fractions are respectively equal to the *differences of the scalars and of the vectors* of those fractions; precisely as, and because, the *projection of a difference* of two lines, whether on a line or on a plane, is equal to the *difference of the projections*.

Addition and Subtraction of Vectors by their Indices.

10. We see, then, that in order to combine by addition or subtraction any two geometrical fractions, it is sufficient to combine separately their scalar and their vector parts. The former parts, namely the scalars, are simply *numbers*, of the kind called commonly real; and are to be added or subtracted among themselves according to the usual rules of algebra. But for effecting with convenience the combination of the latter parts among themselves, namely the vectors, which have been shown in a former article to be of a kind essentially distinct from all stages of the progression of real number from negative to positive infinity (and therefore to be rather *extra-positives* than either positive or *contra-positive* numbers), it is necessary to establish other rules: and it will be found useful for this purpose to employ the consideration of certain connected *lines*, namely the *indices*, of which each is determined by, and in its turn completely characterises, that vector quotient or fraction to which it corresponds, according to the construction assigned in the 7th article. If we apply the rules of that construction to determine the indices of the vector parts of any two fractions and of their sum, we may first, as in recent articles, reduce the two fractions to a common denominator; and may, for simplicity, take this denominator line a of a length equal to that assumed unit of length which is to be employed in the determination of the indices. Then, having projected, as in the last article, the new numerators b and c, and their sum d, on a plane perpendicular to a, and having called these projections b_2, c_2, d_2, as before; we may conceive a right-handed rotation of each of these three projected lines, through a right angle, round the line a as a common axis, which shall transport them without altering their lengths or relative di-

rections, and therefore without affecting their mutual relation as summands and sum, into coincidence with three other lines b_3, c_3, d_3, such that

$$d_3 = c_3 + b_3 \ldots\ldots\ldots\ldots\ldots\ldots (69);$$

and these three new lines will be the three indices required. For a right-handed rotation through a right angle, round the line b_3 as an axis, would bring the line a into the direction originally occupied by b_2; and the length of b_2 is to the length of a in the same ratio as the length of b_3 to the assumed unit of length; therefore b_3 is, in the sense of the 7th article, the index of the vector quotient $\frac{b_2}{a}$, that is, the index of the vector part of the fraction $\frac{b}{a}$, or $\frac{f}{e}$; and similarly for the indices of the two other fractions, in the first equation (59). We may therefore write, as consequences of the construction lately assigned, and of the equations (49) and (52),

$$b_3 = \mathrm{I}\,\frac{f}{e}; \quad c_3 = \mathrm{I}\,\frac{h}{g}; \quad d_3 = \mathrm{I}\left(\frac{h}{g} + \frac{f}{e}\right) \ldots\ldots (70);$$

if we agree for the present to prefix the letter I to the symbol of a geometrical fraction, as the characteristic of the operation of *taking the index of the vector part.* Eliminating now the three symbols b_3, c_3, d_3 between the four equations (69) and (70), we obtain this general formula:

$$\mathrm{I}\left(\frac{h}{g} + \frac{f}{e}\right) = \mathrm{I}\,\frac{h}{g} + \mathrm{I}\,\frac{f}{e} \ldots\ldots\ldots\ldots (71),$$

which may be thus enunciated: the *index of the vector part of the sum* of any two geometrical fractions is equal to the *sum of the indices* of the vector parts of the summands. Combining this result with the formula (63), which expresses that the scalar of the sum is the sum of the scalars, we see that the complex *operation of adding any two geometrical fractions,* of which each is determined by its scalar and by the index of its vector part, may be in general *decomposed into two* very simple but *essentially distinct operations;* namely, *first,* the operation of adding together *two numbers,* positive or negative or null, so as to obtain a third number for their sum, according to the usual rules of elementary algebra; and *second,* the operation of adding together *two lines* in space, so as to obtain a third line, according to the geometrical rules of the composition of motions, or by drawing the diagonal of a parallelogram. In like manner the operation of *taking the difference* of two

fractions may be decomposed into the two operations of taking separately the difference of two numbers, and the difference of two lines; for we can easily prove that

$$\mathrm{I}\left(\frac{\mathrm{h}}{\mathrm{g}}-\frac{\mathrm{f}}{\mathrm{e}}\right)=\mathrm{I}\,\frac{\mathrm{h}}{\mathrm{g}}-\mathrm{I}\,\frac{\mathrm{f}}{\mathrm{e}}\ \ldots\ldots\ldots\ldots(72);$$

or, in words, that the *index* (of the vector part) *of the difference* of any two fractions is equal to the *difference of the indices.* And because it has been seen that not only for numbers but also for lines, considered among themselves, any number of summands may be in any manner grouped or transposed without altering the sum; and that the sum of a scalar and a vector is equal to the sum of the same vector and the same scalar, combined in a contrary order; it follows that the *addition* of any number of geometrical fractions is an *associative* and also a *commutative* operation: in such a manner that we may now write

$$\frac{\mathrm{h}}{\mathrm{g}}+\frac{\mathrm{f}}{\mathrm{e}}=\frac{\mathrm{f}}{\mathrm{e}}+\frac{\mathrm{h}}{\mathrm{g}};\ \frac{\mathrm{k}}{\mathrm{i}}+\left(\frac{\mathrm{h}}{\mathrm{g}}+\frac{\mathrm{f}}{\mathrm{e}}\right)=\left(\frac{\mathrm{k}}{\mathrm{i}}+\frac{\mathrm{h}}{\mathrm{g}}\right)+\frac{\mathrm{f}}{\mathrm{e}}=\frac{\mathrm{f}}{\mathrm{e}}+\frac{\mathrm{h}}{\mathrm{g}}+\frac{\mathrm{k}}{\mathrm{i}},\ \&\mathrm{c}.\ \ldots\ldots\ldots(73),$$

whatever straight lines in space may be denoted by e, f, g, h, i, k, &c. We may also write, concisely,

$$\mathrm{S}\Sigma=\Sigma\mathrm{S};\quad \mathrm{V}\Sigma=\Sigma\mathrm{V};\quad \mathrm{I}\Sigma=\Sigma\mathrm{I}\ldots\ (74);$$

$$\mathrm{S}\Delta=\Delta\mathrm{S};\quad \mathrm{V}\Delta=\Delta\mathrm{V};\quad \mathrm{I}\Delta=\Delta\mathrm{I}\ldots.(75);$$

using Σ, Δ as the characteristics of sum and difference, while S, V, I are still the signs of scalar, vector, index.

Separation of the Scalar and Vector Parts of the Product of any two Geometrical Fractions.

11. The definitions (46), (47) of addition and multiplication of fractions, namely

$$\frac{\mathrm{c}}{\mathrm{a}}+\frac{\mathrm{b}}{\mathrm{a}}=\frac{\mathrm{c}+\mathrm{b}}{\mathrm{a}},\qquad \frac{\mathrm{c}}{\mathrm{a}}\times\frac{\mathrm{a}}{\mathrm{b}}=\frac{\mathrm{c}}{\mathrm{b}},$$

give obviously, for any 4 straight lines a, b, c, a′, the formula

$$\left(\frac{\mathrm{c}}{\mathrm{a}}+\frac{\mathrm{b}}{\mathrm{a}}\right)\times\frac{\mathrm{a}}{\mathrm{a}'}=\frac{\mathrm{c}+\mathrm{b}}{\mathrm{a}'}=\left(\frac{\mathrm{c}}{\mathrm{a}}\times\frac{\mathrm{a}}{\mathrm{a}'}\right)+\left(\frac{\mathrm{b}}{\mathrm{a}}\times\frac{\mathrm{a}}{\mathrm{a}'}\right)\ldots\ldots\ (76);$$

and this other formula of the same kind,

$$\frac{\mathrm{a}'}{\mathrm{a}}\times\left(\frac{\mathrm{c}}{\mathrm{a}}+\frac{\mathrm{b}}{\mathrm{a}}\right)=\cfrac{\mathrm{a}'}{\cfrac{\mathrm{a}}{\mathrm{c}+\mathrm{b}}\times\mathrm{a}}=\left(\frac{\mathrm{a}'}{\mathrm{a}}\times\frac{\mathrm{c}}{\mathrm{a}}\right)+\left(\frac{\mathrm{a}'}{\mathrm{a}}\times\frac{\mathrm{b}}{\mathrm{a}}\right)\ldots.\ (77),$$

[redacted]

may be proved without difficulty to be a consequence of the same definitions; the operation of multiplying a line, by the quotient of two others with which it is co-planar, being interpreted by the definition (23), so as to give, in the present notation,

$$\frac{e}{a} \times a = e \;\ldots\ldots\ldots\ldots\ldots\ldots \;(78).$$

In fact, if we assume, as we may, seven new lines, $d b' c' d' b'' c'' d''$, so as to satisfy the seven conditions

$$\left.\begin{array}{l} c + b = d,\ \dfrac{b}{a} = \dfrac{a}{b'},\ \dfrac{c}{a} = \dfrac{a}{c'},\ \dfrac{d}{a} = \dfrac{a}{d'}, \\ \dfrac{b''}{a'} = \dfrac{a'}{b'},\ \dfrac{c''}{a'} = \dfrac{a'}{c'},\ \dfrac{d''}{a'} = \dfrac{a'}{d'}, \end{array}\right\} \ldots\ldots(79),$$

we shall have the first member of the formula (77) equal to $\frac{a'}{a} \times \frac{a}{d'} = \frac{a'}{d'} =$ the second member of that formula; it will therefore be equal to $\frac{d''}{a'}$, and consequently will be shown to be $= \frac{c''}{a'} + \frac{b''}{a'} = \frac{a'}{c'} + \frac{a'}{b'} =$ the third member of that formula, if we can show that the conditions (79) give the relation

$$d'' = c'' + b'' \;\ldots\ldots\ldots\ldots\ldots\ldots \;(80).$$

Now those conditions show that the line a is common to the planes of b, b', and c, c', and that it bisects the angle between b and b', and also the angle between c and c'; therefore the mutual inclination of the lines b' and c' is equal to the mutual inclination of b and c; while the lengths of the two former lines are, by the same conditions, inversely proportional to those of the two latter. And on pursuing this geometrical reasoning, in combination with the definitional meanings of the symbolic equations (79), it appears easily that the mutual inclinations of the lines b'', c'', d'', are equal to those of b', c', d', and therefore to those of b, c, d; while the lengths of b'', c'', d'' are inversely proportional to those of b', c', d', and therefore directly proportional to the lengths of b, c, d: since then the line d is the symbolic sum of b and c, or the diagonal of a parallelogram described with those two lines as adjacent sides, it follows that the line d'' is similarly related to b'' and c'', or that the relation (80) holds good. The formula (77) is therefore shown to be true: and although we have not *yet* proved that the multiplication of two geometrical fractions is *always* a *distributive* operation, we see at least that either

factor may be distributed into two partial factors, and that the sum of the two partial products will give the total product, whenever either total factor and the two parts of the other factor are *co-linear*; that is, whenever the planes of these three fractions are *parallel to any common line*, such as the line a in the formulæ (76) (77): the *plane* of a geometrical fraction being one which contains or is parallel to the numerator and denominator thereof. A *scalar* fraction, being the quotient of two parallel lines, of which either may be transported without altering its direction to any other position in space while both may revolve together, may be regarded as having an entirely *indeterminate plane*, which may thus be rendered parallel to any arbitrary line; we shall therefore always satisfy the condition of *co-linearity*, by distributing either or both of two factors into their scalar and vector parts, and may consequently write,

$$\frac{\mathrm{h}}{\mathrm{g}} \times \frac{\mathrm{f}}{\mathrm{e}} = \left(\mathrm{V}\,\frac{\mathrm{h}}{\mathrm{g}} \times \frac{\mathrm{f}}{\mathrm{e}}\right) + \left(\mathrm{S}\,\frac{\mathrm{h}}{\mathrm{g}} \times \frac{\mathrm{f}}{\mathrm{e}}\right)$$

$$= \left(\frac{\mathrm{h}}{\mathrm{g}} \times \mathrm{V}\,\frac{\mathrm{f}}{\mathrm{e}}\right) + \left(\frac{\mathrm{h}}{\mathrm{g}} \times \mathrm{S}\,\frac{\mathrm{f}}{\mathrm{e}}\right)$$

$$= \left(\mathrm{V}\,\frac{\mathrm{h}}{\mathrm{g}} \times \mathrm{V}\,\frac{\mathrm{f}}{\mathrm{e}}\right) + \left(\mathrm{V}\,\frac{\mathrm{h}}{\mathrm{g}} \times \mathrm{S}\,\frac{\mathrm{f}}{\mathrm{e}}\right) + \left(\mathrm{S}\,\frac{\mathrm{h}}{\mathrm{g}} \times \mathrm{V}\,\frac{\mathrm{f}}{\mathrm{e}}\right) + \left(\mathrm{S}\,\frac{\mathrm{h}}{\mathrm{g}} \times \mathrm{S}\,\frac{\mathrm{f}}{\mathrm{e}}\right) \quad \ldots\ldots (81);$$

or more concisely,

$$(\beta + b)(\alpha + a) = \beta\alpha + \beta a + b\alpha + ba \ \ldots\ldots (82),$$

if we denote, as in a former article, vectors by greek and scalars by italic letters, and omit the mark of multiplication between any two successive letters of these two kinds, or between sums of such letters, when those sums are enclosed in parentheses. But the multiplication of scalars is effected, as we have seen, by the ordinary rules of algebra; and to multiply a vector by a scalar, or a scalar by a vector, is easily shown, by the definitions already laid down, to be equivalent to multiplying by the scalar, on the plan of the sixth article, either the index or the numerator of the vector, without altering the denominator of that vector: thus, in the second member of (82), the term ba is a known scalar, and the terms $b\alpha$, βa are known vectors, if the partial factors α, b, a, β be known; in order therefore to apply the equation (82), which in its form agrees with ordinary algebra, to any question of multiplication of any two geometrical fractions, it is sufficient to know how to interpret generally the remaining term

$\beta\alpha$, or the product of one vector by another. For this purpose we may always conceive the index $\mathrm{I}\beta$ of the vector β to be the sum of two other indices, which shall be respectively parallel and perpendicular to the index $\mathrm{I}\alpha$ of the other vector α, as follows:

$$\mathrm{I}\beta' \parallel \mathrm{I}\alpha, \quad \mathrm{I}\beta'' \perp \mathrm{I}\alpha, \quad \mathrm{I}\beta'' + \mathrm{I}\beta' = \mathrm{I}\beta \;\ldots\ldots (83);$$

and then the vector β itself will be, by the last article, the sum of the two new vectors β' and β'', and the planes of these two new vector fractions will be respectively parallel and perpendicular to the plane of the vector fraction α; consequently, the three fractions β', β'', α will be co-linear, and we shall have, by the principle (76),

$$\beta\alpha = (\beta'' + \beta')\,\alpha = \beta''\alpha + \beta'\alpha \;\ldots\ldots\ldots (84).$$

The problem of the multiplication of *any two* vectors is thus decomposed into the two simpler problems, of multiplying first *two parallel*, and secondly *two rectangular, vectors* together. If then we merely wish to separate the scalar and the vector parts, it is sufficient to observe that if, in the general formula (47), for the multiplication of any two fractions, we suppose the factors to be parallel vectors, then the line a is perpendicular to both b and c, and is also co-planar with them, so that they are necessarily parallel to each other, and the product $\frac{\mathrm{c}}{\mathrm{b}}$ is a scalar; but if, in the same general formula, we suppose the factors to be rectangular vectors, then the three lines a, b, c are themselves mutually rectangular, and the product of the fractions is a vector. Thus, in the formula (84), the partial product $\beta'\alpha$ is a scalar, but the other partial product $\beta''\alpha$ is a vector: and we may write

$$\mathrm{S}\,.\,\beta\alpha = \beta'\alpha\,; \;\; \mathrm{V}\,.\,\beta\alpha = \beta''\alpha \;\ldots\ldots\ldots\ldots (85).$$

We may therefore, more generally, under the conditions (83), decompose the formula of multiplication (82) into the two following equations:

$$\left.\begin{array}{l} \mathrm{S}\,.\,(\beta + b)\,(\alpha + a) = \beta'\alpha + ba\,; \\ \mathrm{V}\,.\,(\beta + b)\,(\alpha + a) = \beta''\alpha + \beta a + b\alpha \end{array}\right\} \ldots\ldots\ldots (86).$$

Or we may write, for abridgment,

$$c = \beta'\alpha + ba\,; \;\; \gamma = \beta''\alpha + \beta a + b\alpha \ldots\ldots (87);$$

and then we shall have this other equation of multiplication,

$$\gamma + c = (\beta + b)(\alpha + a) \ldots\ldots\ldots\ldots (88).$$

And thus the general *separation of the scalar and vector*

parts of the product of any two geometrical fractions may be effected. But it seems proper to examine more closely into the separate meanings of the two partial products of vectors, denoted here by the two terms $\beta' a$ and $\beta'' a$; which will be done in the two following articles.

Products of two Parallel Vectors; Geometrical Representations of the Square Roots of Negative Scalars.

12. It was shown, in the last article, that the product of any two parallel vectors, such as a and β', that is, the product of any two vectors of which the planes or the indices are parallel, is equal to a scalar. By pursuing the reasoning of that article, it is easy to show, farther, that this *scalar product of two parallel vectors* is equal to the *product of the numbers* which express the lengths of the two parallel indices; this numerical product being taken with a *negative* or with a *positive* sign, according as these indices are *similar* or *opposite* in direction. In fact, in the general formula $\frac{\mathrm{c}}{\mathrm{a}} \times \frac{\mathrm{a}}{\mathrm{b}} = \frac{\mathrm{c}}{\mathrm{b}}$, we have now $\mathrm{b} \perp \mathrm{a}$, $\mathrm{c} \parallel \mathrm{b}$; the length of c is to the length of b, in a ratio compounded of the ratio of the length of c to that of a, and of the ratio of the length of a to that of b; and the direction of c is opposite or similar to that of b, according as the two quadrantal rotations in one common plane, from b to a, and from a to c, are performed right-handedly round the same index, or round opposite indices.

We know then perfectly how to interpret the product of any two parallel vectors; and, as a case of such interpretation, if we agree to say that the product of any two equal fractions is the *square* of either, and to write

$$\frac{\mathrm{b}}{\mathrm{a}} \times \frac{\mathrm{b}}{\mathrm{a}} = \left(\frac{\mathrm{b}}{\mathrm{a}}\right)^2 \ldots\ldots\ldots\ldots\ldots(89),$$

whatever two lines may be denoted by a and b, we see that, in the present system, the *square of a vector is always a negative scalar*, namely the negative of the square of the number which denotes the length of the index of the vector; in such a manner that, for any vector a, we shall have the equation

$$a^2 = -\bar{a}^2 \ldots\ldots\ldots\ldots\ldots\ldots(90),$$

if we agree to denote by the symbol $\bar{a}$ that positive or absolute number which expresses the *length of the index* Ia. We have then, reciprocally,

$$\bar{a}^2 = -a^2 \ldots\ldots\ldots\ldots\ldots\ldots(91);$$

and may therefore write

$$\bar{a} = \surd(-a^2) \ldots\ldots\ldots\ldots\ldots (92),$$

$-a^2$ being here a positive number (because a^2 is negative), and $\surd(-a^2)$ being its positive or absolute *square root*, which is an entirely *determined* (and real) *number*, when the vector a, or even when the length of its index, is determined. But although we might be led to write, in like manner, from (90), the equation

$$a = \surd(-\bar{a}^2) \ldots\ldots\ldots\ldots\ldots (93),$$

yet the same principles prove that this expression, which may denote generally any *square root of a negative number*, by a suitable choice of the positive number $\bar{a}$, is equal to a *vector* a, of which the index Ia has indeed a *determined length*, but has an entirely *undetermined direction;* the symbol in the second member of the equation (93) may therefore receive (in the present system) infinitely many different geometrical representations, or constructions, though they have all one common character: and it will be a little more consistent with the analogies of ordinary algebra to write the equation under the form

$$a = (-\bar{a}^2)^{\frac{1}{2}} \ldots\ldots\ldots\ldots\ldots (94),$$

using a fractional exponent which suggests a certain degree of indeterminateness, rather than a radical sign which it is often convenient to restrict to one determined value. Thus, for example, the symbol $(-1)^{\frac{1}{2}}$, or the *square root of negative unity*, will, in the present system, denote, or be geometrically constructed by, *any vector of which the index is equal to the unit of length;* that is, any geometrical fraction of which the numerator and the denominator are lines equal to each other in length, but perpendicular to each other in direction. And we see that the geometrical principle, on which this conclusion ultimately depends, is simply this: that *two successive and similar quadrantal rotations, in any arbitrary plane, reverse the direction* of any straight line in that plane. Mr. Warren, confining himself to the consideration of lines in *one fixed plane*, has been led to attribute to his geometrical representations of the square roots of negative numbers, *one fixed direction*, or rather axis, perpendicular to that other axis on which he represents square roots of positive numbers. And other authors, both before and since the publication of Mr. Warren's work,* seem to have been in like manner

* *Treatise on the Geometrical Representation of the Square Roots of Negative Quantities*, by the Rev. John Warren, Cambridge, 1828. See also Dr. Peacock's Treatises on Algebra, and his Rep[illegible] Association, containing references to other works.

disposed to represent positive or negative numbers by lines in some one direction, or in the direction opposite, but symbols of the form $a\sqrt{(-1)}$ by lines perpendicular thereto. Such is at least the impression on the mind of the present writer, produced perhaps by an insufficient acquaintance with the works of those who have already written on this class of subjects. It will however be attempted to show, in a future article of this paper, that the geometrical fractions which have been called *vectors,* in the present and in former articles, may be symbolically equated to their own indices; and that thus *every straight line having direction in space* may properly be looked upon *in the present system* as *a geometrical representation of a square root of a negative number;* while positive and negative numbers are in the same system regarded indeed as belonging to one common *scale* of progression, from $-\infty$ to $+\infty$, but to a scale which is not to be considered as having any one direction rather than any other, in tridimensional space.

Products of two Rectangular Vectors; Non-commutativeness of the Factors, in the general Multiplication of two Geometrical Fractions.

13. The reasoning by which it was shown, in the 11th article, that the *product* $\beta''\alpha$ *of any two rectangular vectors,* α and β'', is *itself a vector,* may be continued so as to show that the number expressing the length of the index of this vector product is the product of the numbers which express the lengths of the indices of the factors; or that, in a notation similar to one employed in the last article,

$$\overline{\beta''\alpha} = \overline{\beta''}\,\overline{\alpha}, \quad \text{when } \mathrm{I}\beta'' \perp \mathrm{I}\alpha \ldots\ldots (95);$$

and therefore that, by the principle (92), for the same case of *rectangular vectors,* we have the formula

$$\sqrt{\{-(\beta''\alpha)^2\}} = \sqrt{(-\beta''^2)}\,\sqrt{(-\alpha^2)} \ldots\ldots\ldots (96).$$

Also in the general formula of multiplication $\frac{\mathrm{c}}{\mathrm{a}} \times \frac{\mathrm{a}}{\mathrm{b}} = \frac{\mathrm{c}}{\mathrm{b}}$, the three lines a, b, c compose here a rectangular system; and therefore the *index of the product* is parallel to the line a, and is consequently *perpendicular to the indices of the two factors;* $\mathrm{I}.\beta''\alpha$ is therefore perpendicular to both $\mathrm{I}\beta''$ and $\mathrm{I}\alpha$; a conclusion which may be extended by (83) and (85) to the multiplication of *any two vectors,* so that we may write generally,

$$\mathrm{I}.\beta\alpha \perp \mathrm{I}\beta; \quad \mathrm{I}.\beta\alpha \perp \mathrm{I}\alpha \ldots\ldots\ldots\ldots (97).$$

Again, we are allowed to suppose, in applying the same general formula of multiplication to the same case of rectangular vectors, that the index $\mathrm{I}a$ of the multiplicand $\frac{\mathrm{a}}{\mathrm{b}}$ is not only parallel to the line c, but similar (and not opposite) in direction to that line; in such a manner that the rotation round c from b to a is positive: and then the rotation round b from a to c is positive, and so is the rotation round a from c to b, and also that round $-$ a from b to c; therefore the index $\mathrm{I}\beta''$ of the multiplier is similar in direction to $+$ b, and the index $\mathrm{I}.\beta''a$ of the product is similar in direction to $-$ a; consequently *the rotation round the index of the product, from the index of the multiplier to that of the multiplicand, is positive.* And although this last result has only been proved here for the case of two rectangular vectors, yet it may easily be shown, by the principles of the 11th article, to extend to the multiplication of two general geometrical fractions. For, in the notation of that article, γ denoting the vector part of the product of any two such fractions, we have, by (87),

$$\mathrm{I}\gamma = \mathrm{I}.\beta''a + a\mathrm{I}\beta + b\mathrm{I}a \dots\dots\dots\dots (98);$$

$\mathrm{I}\gamma$ is therefore the symbolic sum of $\mathrm{I}.\beta''a$ and of two other lines which are respectively parallel to the indices of the vector parts of the two factors, and which consequently have their sum co-planar with those indices, and therefore also co-planar, by (83), with $\mathrm{I}\beta''$ and $\mathrm{I}a$; consequently $\mathrm{I}\gamma$ and $\mathrm{I}.\beta''a$ both lie at the same side of the plane of $\mathrm{I}a$ and $\mathrm{I}\beta''$; and therefore the rotation round $\mathrm{I}\gamma$, like that round $\mathrm{I}.\beta''a$, from $\mathrm{I}\beta''$ to $\mathrm{I}a$, and consequently from $\mathrm{I}\beta$ to $\mathrm{I}a$, is positive. Hence also the rotation round $\mathrm{I}\beta$ from $\mathrm{I}a$ to $\mathrm{I}\gamma$ is positive; that is to say, in the multiplication of two general geometrical fractions, *the rotation round the index of the vector part of the multiplier, from that of the multiplicand to that of the product, is positive;* from which may immediately be deduced a remarkable consequence, already alluded to by anticipation in the 8th article, namely—that the *multiplication of two general geometrical fractions is not a commutative operation,* or that the *order of the factors is not in general indifferent;* since the index of the vector part of the product lies at one or at the other side of the plane of the indices of the vector parts of the two factors, according as those factors are taken in one or in the other order. We have, for example, by the present article, a relation of *opposition* of signs between the products of two *rectangular* vectors, taken in two opposite

orders; which relation may be expressed by the following *equation of perpendicularity*,

$$\alpha\beta'' = -\beta''\alpha, \quad \text{when } \mathrm{I}\beta'' \perp \mathrm{I}\alpha \; \ldots\ldots (99).$$

But in the case where the indices of the vector parts α and β of two fractional factors are *parallel* (which includes the case where either of those indices vanishes, the corresponding factor becoming then a scalar), the part β'' of the vector β vanishes, and the latter vector reduces itself by (83) to its other part β'; so that in *this* case, by the results of the last article, the order of the factors is indifferent, and the operation of multiplication is commutative: and thus we may write, as the *equation of parallelism* between two vectors,

$$\alpha\beta' = \beta'\alpha, \quad \text{when } \mathrm{I}\beta' \parallel \mathrm{I}\alpha \ldots\ldots\ldots (100).$$

It is easy to infer hence, by (84) and (77), that in the more general case of the multiplication of any two vectors α and β, we may write, instead of (85), the following formulæ for the separation of the scalar and vector parts of the product:

$$\left.\begin{aligned} \mathrm{S}.\beta\alpha &= \tfrac{1}{2}(\beta\alpha + \alpha\beta) = \mathrm{S}.\alpha\beta \\ \mathrm{V}.\beta\alpha &= \tfrac{1}{2}(\beta\alpha - \alpha\beta) = -\mathrm{V}.\alpha\beta \end{aligned}\right\} \ldots\ldots (101),$$

with corresponding formulæ instead of (86), which give

$$(\beta + b)(\alpha + a) - (\alpha + a)(\beta + b) = \beta\alpha - \alpha\beta \,..(102),$$

the second member of this last equation being a vector different from 0, unless it happen that the planes (or the indices) of the vectors α and β are parallel to each other. Finally, we may here observe that in virtue of the principles and definitions already laid down, *the length of the index* ($\mathrm{I}.\beta\alpha$) *of the vector part of the product of any two vectors bears to the unit of length the same ratio which the area of the parallelogram under the indices* ($\mathrm{I}\beta$ and $\mathrm{I}\alpha$) *of the factors bears to the unit of area;* the *direction* of this index of the product being also (as we have seen) *perpendicular to the plane* of the indices of the factors, and therefore to the plane of the parallelogram under them; and being changed to its own *opposite* when the order of the factors is inverted, which *inversion* of their order may be considered as corresponding to a *reversal of the face* of the parallelogram: for all which reasons, there appears to be a propriety in considered this index of the vector part of a product of any two vectors as a symbolical representation of this parallelogram under the indices of the factors, and in writing the symbolical equation

$$\mathrm{I}.\beta\alpha = \square(\mathrm{I}\beta, \mathrm{I}\alpha) \ldots\ldots\ldots\ldots (103).$$

[redacted]

It will be remembered that the indices $I(\beta + \alpha)$, $I(\beta - \alpha)$, of the sum and difference of the same two vectors, are symbolically equal to two different diagonals of the same parallelogram, by former articles of this paper.

(To be continued.)

ERRATA IN THE PRECEDING PORTION OF THIS PAPER.

In second note, page 47, *for* ordinarily *read* ordinally.
In page 50, after equation (11), *for* theory *read* theorem.
In formula (27), page 53, *for* + *read* ÷.
In page 57, *for* co-planal *read* co-planar.

ON THE INTEGRATION OF CERTAIN DIFFERENTIAL EQUATIONS.

By the Rev. BRICE BRONWIN.

THE equations integrated in this paper are linear, and of the second order. The mode of integration is a little different from the methods hitherto employed. But the object chiefly aimed at is to give the most simple and elegant form to the more complex of the two particular integrals. For this will often admit of very different forms.

The formula $\epsilon^{a\theta} f\left(\frac{d}{dx} + a\right) y = f\left(\frac{d}{dx}\right) \epsilon^{a\theta} y$ is well known.

Change ϵ^{θ} into x, $\frac{d}{dx}$ into $x\frac{d}{dx}$, and put D for $\frac{d}{dx}$; it becomes

$$x^a f(xD + a)\, y = f(xD)\, x^a y \,\ldots\ldots\ldots\ldots (a).$$

By this formula

$$Dy = x^{-1}(xD)\, y = (xD + 1)\, x^{-1} y;$$

$$D^2 y = (xD + 1)\, x^{-1} Dy = (xD + 1)\, x^{-1}(xD + 1)\, x^{-1} y$$
$$= (xD + 1)(xD + 2)\, x^{-2} y,$$

$$D^3 y = (xD + 1)(xD + 2)(xD + 3)\, x^{-3} y, \text{ \&c.; and generally}$$

$$D^n y = (xD + 1)(xD + 2) \ldots\ldots (xD + n)\, x^{-n} y^{*} \ldots\ldots (b).$$

Put in this last $D^{-n} y$ for y, and divide by the operating factors; there results

$$x^{-n} D^{-n} y = (xD + n)^{-1}(xD + n - 1)^{-1} \ldots\ldots (xD + 1)^{-1} y.$$

Multiply this by x^n; and it becomes in virtue of (a)

$$D^{-n} y = (xD)^{-1}(xD - 1)^{-1} \ldots . (xD - n + 1)^{-1} x^n y \ldots\ldots (c).$$

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* [This agrees with a formula proved in Vol. I. of the 1st Series, p. 282; also in Gregory's *Examples*, p. 31.]

Multiply (b) and (c) by x^p, and change y into $x^q y$; they become

$$\left.\begin{aligned} x^p D^n x^q y &= (xD-p+1)(xD-p+2)\dots(xD-p+n)x^{p+q-n}y \\ x^p D^{-n} x^q y &= (xD-p)^{-1}(xD-p-1)^{-1}\dots(xD-p-n+1)x^{p+q+n}y \end{aligned}\right\}\dots(d).$$

These may be convenient. We will now proceed to integrate a few equations.

Let $\dfrac{d^2y}{dx^2} + qx\dfrac{dy}{dx} + qmy = 0$, ($m$ a positive integer). . . . (1).

Multiplying by x^2, this may be written

$$x^2D^2y + qx^2(xD+m)y = 0;$$

by (a) and (b), or by (d), at once this becomes

$$xD(xD-1)y + (xD+m-2)qx^2y = 0.$$

Make $y = (xD+1)(xD+2)\dots(xD+m-1)x^{-m+1}u = D^{m-1}u$.

Then $x^2y = (xD-1)(xD)\dots\dots(xD+m-3)x^{-m+3}u$.

These values of y and x^2y being put in the preceding equation, after dividing by the factors common to both the terms, it becomes

$$(xD+m-1)x^{-m+1}u + qx^{-m+3}u = 0\dots\dots\dots(2).$$

I shall recur to this equation presently. It is equivalent to $\dfrac{du}{dx} + qxu = 0$, or $u = C\varepsilon^{-\frac{1}{2}qx^2}$, which gives a particular integral of (1). To find the other particular integral, make $y = \varepsilon^{-\frac{1}{2}qx^2}z$; and (1) will be transformed into

$$\frac{d^2z}{dx^2} - qx\frac{dz}{dx} + qmz = 0\dots\dots\dots\dots(3).$$

Treating this in the same manner, we have

$$xD(xD-1)z - (xD-m-2)qx^2z = 0.$$

Make $z = (xD)^{-1}(xD-1)^{-1}\dots(xD-m)^{-1}x^{m+1}u = D^{-m-1}u$:
then also $x^2z = (xD-2)^{-1}(xD-3)^{-1}\dots(xD-m-2)^{-1}x^{m+3}u$.

With these values, the common factors being expunged, the preceding becomes

$$(xD-m-1)x^{m+1}u - qx^{m+3}u = 0,$$

or $\dfrac{du}{dx} - qxu = 0$, $u = C_1\varepsilon^{\frac{1}{2}qx^2}$; and therefore $z = C_1D^{-m-1}\varepsilon^{\frac{1}{2}qx^2}$.

Hence the complete integral of (1) is

$$y = C\left(\frac{d}{dx}\right)^{m-1}\varepsilon^{-\frac{1}{2}qx^2} + C_1\varepsilon^{-\frac{1}{2}qx^2}\left(\frac{d}{dx}\right)^{-m-1}\varepsilon^{\frac{1}{2}qx^2};$$

that of (3) is given by $z = \varepsilon^{\frac{1}{2}qx^2}y$.

We now return to (2). By making the factors divided by to operate upon the second member; that would have been

$$(xD - n - 1)x^{n-1}u - qx^{n-2}u = (xD - n - 2)^{-1}(xD - n - 3)^{-1}\ldots$$
$$(xD - 1)^{-1}0 = x^{n-2}D^{n}x^{-2}0 = x^{n-2}D^{n-1}C.$$

This reduction is made by the second of (d) and one integration. Now this would give at once the complete integral of (1). But it would not be so simple and elegant as the above by a great deal, and simplicity of form is often a matter of great importance

Let $\quad x^2\dfrac{d^2y}{dx^2} - m\dfrac{dy}{dx} - ry = 0,\ r = p(p+1)\ \ldots\ldots\ (4).$

Multiply by x, and as before by the formulæ (a), (b), or (d); we find

$$x\,\{xD(xD-1) - p(p+1)\}\,y - xDmy = 0,$$

or $$x\,(xD+p)(xD-p-1)\,y - xDmy = 0,$$

or $$(xD+p-1)(xD-p-2)\,xy - (xD)\,my = 0\ \ldots\ (5),$$

where we may observe that the case of $p = 1$ is integrable immediately.

Make $\quad y = (xD+1)(xD+2)\ldots(xD+p-1)\,x^{-p+1}u = D^{p-1}u,$

and $\quad xy = (xD)(xD+1)\ldots(xD+p-2)\,x^{-p+2}u.$

These values put in (5), it will become, dividing by the common factors,

$(xD-p-2)\,x^{-p+2}u - mx^{-p+1}u = 0$, or $x^2\dfrac{du}{dx} - (m + 2px)\,u = 0,$

and $$u = Cx^{2p}\varepsilon^{-\frac{m}{x}}.$$

Again make

$$y = (xD)^{-1}(xD-1)^{-1}\ldots(xD-p-1)^{-1}x^{p+2}u = D^{-p-2}u,$$
$$xy = (xD-1)^{-1}\ldots(xD-p-2)^{-1}x^{p+3}u.$$

With these values (5) becomes

$$(xD+p-1)\,x^{p+3}u - mx^{p+2}u = 0,$$

or $x^2\dfrac{du}{dx} - \{m - (2p+2)\,x\}\,u = 0$; which gives $u = C_1x^{-2p-2}\varepsilon^{-\frac{m}{x}}.$

The complete integral of (5) therefore is

$$y = C\left(\frac{d}{dx}\right)^{p-1}x^{2p}\varepsilon^{-\frac{m}{x}} + C_1\left(\frac{d}{dx}\right)^{-p-2}x^{-2p-2}\varepsilon^{-\frac{m}{x}}.$$

Here again the integral is much more simple than it would be if found at once by operating upon 0 with the expunged factors.

Let $x^2\dfrac{d^2y}{dx^2} + (m-x)x\dfrac{dy}{dx} + n(m-n-1)y = 0$, n integer...(6).

From this we derive

$$\{xD(xD-1) + mxD + n(m-n-1)\}y - x(xD)y = 0:$$

or by further reduction

$$(xD+n)(xD+m-n-1)y - (xD-1)xy = 0.$$

Make $y = (xD+n)^{-1}(xD+n-1)^{-1}\dots(xD)^{-1}u = x^{-n}D^{-n-1}x^{-1}u$,

$$xy = (xD+n-1)^{-1}(xD+n-2)^{-1}\dots\dots(xD-1)^{-1}xu.$$

By the substitution of these values, and dividing by the factors common to both the terms, the last equation becomes

$$(xD+m-n-1)u - xu = 0, \quad \text{or } x\frac{du}{dx} + (m-n-1-x)u = 0,$$

and $u = Cx^{n-m+1}\epsilon^x$. There are other forms by which this particular integral might be found, but this is the simplest.

Now make $y = x^{-m}\epsilon^x z$, and with this value of y (6) will be transformed into

$$x^2\frac{d^2z}{dx^2} + (x-m)x\frac{dz}{dx} + (m-n)(n+1)z = 0 \dots\dots (7).$$

This gives

$$\{xD(xD-m-1) + (m-n)(n+1)\}z + x(xD)z = 0,$$

or $\qquad (xD+n-m)(xD-n-1)z + (xD-1)xz = 0.$

In the last make

$$z = (xD-n)(xD-n+1)\dots\dots(xD-1)u = x^{n+1}D^n x^{n-1}u,$$

$$xz = (xD-n-1)(xD-n-2)\dots\dots(xD-2)xu.$$

These values, substituted in the preceding, change it into

$$(xD+n-m)u + xu = 0, \quad \text{or } x\frac{du}{dx} + (n-m+x)u = 0,$$

$$\text{and } u = C_1x^{m-n}\epsilon^{-x}, \; z = C_1x^{n+1}\left(\frac{d}{dx}\right)^n x^{m-n-1}\epsilon^{-x};$$

therefore $y = Cx^{-n}\left(\dfrac{d}{dx}\right)^{-n-1} x^{n-m}\epsilon^x + C_1x^{n-m+1}\epsilon^x\left(\dfrac{d}{dx}\right)^n x^{m-n-1}\epsilon^{-x}.$

This is the complete integral of (6), and $z = x^m\epsilon^{-x}y$ will give that of (7). We might of course have found the complete integral at once here, as before indicated; but our object is to exhibit it in the simplest form.

Let $\qquad (1-x^2)\dfrac{d^2y}{dx^2} + p(p+1)y = 0 \dots\dots\dots (8).$

[redacted]

Multiply by x^2, and as before we find

$$xD(xD-1)y - x^2\{xD(xD-1) - p(p+1)\}y = 0,$$

or $$xD(xD-1)y - x^2(xD+p)(xD-p-1)y = 0,$$

and $$xD(xD-1)y - (xD+p-2)(xD-p-3)x^2y = 0 \dots(9).$$

Make $y = (xD+1)(xD+2)\dots(xD+p-1)x^{-p+1}u = D^{p-1}u,$

$$x^2y = (xD-1)(xD)\dots(xD+p-3)x^{-p+3}u.$$

Substitute these values in the last, and it becomes

$$(xD+p-1)x^{-p+1}u - (xD-p-3)x^{-p+3}u = 0,$$

or $$(1-x^2)\frac{du}{dx} + 2pxu = 0, \text{ and } u = C(1-x^2)^p.$$

Again make

$$y = (xD)^{-1}(xD-1)^{-1}\dots(xD-p-1)^{-1}x^{p+2}u = D^{-p-2}u,$$

$$x^2y = (xD-2)^{-1}(xD-3)^{-1}\dots(xD-p-3)^{-1}x^{p+4}u.$$

Put these values in (9), and it will give

$$(xD-p-2)x^{p+2}u - (xD+p-2)x^{p+4}u = 0,$$

or $$(1-x^2)\frac{du}{dx} - (2p+2)xu = 0, \text{ and } u = C_1(1-x^2)^{-p-1}.$$

Consequently $y = C\left(\frac{d}{dx}\right)^{p-1}(1-x^2)^p + C_1\left(\frac{d}{dx}\right)^{-p-2}(1-x^2)^{-p-1}$, the complete integral of (8). And this also is much more simple than it would be if found at once. We shall give one example more.

Let $x\frac{d^2y}{dx^2} + qx\frac{dy}{dx} + qmy = 0$, m a positive integer...(10).

This gives, after multiplying by x,

$$xD(xD-1)y + qx(xD+m)y = 0,$$

or $$xD(xD-1)y + (xD+m-1)qxy = 0.$$

Make $y = (xD+1)(xD+2)\dots(xD+m-1)x^{-m+1}u = D^{m-1}u,$

$$xy = (xD)(xD+1)\dots(xD+m-2)x^{-m+2}u.$$

Substitute these expressions of y and xy in the last, and it gives

$$(xD-1)x^{-m+1}u + qx^{-m+2}u = 0, \quad \text{or } x\frac{du}{dx} + (qx-m)u = 0,$$

and $$u = Cx^m\epsilon^{-qx}.$$

To find the other particular integral make $y = \epsilon^{-qx}z$, and (10) will be transformed into

$$x\frac{d^2z}{dx^2} - qx\frac{dz}{dx} + qmz = 0 \dots\dots\dots (11).$$

From this we deduce, first

$$xD(xD-1)z - x(xD-m)qz = 0,$$

and then $\quad xD(xD-1)z - (xD-m-1)qxz = 0.$

Make $z = (xD)^{-1}(xD-1)^{-1}\dots(xD-m)^{-1}x^{m+1}u = D^{-m-1}u,$

$$xz = (xD-1)^{-1}(xD-2)^{-1}\dots(xD-m-1)^{-1}x^{m+2}u.$$

Put these values in the above, and we find

$$(xD-1)x^{m+1}u - qx^{m+2}u = 0, \text{ or } x\frac{du}{dx} + (m-qx)u = 0,$$

and $\quad u = C_1 x^{-m}\epsilon^{qx}:$

whence $\quad y = C\left(\frac{d}{dx}\right)^{m-1} x^{m}\epsilon^{-qx} + C_1\epsilon^{-qx}\left(\frac{d}{dx}\right)^{-m-1} x^{-m}\epsilon^{qx},$

the complete integral of (10). And $z = \epsilon^{qx}y$ will give that of (11).

The method of integration employed in this paper is well adapted to the integration of a certain class of partial differential equations. I shall give an example in concluding this paper. And here let D_x, D_y stand for $\frac{d}{dx}$ and $\frac{d}{dy}$ respectively.

Let $$\frac{d^2z}{dx^2} - \frac{d^2z}{dy^2} - \frac{2}{x}\frac{dz}{dx} = 0 \quad \dots\dots\dots\dots (12).$$

By (*a*) and (*b*) this may be put under the form

$$(xD_x+1)(xD_x+2)x^{-2}z - (yD_y+1)(yD_y+2)y^{-2}z$$
$$- 2(xD_x+2)x^{-2}z = 0,$$

or $\quad (xD_x-1)(xD_x+2)x^{-2}z - (yD_y+1)(yD_y+2)y^{-2}z = 0.$

Make $\quad z = (xD_x-1)u;$

then $\quad x^{-2}z = (xD_x+1)x^{-2}u, \; y^{-2}z = (xD_x-1)y^{-2}u.$

Putting these values in the preceding, and dividing by $xD_x - 1$, it becomes

$$(xD_x+1)(xD_x+2)x^{-2}u - (yD_y+1)(yD_y+2)y^{-2}u$$
$$= (xD_x-1)^{-1}0 = xD^{-1}x^{-2}0 = Cx = xf(y),$$

by (*d*) and by integration. But this is equivalent to

$$\frac{d^2u}{dx^2} - \frac{d^2u}{dy^2} = xf(y).$$

Make $u = v - xD_y^{-2}f(y)$, and we have $\frac{d^2v}{dx^2} - \frac{d^2v}{dy^2} = 0.$

Consequently $\quad v = \phi(y+x) + \psi(y-x), \; z = (xD_x-1)u$

$$= (xD_x-1)v - (xD_x-1)xD_y^{-2}f(y) = (xD_x-1)v = x\frac{dv}{dx} - v;$$

or $z = x\{\phi'(y+x) - \psi'(y-x)\} - \{\phi(y+x) + \psi(y-x)\}.$

If we had not operated upon the cypher, we should have obtained the same result; but we should not have had the same assurance of the generality of the solution.

I presume this method may be applied to similar equations in finite differences.

Gunthwaite Hall, Penistone, March, 1846.

ON THE THEORY OF MAGIC SQUARES, CUBES, &c.

By R. Moon, M.A., Fellow of Queens' College.

In a former paper, published in this *Journal*, I endeavoured to develope a new method of treating the subject of Magic Squares, and exhibited more or less fully the mode of its application to the case of squares containing an odd number of places. On the present occasion I purpose to shew that the same method may be applied to the composition of Magic Cubes, and I shall in conclusion say a few words on the extension of the theory to squares of even numbers.

For the sake of simplicity I shall confine myself to the cube made up of the natural numbers from 0 to 26, both inclusive; which may be derived from the formula

$$x + 3y + 3^2z,$$

by giving successively to xyz the values 0.1.2 respectively.

The numbers represented by the nine following columns properly arranged, *i.e.* the second line being placed behind the first and the third behind the second, will form a magic cube; except as regards the diagonals, to which I shall afterwards direct attention.

A	*B*	*C*
$x_0 + 3y_0 + 3^2z_0$	$x_1 + 3y_2 + 3^2z_1$	$x_2 + 3y_1 + 3^2z_2$
$x_1 + 3y_1 + 3^2z_1$	$x_2 + 3y_0 + 3^2z_2$	$x_0 + 3y_2 + 3^2z_0$
$x_2 + 3y_2 + 3^2z_2$	$x_0 + 3y_1 + 3^2z_0$	$x_1 + 3y_0 + 3^2z_1$

D	*E*	*F*
$x_1 + 3y_1 + 3^2z_2$	$x_2 + 3y_0 + 3^2z_0$	$x_0 + 3y_2 + 3^2z_1$
$x_2 + 3y_2 + 3^2z_0$	$x_0 + 3y_1 + 3^2z_1$	$x_1 + 3y_0 + 3^2z_2$
$x_0 + 3y_0 + 3^2z_1$	$x_1 + 3y_2 + 3^2z_2$	$x_2 + 3y_1 + 3^2z_0$

G	*H*	*K*
$x_2 + 3y_2 + 3^2z_1$	$x_0 + 3y_1 + 3^2z_2$	$x_1 + 3y_0 + 3^2z_0$
$x_0 + 3y_0 + 3^2z_2$	$x_1 + 3y_2 + 3^2z_0$	$x_2 + 3y_1 + 3^2z_1$
$x_1 + 3y_1 + 3^2z_0$	$x_2 + 3y_0 + 3^2z_1$	$x_0 + 3y_2 + 3^2z_2$

In the first place the same number never recurs, so that the above comprise *all* the numbers from 0 to 26; secondly, the sum of each vertical column

$$= (x_0 + x_1 + x_2) + 3\,(y_0 + y_1 + y_2) + 3^2\,(z_0 + z_1 + z_2)$$
$$= 3\,(1 + 3 + 3^2)$$
$$= 39 \text{ (three times the mean number).}$$

Thirdly, if we take any number in A, and the corresponding numbers in B and C, we obtain the same result for the sum of the three; and similarly of the columns DEF, GHK, respectively.

Lastly, if we take any number in A and add to it the corresponding numbers in D and G respectively, we again have the same result, and similarly of the columns BEH, CFK respectively.

The diagonals are four in number. One of them may be found by taking the first of A (A_1), the second of E (E_2), and the third of K (K_3), which gives the sum

$$(a) \qquad x_0 + x_0 + x_0 + 3\,(y_0 + y_1 + y_2) + 3^2\,(z_0 + z_1 + z_2),$$

which, but for our having x_0 three times repeated instead of $x_0 + x_1 + x_2$, would likewise $= 39$. This defect may be remedied however by interchanging *throughout*, as we are at full liberty to do, x_0 and x_1; in which case we should have in (a) $(x_1 + x_1 + x_1)$ instead of $(x_0 + x_0 + x_0)$: and since

$$x_1 + x_1 + x_1 = x_0 + x_1 + x_2,$$

it is obvious that this diagonal will have the same value as any column of the square.

Another diagonal is $A_3 + E_2 + K_1$

$$= (x_2 + x_0 + x_1) + 3\,(y_2 + y_1 + y_0) + 3^2\,(z_2 + z_1 + z_0)$$
$$= 39.$$

A third is $C_1 + E_2 + G_3$

$$= (x_2 + x_0 + x_1) + 3\,(y_1 + y_1 + y_1) + 3^2\,(z_2 + z_1 + z_0)$$
$$= 39.$$

The last is $C_3 + E_2 + G_1$

$$= (x_1 + x_0 + x_2) + 3\,(y_0 + y_1 + y_2) + 3^2\,(z_1 + z_0 + z_2)$$
$$= 39.$$

Hence, as it will be shewn that the whole system may be formed by an invariable method from the first number of the first column, it follows that by properly choosing that number we shall always obtain a perfect cube.

I next come to the principle of formation. It will be seen that the numbers in any column are formed successively

from each other according to the order of their indices: thus in the column A the indices of x are 0.1.2, in D they are 1.2.0, but though the initial figure is different, the figures occur *in the same order;* and the same holds of the other columns.

Again, B is formed from A, as regards the x's and z's, by rejecting the head of each column and throwing it to the base: the column of y's is formed by elevating the base to the top: C is formed from B in the same manner as B from A, and E, F: H, K are formed from D and G respectively in the same manner as B and C are formed from A.

D is formed from A by treating the x's and y's in exactly the same way as the x's and z's are treated in the formation of B from A, and treating the z's in the former case as y in the latter; which we may, if we choose, express by saying that y and z are to be interchanged: but in practice it is better not to adopt this latter view.

G is formed from D in the same way as D from A; and E H, F K respectively might be formed from B and C respectively in a similar manner.

With regard to the number of different cubes to be obtained by the above method, I would observe that in the above example we may interchange at pleasure z_0 and z, z_1 and z_2, z_2 and z_0, so that if n be the number of cubes we should obtain independently of this consideration, we may by means of it increase the number to $3n$. Also we may interchange y_0 and y_2 (but not $y_1 . y_2 : y_1 . y_0$, as is easily seen from the consideration of the diagonals), and we may interchange x_0 and x_2, so that on the whole we shall obtain 2.2.3 cubes. But it is evident that in the above method z is, so to speak, the centre of the system, and as x and y have each equal claims in that respect the entire number of different cubes is $2^2.3^2$ or $1^2.2^2.3^2$.

It will be perceived that any one vertical row may be changed for another: thus we may interchange ADG with CFH, and so on; and in like manner any one horizontal row may be changed for another. The only effect of these changes will be on the diagonals of the cube, which however may always be adjusted by properly assuming the initial column.

The next remark I shall make bears on the subject of magic squares, as well as on that of magic cubes. The above method applies independently of the absolute values of $x_0y_0z_0$, $x_1y_1z_1$, $x_2y_2z_2$, provided only that, as regards one class of those quantities, the x's for example, one of the three values $x_0x_1x_2$ is the mean of the other two. Hence any series

of numbers which can be formed by giving to each of the quantities xyz respectively in the formula

$$x + 3y + 3^2z,$$

any three values whatever, consistently with the restriction that one of the three values of one of those variables is the mean of the other two, may be formed with a magic cube. I believe that in a cube containing $5^3 . 7^3$, &c. places, no restriction whatever is necessary, as to the values of any of the variables, beyond this, that they must not exceed 5.7, &c. in number respectively. A similar remark applies to magic squares, and thus forms a generalization of that part of the theory which I believe has not hitherto been adverted to.

If, instead of $x + 3y + 3^2z$, the series of numbers be represented by the formula

$$x + my + nz,$$

the same reasoning applies.

It is obvious that the theory may be extended, analytically, beyond the case of cube numbers. It might also not be impossible to contrive rectangles having the sums of their sides in a given ratio.

Instead of entering into any detailed account of the application of the above method to the case of squares of an even number of places, I shall subjoin two such squares, taken from an old French work, expressed according to the above system, which I analysed with a view to following up the theory; an intention however which, partly from lack of opportunity and partly inclination, I have not been able to carry out. To any person desirous of entering more fully into the subject, they may not be unserviceable.

The following is a magic square, composed of the natural numbers from 0 to 35 both inclusive. For the sake of convenience I put down only the indices of x and y,

(1)	4 + 4	(2)	5 + 1	(3)	3 + 4
	1 + 5		0 + 5		2 + 0
	4 + 3		5 + 2		2 + 2
	1 + 2		5 + 3		2 + 3
	1 + 0		0 + 0		3 + 5
	4 + 1		0 + 4		3 + 1
(4)	2 + 1	(5)	0 + 1	(6)	1 + 4
	3 + 0		5 + 5		4 + 0
	3 + 2		0 + 3		1 + 3
	3 + 3		0 + 2		4 + 2
	2 + 5		5 + 0		4 + 5
	2 + 4		5 + 4		1 + 1.

The next is composed of the numbers from 0 to 63 inclusive.

(1)	(2)	(3)	(4)
0 + 6	5 + 1	4 + 6	1 + 1
7 + 2	2 + 5	3 + 2	6 + 5
0 + 4	5 + 3	4 + 4	1 + 3
7 + 0	2 + 7	3 + 0	6 + 7
7 + 7	2 + 0	3 + 7	6 + 0
1 + 3	5 + 4	4 + 3	1 + 4
7 + 5	2 + 2	3 + 5	6 + 2
1 + 1	5 + 6	4 + 1	1 + 6

(5)	(6)	(7)	(8)
6 + 1	3 + 6	2 + 1	7 + 6
1 + 5	4 + 2	5 + 5	0 + 2
6 + 3	3 + 4	2 + 3	7 + 4
1 + 7	4 + 0	5 + 7	0 + 0
1 + 0	4 + 7	5 + 0	0 + 7
6 + 4	3 + 3	2 + 4	7 + 3
1 + 2	4 + 5	5 + 2	0 + 5
6 + 6	3 + 1	2 + 6	7 + 1.

I may observe that in my previous paper the number of magic squares, produced by the methods there indicated, is considerably under-estimated. It is there stated that in the case of a square of twenty-five places the effect produced by rejecting the *three* first x's from the top of the column would be to give the same result as would be obtained by rejecting the *two* first, but in the reverse order, which is not the fact. The squares obtained on the former principle of formation are distinct from those composed in the latter. Also the number of squares may be increased by combining the principle of the two methods explained in the paper alluded to. Thus the column of x's may be formed by rejecting the first, and that of the y's by rejecting the two first members (the initial x being always the mean value), and *vice versâ*.

Liverpool, December 29, 1845.

ON THE GEOMETRICAL REPRESENTATION OF THE MOTION OF A SOLID BODY.

By Arthur Cayley.

Let $P, Q, R, \ldots$ be consecutive generating lines of a skew surface, and on these take points p', p; q', q; $r', r \ldots$ such that $pq', qr' \ldots$ are the shortest distances between P and Q, Q and R, &c. Then for the generating line P, the ratio of

the inclination of the lines P, Q to the distance pq' is said to be "the torsion," the angle $q'pq$ is said to be the deviation, and the ratio of the inclination of the planes Qpq' and Qqr' to the inclination of P and Q is said to be the "skew curvature." And similarly for any other generating line; so that the torsion and deviation depend on the position of the consecutive line, and the skew curvature on the position of the two consecutive lines. The curve pqr.... is said to be the minimum distance curve. [When the skew surface degenerates into a developable surface, the torsion is infinite, the deviation a right angle, the skew curvature proportional to the curvature of the principal section, *i.e.* it is the distance of a point from the edge of regression, multiplied into the reciprocal of the radius of curvature, a product which is evidently constant along a generating line. Also the curve of minimum distance becomes the edge of regression.] A skew surface, considered independently of its position in space, is determined when for each generating line we know the torsion, deviation, and skew curvature. For, assuming arbitrarily the line P and the point p, also the plane in which pq' lies, the position of Q is completely determined from the given torsion and deviation; and then Q being known, the position of R is completely determined from the skew curvature for P, and the torsion and deviation for Q; and similarly the consecutive generating lines are to be determined.

Two skew surfaces are said to be "deformations" of each other, when for generating corresponding lines the torsion is always the same. Thus a surface will be deformed if considering the elements between the successive generating lines P, Q .... as rigid, these elements be made to revolve round the successive generating lines P, Q .... and to slide along them. [They are transformations, when not only the torsions but also the deviations are equal at corresponding generating lines: thus, if the sliding of the elements along P, Q....be omitted, the new surface will be, not a deformation, but a transformation of the other.] No two skew surfaces can be made to roll and slide one upon the other, so that their successive generating lines coincide, unless one of them is a deformation of the other: and when this is the case, the rolling and sliding motions are *completely determined.* In fact the angular velocity of the generating line is the angular velocity round this line, into the difference of the skew curvatures of the two surfaces; the velocity of translation of the generating line in its own direction is to the angular velocity of the generating line, as the difference of the deviations is to

[redacted]

the torsion. [This includes also the case in which one surface is a transformation of the other, where the motion is evidently a rolling one.] A skew surface moving in this manner upon another of which it is the deformation, may be said to "glide" upon it. We may now state the kinematical theorem.

"Any motion whatever of a solid body in space may be represented as the 'gliding' motion of one skew surface upon another fixed in space, and of which it is the deformation."

A theorem which is to be considered as the generalization of the well known one—

"Any motion of a solid body round a fixed point may be represented as the rolling motion of a conical surface upon a second conic surface fixed in space."

And of the supplementary theorem—

"The angular velocity round the line of contact (the instantaneous axis) is to the angular velocity of this line as the difference of curvatures of the two cones at any point in the same line, to the reciprocal of the distance of the point from the vertex."

The analytical demonstration of this last theorem is rather interesting: it depends on the following formulæ. Forming two determinants, the first with the angular velocities round three axes fixed in space, and the first and second derived coefficients with respect to the time of these velocities; the other in the same way with the angular velocities round axes fixed in the body; the difference of these determinants is equal to the fourth power of the angular velocity into the square of the angular velocity of the instantaneous axis.

To show this, let p, q, r be the angular velocities round the axes fixed in the body; u, v, w those round axes fixed in space; ω the angular velocity round the instantaneous axis; ∇, Ω the two determinants: the theorem comes to

$$\nabla - \Omega = M,$$

where $M = \omega^2 (p'^2 + q'^2 + r'^2 - \omega'^2)$, or $\omega^2 (u'^2 + v'^2 + w'^2 - \omega'^2)$.

Here
$$\begin{aligned} u &= \alpha p + \beta q + \gamma r, \\ v &= \alpha' p + \beta' q + \gamma' r, \\ w &= \alpha'' p + \beta'' q + \gamma'' r. \end{aligned}$$

Whence
$$\begin{aligned} u' &= \alpha p' + \beta q' + \gamma r', \\ v' &= \alpha' p' + \beta' q' + \gamma' r', \\ w' &= \alpha'' p' + \beta'' q' + \gamma'' r', \end{aligned}$$

(the remaining terms vanishing as is well known); and therefore

$$vw' - v'w = \alpha\,(qr' - q'r) + \beta\,(rp' - r'p) + \gamma\,(pq' - p'q),$$
$$wu' - w'u = \alpha'\,(qr' - q'r) + \beta'\,(rp' - r'p) + \gamma'\,(pq' - p'q),$$
$$uv' - u'v = \alpha''\,(qr' - q'r) + \beta''\,(rp' - r'p) + \gamma''\,(pq' - p'q).$$

And hence

$$vw'' - v''w = \alpha\,(qr'' - q''r) + \beta\,(rp'' - r''p) + \gamma\,(pq'' - p''q) + u'\omega^2 - u\omega\omega',$$
$$wu'' - w''u = \alpha'\,(qr'' - q''r) + \beta'\,(rp'' - r''p) + \gamma'\,(pq'' - p''q) + v'\omega^2 - v\omega\omega',$$
$$uv'' - u''v = \alpha''(qr'' - q''r) + \beta''(rp'' - r''p) + \gamma''(pq'' - p''q) + w'\omega^2 - w\omega\omega'.$$

And multiplying these by u', v', w', and adding, the required equation is immediately obtained.

In fact, if r be the distance of a point in the instantaneous axis from the vertex, and ρ, σ the radii of curvature of the two cones at that point, then

$$\frac{r}{\rho} = \frac{\omega^3}{M^{\frac{3}{2}}}\,\Omega,\quad \frac{r}{\sigma} = \frac{\omega^3}{M^{\frac{3}{2}}}\,\nabla.$$

As may be shown without difficulty, and the angular velocity of the instantaneous axis is given by the equation $\varpi = \dfrac{M^{\frac{1}{2}}}{\omega^2}$, whence the relation between the two angular velocities is

$$\omega : \varpi = \frac{1}{\rho} - \frac{1}{\sigma} : \frac{1}{r}.$$

ON THE ROTATION OF A SOLID BODY ROUND A FIXED POINT.

By Arthur Cayley.

The difficulty of completing elegantly the solution of this problem, in the case where no forces act upon the body, arises from the complexity and want of symmetry of the ordinary formulæ for determining the position of one set of rectangular axes with respect to another set; in consequence of which it has hitherto been considered necessary to make a particular supposition relative to the position of the fixed axes in space, viz. that one of them shall be perpendicular to the "invariable plane" of the rotating body. But some formulæ for the above purpose, given also by Euler, are entirely free from

these objections. Imagine two sets of axes Ax, Ay, Az, $Ax_,$, $Ay_,$, $Az_,$. The former set can be made to coincide with the second set, by a rotation θ round a certain axis AR, inclined to Ax, Ay, Az at angles f, g, h. (As usual f, g, h are the angles RAx, RAy, RAz considered as positive, and the rotation is in the same direction as a rotation round Az from x towards y). This axis may be termed the resultant axis, and the angle θ the resultant rotation. The formulæ of Euler express the coefficients of the transformation in terms of the resultant rotation and of the position of the resultant axis, *i.e.* in terms of θ and of the angles f, g, h, whose cosines are connected by the equation

$$\cos^2 f + \cos^2 g + \cos^2 h = 1.$$

This idea was improved upon by M. Rodrigues (Liouv. tom. v. p. 404), who introduced the quantities

$$\tan \tfrac{1}{2}\theta \cos f, \quad \tan \tfrac{1}{2}\theta \cos g, \quad \tan \tfrac{1}{2}\theta \cos h,$$

(quantities which will be represented by λ, μ, ν,) by means of which he expressed the coefficients as fractions, the numerators of which are very simple rational functions of the second order of λ, μ, ν, and which have the common denominator $(1 + \lambda^2 + \mu^2 + \nu^2)$. These quantities may conveniently be termed the "coordinates of the resultant rotation," and the denominator or the square of the secant of the semiangle of resultant rotation will be the "modulus" of the rotation. The elegance of these results led me to apply them to the mechanical question, and I gave in the *Journal* (vol. III. p. 224) the differential equations of motion obtained in terms of λ, μ, ν: which I integrated as in the common theory, by supposing one of the fixed axes to be perpendicular to the invariable plane. Though my attention was again called to the subject, by the connexion of some of these formulæ with Sir William Hamilton's theory of quaternions, no other way of performing the integration occurred to me. The grand discovery however of Jacobi, of the possibility of reducing to quadratures the two final differential equations of any mechanical problem, when the remaining integrals are known, induced me to resume the problem, and at least attempt to bring it so far as to obtain a differential equation of the first order between two variables only, the multiplier of which could be obtained theoretically by Jacobi's discovery. The choice of two new variables to which the equations of the problem led me, enabled me to effect this with the greatest simplicity; and the differential equation which I finally obtained, turned out

to be integrable *per se*, so that the laborious process of finding the multiplier became unnecessary. The new variables Ω, υ have the following geometrical interpretations, $\Omega = k \tan \frac{1}{2}\theta \cos I$, where k is the principal moment, θ as before the angle of resultant rotation, and I is the inclination of the resultant axis to the perpendicular upon the invariable plane, and $\upsilon = k^2 \cos^2 \frac{1}{2} J$; where, if we imagine a line AQ having the same position relatively to the axes in fixed space, that the perpendicular upon the invariable plane has to the principal axes of the rotating body, then J is the inclination of this line to the above perpendicular. To the choice of these variables I was led by the analysis only. It will be seen that p, q, r are functions of υ only, while λ, μ, ν contain besides the variable Ω. In obtaining these relations a singular equation $\Omega^2 = k\upsilon - k^2$ occurs (equation 13), which may also be written $1 + \tan^2 \frac{1}{2}\theta \cos^2 I = \sec^2 \frac{1}{2}\theta \cos^2 \frac{1}{2} J$, in which form the interpretation of the quantities I, J has just been given. The equation (17), it may be remarked, is self-evident: it expresses that the inclination of the resultant axis to the normal of the invariable plane, is equal to the inclination of the same axis to the line AQ. Now the resultant axis having the same inclination to the axes fixed in space as it has to the principal axes, and the line AQ the same inclinations to these fixed axes that the normal to the invariable plane has to the principal axes, the truth of the proposition becomes manifest. The correspondence in form between the systems (10) and (14) is also worth remarking. The final results at which I arrive are, that the time and the arc whose tangent is $\Omega \div k$, are each of them expressible as the integrals of certain algebraical functions of υ. The notation throughout is the same as that made use of in the paper already quoted.

The equations of rotatory motion are

$$dt = \frac{dp}{P} = \frac{dq}{Q} = \frac{dr}{R} = \frac{d\lambda}{\Lambda} = \frac{d\mu}{\mathrm{M}} = \frac{d\nu}{\mathrm{N}} \,\ldots\ldots\ldots\, (1),$$

where

$$\left.\begin{aligned}
P &= \frac{1}{A}\left[(B-C)qr + \tfrac{1}{2}\left\{(1+\lambda^2)\frac{dV}{d\lambda} + (\lambda\mu+\nu)\frac{dV}{d\mu} + (\lambda\nu-\mu)\frac{dV}{d\nu}\right\}\right] \\
Q &= \frac{1}{B}\left[(C-A)rp + \tfrac{1}{2}\left\{(\mu\lambda-\nu)\frac{dV}{d\lambda} + (1+\mu^2)\frac{dV}{d\mu} + (\mu\nu+\lambda)\frac{dV}{d\nu}\right\}\right] \\
R &= \frac{1}{C}\left[(A-B)pq + \tfrac{1}{2}\left\{(\nu\lambda+\mu)\frac{dV}{d\lambda} + (\mu\nu-\lambda)\frac{dV}{d\mu} + (1+\nu^2)\frac{dV}{d\nu}\right\}\right]
\end{aligned}\right\}$$

$$\ldots\ldots\ldots\, (2).$$

$$\left.\begin{aligned} \Lambda &= \tfrac{1}{2}\{(1+\lambda^2)p + (\lambda\mu - \nu)q + (\lambda\nu + \mu)r\} \\ \mathrm{M} &= \tfrac{1}{2}\{(\mu\lambda + \nu)p + (1+\mu^2)q + (\mu\nu - \lambda)r\} \\ \mathrm{N} &= \tfrac{1}{2}\{(\nu\lambda - \mu)p + (\mu\nu + \lambda)q + (1+\nu^2)r\} \end{aligned}\right\}\ldots\ldots(3).$$

And in the case where the forces vanish, the first three equations become simply

$$\left.\begin{aligned} P &= \frac{1}{A}(B-C)\,qr, \\ Q &= \frac{1}{B}(C-A)\,rp, \\ R &= \frac{1}{C}(A-B)\,pq. \end{aligned}\right\}\ldots\ldots\ldots\ldots(4).$$

In which case the usual four integrals of the system are

$$Ap^2 + Bq^2 + Cr^2 = h \ldots\ldots\ldots\ldots(5),$$

$$\left.\begin{aligned} Ap(1+\lambda^2-\mu^2-\nu^2) + 2Bq(\lambda\mu-\nu) + 2Cr(\nu\lambda+\mu) &= \mathrm{a}(1+\lambda^2+\mu^2+\nu^2) \\ 2Ap(\lambda\mu+\nu) + Bq(1+\mu^2-\nu^2-\lambda^2) + 2Cr(\mu\nu-\lambda) &= \mathrm{b}(1+\lambda^2+\mu^2+\nu^2) \\ 2Ap(\nu\lambda-\mu) + 2Bq(\mu\nu+\lambda) + Cr(1+\nu^2-\lambda^2-\mu^2) &= \mathrm{c}(1+\lambda^2+\mu^2+\nu^2) \end{aligned}\right\}\ldots\ldots(6).$$

Or as they may also be written,

$$\left.\begin{aligned} \mathrm{a}(1+\lambda^2-\mu^2-\nu^2) + 2\mathrm{b}(\lambda\mu+\nu) + 2\mathrm{c}(\nu\lambda-\mu) &= Ap(1+\lambda^2+\mu^2+\nu^2) \\ 2\mathrm{a}(\lambda\mu-\nu) + \mathrm{b}(1+\mu^2-\nu^2-\lambda^2) + 2\mathrm{c}(\mu\nu+\lambda) &= Bq(1+\lambda^2+\mu^2+\nu^2) \\ 2\mathrm{a}(\nu\lambda+\mu) + 2\mathrm{b}(\mu\nu-\lambda) + \mathrm{c}(1+\nu^2-\lambda^2-\mu^2) &= Cr(1+\lambda^2+\mu^2+\nu^2) \end{aligned}\right\}\ldots\ldots(6\ bis).$$

To which we may add,

$$A^2p^2 + B^2q^2 + C^2r^2 = k^2 \ldots\ldots\ldots\ldots(7);$$

where $$k^2 = \mathrm{a}^2 + \mathrm{b}^2 + \mathrm{c}^2 \ldots\ldots\ldots\ldots(8).$$

Introducing the quantities κ, Ω, (the former of which has been already made use of) given by the equations

$$\left.\begin{aligned} \kappa &= 1 + \lambda^2 + \mu^2 + \nu^2, \\ \Omega &= \lambda Ap + \mu Bq + \nu Cr \end{aligned}\right\}\ldots\ldots\ldots\ldots(9).$$

The equations (6) may be written under the form

$$\left.\begin{aligned} 2\lambda\Omega + 2\mu Cr - 2\nu Bq &= \kappa(Ap + \mathrm{a}) - 2Ap \\ -2\lambda Cr + 2\mu\Omega + 2\nu Ap &= \kappa(Bq + \mathrm{b}) - 2Bq \\ 2\lambda Bq - 2\mu Ap + 2\nu\Omega &= \kappa(Cr + \mathrm{c}) - 2Cr \end{aligned}\right\}\ldots(10).$$

Whence also, multiplying by Ap, Bq, Cr, and adding,

$$2\Omega^2 = \kappa\{k^2 + (Ap\mathrm{a} + Bq\mathrm{b} + Cr\mathrm{c})\} - 2k^2 \ldots(11),$$

or writing $$k^2 + (Ap\mathrm{a} + Bq\mathrm{b} + Cr\mathrm{c}) = 2\upsilon \;\ldots\ldots\; (12),$$
this becomes $$\Omega^2 = k\upsilon - k^2 \;\ldots\ldots\ldots\ldots\; (13);$$
an equation, the geometrical interpretation of which has already been given.

From the equations (10) we deduce the inverse system

$$\left.\begin{aligned} \mathrm{a}\Omega - \mathrm{b}Cr + \mathrm{c}Bq &= 2\lambda\upsilon - \Omega Ap \\ \mathrm{a}Cr + \mathrm{b}\Omega - \mathrm{c}Ap &= 2\mu\upsilon - \Omega Bq \\ -\mathrm{a}Bq + \mathrm{b}Ap + \mathrm{c}\Omega &= 2\nu\upsilon - \Omega Cr \end{aligned}\right\}\ldots\ldots(14),$$

which are easily verified by multiplying by Ω, Cr, $-Bq$; or by $-Cr$, Ω, Ap; or Bq, $-Ap$, Ω: adding and reducing, by which means the equations (10) are re-obtained. Hence also, if for shortness

$$\left.\begin{aligned} \Phi &= \mathrm{a}p + \mathrm{b}q + \mathrm{c}r \\ \nabla &= \mathrm{a}qr\,(B - C) + \mathrm{b}rp\,(C - A) + \mathrm{c}pq\,(A - B) \end{aligned}\right\}\ldots\,(15),$$

we have, multiplying by p, q, r, and adding,

$$\Omega\Phi - \nabla = 2\upsilon\,(\lambda p + \mu q + \nu r) - \Omega h \;\ldots\ldots\; (16).$$

To which may be added the equation

$$\Omega = \mathrm{a}\lambda + \mathrm{b}\mu + \mathrm{c}\nu \;\ldots\ldots\ldots\ldots\; (17),$$

which follows immediately from either of the systems (10) or (14).

We may also put the equations (10) under this other form,

$$\left.\begin{aligned} 2\lambda\Omega - 2\mu\mathrm{c} + 2\nu\mathrm{b} &= \kappa\,(Ap + \mathrm{a}) - 2\mathrm{a} \\ -2\lambda\mathrm{c} + 2\mu\Omega - 2\nu\mathrm{a} &= \kappa\,(Bq + \mathrm{b}) - 2\mathrm{b} \\ -2\lambda\mathrm{b} + 2\mu\mathrm{a} + 2\nu\Omega &= \kappa\,(Cr + c) - 2\mathrm{c} \end{aligned}\right\}\ldots\; (10\ \mathit{bis}).$$

It may be remarked now, that p, q, r are functions of υ; since we have to determine these quantities, the three equations

$$\left.\begin{aligned} Ap^2 + Bq^2 + Cr^2 &= h, \\ A^2p^2 + B^2q^2 + C^2r^2 &= k^2, \\ Ap\mathrm{a} + Bq\mathrm{b} + Cr\mathrm{c} &= 2\upsilon - k^2 \end{aligned}\right\}\ldots\ldots\ldots\ldots(18).$$

Also λ, μ, ν are given by the equations (14) as functions of p, q, r, Ω, *i.e.* of υ, Ω. So that every thing is prepared for the investigation of the differential equation between υ, Ω. To find this we have immediately

$$d\upsilon = \tfrac{1}{2}\,(A\mathrm{a}dp + B\mathrm{b}dq + C\mathrm{c}dr) = \tfrac{1}{2}\,\nabla dt \;\ldots\ldots\; (19),$$

from the equations (4) and (15). ∇ is of course to be considered as a given function of υ. Again,

$$\Omega d\Omega = \tfrac{1}{2}\,(\kappa d\upsilon + \upsilon d\kappa) \;\ldots\ldots\ldots\; (20),$$

where $$d\kappa = 2\,(\lambda d\lambda + \mu d\mu + \nu d\nu) \;\ldots\ldots\ldots\; (21);$$

or from the equations (1), (3),

$$d\kappa = \kappa\,(\lambda p + \mu q + \nu r)\,dt \;\ldots\ldots\ldots\; (22).$$

Whence, from (16),

$$2\upsilon d\kappa = \kappa\,\{\Omega\,(h + \Phi) - \nabla\}\,dt \;\ldots\ldots\; (23);$$

or

$$2\,(\upsilon d\kappa + \kappa d\upsilon) = \kappa\Omega\,(h + \Phi)\,dt \;\ldots\ldots\; (24).$$

Whence

$$d\Omega = \tfrac{1}{4}\,\kappa\,(h + \Phi)\,dt$$

$$= \tfrac{1}{4}\,\frac{\Omega^2 + k^2}{\upsilon}\,(h + \Phi)\,dt \;\ldots\ldots\; (25).$$

And therefore, from (19),

$$\frac{2d\Omega}{\Omega^2 + k^2} = \frac{h + \Phi}{\upsilon\nabla}\,d\upsilon \;\ldots\ldots\ldots\; (26),$$

the required differential equation, in which Φ, ∇ are given functions of (υ), *i.e.* they are functions of p, q, r by the equations (15), and these quantities are functions of υ by (18). The variables in (26) are therefore separated, and we have the integral equation

$$2\tan^{-1}\frac{\Omega}{k} = \delta + k\int\frac{(h + \Phi)\,d\upsilon}{\upsilon\nabla} \;\ldots\ldots\; (27),$$

where δ is the constant of integration. The equation (19) gives also

$$t - \epsilon = 2\int\frac{d\upsilon}{\nabla} \;\ldots\ldots\ldots\ldots\; (28);$$

and thus the solution of the problem is completely effected. The integrals may be taken from any particular value υ_0 of υ. The variable Ω may be exhibited as the integral of an *explicit* algebraical function, by recurring to the variable ϕ of the paper quoted.

Thus if

$$Ap_0^2 + Bq_0^2 + Cr_0^2 = h,$$

$$A^2p_0^2 + B^2q_0^2 + C^2r_0^2 = k^2,$$

$$Ap_0\mathrm{a} + Bq_0\mathrm{b} + Cr_0\mathrm{c} = 2\upsilon_0 - k^2;$$

then

$$\sqrt{\left\{p_0^2 - \frac{1}{A}\,(C - B)\,\phi\right\}},\quad \sqrt{\left\{q_0^2 - \frac{1}{A}\,(A - C)\,\phi\right\}}$$

$$\sqrt{\left\{r_0^2 - \frac{1}{C}\,(B - A)\,\phi\right\}},$$

$$dt = \tfrac{1}{2}\,\frac{d\phi}{pqr} = \frac{2d\upsilon}{\nabla},\quad\text{or}\quad \frac{d\upsilon}{\nabla} = \tfrac{1}{4}\,\frac{d\phi}{pqr};$$

Formulæ for the variation of the arbitrary constants, in the case of any distributing forces acting upon the body, will be given in a subsequent paper.

whence $4 \tan^{-1} \frac{\Omega}{k} = 2\delta + k \int_0 \frac{(h + \mathrm{a}p + \mathrm{b}q + cr)\, d\phi}{(k^2 + Ap\mathrm{a} + Bq\mathrm{b} + Cr\mathrm{c})\, pqr}$.

In which form it is exactly analogous to the equation there obtained, p. 230,

$$4 \tan^{-1} \nu_0 = \int \frac{(h + kr)\, d\phi}{(k + Cr)\, pqr}.$$

(*To be continued.*)

ON THE LAWS OF EQUILIBRIUM AND MOTION OF SOLID AND FLUID BODIES.

By SAMUEL HAUGHTON, Fellow of Trinity College, Dublin.

A GENERAL investigation of the laws of equilibrium and motion of solid and fluid bodies must consist essentially of two parts: the investigation of the differential equations of equilibrium and motion, and the subsequent integration of these equations. In this paper I propose to treat of both these subjects. The principles from which I set out are extremely simple and do not involve any assumptions, except such as will readily appear to be natural consequences of our conception of the nature of the bodies which surround us, and the results to which the investigation leads are in accordance with all the known mathematical laws of solid and fluid bodies. Some of these results are also, so far as I am aware, new, and seem to throw light on the difficult problem of the equilibrium and motion of solid elastic bodies.

The *method* which I have followed is that of the '*Mécanique Analytique*' of Lagrange, of which such a successful application has been made by Professor M^cCullagh to the investigation of the mechanical laws of light, and consists in the application of the Calculus of Variations to the principles of rational mechanics. One great difficulty of this method arises from its comprehensiveness, and the labour of the mathematician is more frequently to *distinguish* between different cases included in the formulæ, than to show that these cases are all *contained* in them. Previously, therefore, to entering upon the investigation of the differential equations, I shall endeavour to establish a distinction between solids and fluids; on which subject much confusion and contradiction seems to exist between the writers who have treated most of these subjects.

The most general conception of solids and fluids, is that of 'an immense assemblage of molecules separated from each

[redacted]

other by indefinitely small distances'; if we add to this general notion, the assertion, that '*these molecules act on each other only in the line joining them,*' we shall have a definition of the medium whose laws I propose to investigate. Suppose now that this medium is acted on by *no external* forces, and abandoned solely to the action of its molecules on each other; the distinction which I conceive to exist between solids and fluids is the following:

That in solid bodies the resultant of all the forces exerted by all the surrounding molecules on any molecule (m), *is zero. That in fluids, whether liquid or gaseous, this is not the case, and that consequently the fluid (no external pressures or forces acting) would be dissipated.*

That this is the correct distinction of these two classes of bodies, will, I hope, be made clear by the following investigation, and at present it will be sufficient to observe that it agrees with the ordinary notion of a fluid: in such a body we suppose a pressure (p) to exist at each point (x, y, z), which equilibrates the external forces, such as the forces arising from the sides of the vessel containing the fluid, &c. Hence; if the *external forces cease to act*, the pressure being transmitted to the external surface of the fluid would dissipate it.

The case of a fluid in a closed vessel is not the case here considered, for the pressures of the sides of the vessel are, in this case, the external forces, which, together with the molecular forces, produce equilibrium: but the molecular forces themselves are not, I conceive, in equilibrium. This is manifestly true of gases, and I consider the same to be true of liquids. The distinction between liquids and gases is, probably, *relative* to the ordinary external forces in action at the surface of the earth, such as gravity, the pressure of the atmosphere, &c.

I proceed, without further delay, to the laws of the medium, whether solid or fluid.

The general equation of equilibrium of the points composing a medium is

$$\int\!\!\int\!\!\int(X\delta\xi + Y\delta\eta + Z\delta\zeta)\,dm = \int\!\!\int\!\!\int\delta V\,dx\,dy\,dz \;\ldots\ldots (1),$$

in which equation the left-hand member is the sum of the '*moments*' of the external forces, and the right-hand member is the sum of the *moments* of the internal forces. I use the term *moments* as defined by Lagrange.

The function V depends, in general, on the particular nature of the medium considered, and its form must, in the present case, be deduced from the definition already given. x, y, z are the coordinates of the position of rest of any

molecule (m); and $x + \xi$, $y + \eta$, $z + \zeta$ are the coordinates of the same molecule when displaced by the action of external forces.

When the molecules are in a state of rest, not acted upon by any external forces, the force exerted by any molecule (m') on (m) will be, in general, a function of the *distance* (mm'), and of the *direction* of the line (mm'): if now, by means of external forces, the molecules $(m, m'$, &c.) assume new positions, the force exerted by m' on m will be represented in general by the expression

$$f(\rho, \alpha, \beta, \gamma, \rho'),$$

$\rho, \alpha, \beta, \gamma$ being the *original* length and direction of the line (mm'), and ρ' being the alteration in ρ; ρ' being small, the function (f) may be represented by

$$f = F_0 + 2F_1\rho' + 3F_2\rho'^2 + \&c. \dots\dots\dots (2),$$

F_0, F_1, &c. being functions of $(\rho, \alpha, \beta, \gamma)$.

As the force (f) is a force tending to alter $\rho + \rho'$, its *moment* will be $f.\delta\rho'$, or neglecting ρ'^2, &c.

$$F_0\delta\rho' + 2F_1\rho'\delta\rho'.$$

Hence $$\delta V = \Sigma \left\{F_0\delta\rho' + F_1\delta(\rho'^2)\right\};$$

and therefore

$$V = \int_0^\infty \int_0^\pi \int_0^{2\pi} (F_0\rho' + F_1\rho'^2)\, \rho^2 \sin\theta d\rho\, d\theta\, d\phi \dots . (3).$$

The value of (ρ'), to be substituted in this expression for V, is thus found. Let $x, y, z, x + a, y + b, z + c$, be the coordinates of rest of m and m'; then in the changed position, if x, y, z, become $x + \xi$, $y + \eta$, $z + \zeta$, the coordinates of m' will become

$$\left.\begin{aligned} &x + \xi + a + \frac{d\xi}{dx}a + \frac{d\xi}{dy}b + \frac{d\xi}{dz}c, \\ &y + \eta + b + \frac{d\eta}{dx}a + \frac{d\eta}{dy}b + \frac{d\eta}{dz}c, \\ &z + \zeta + c + \frac{d\zeta}{dx}a + \frac{d\zeta}{dy}b + \frac{d\zeta}{dz}c, \end{aligned}\right\}.$$

$\rho + \rho'$ is equal to the square root of the sum of the squares of the differences of the coordinates of its extreme points, *i.e.*

$$\rho + \rho' = \sqrt{\left\{\left(a + \frac{d\xi}{dx}a + \frac{d\xi}{dy}b + \frac{d\xi}{dz}c\right)^2 + \left(b + \frac{d\eta}{dx}a + \frac{d\eta}{dy}b + \frac{d\eta}{dz}c\right)^2 + \left(c + \frac{d\zeta}{dx}a + \frac{d\zeta}{dy}b + \frac{d\zeta}{dz}c\right)^2\right\}};$$

and, neglecting the smaller quantities, we obtain

$$\rho + \rho' = \rho + \rho\left(\frac{d\xi}{dx}\cos^2 a + \frac{d\eta}{dy}\cos^2\beta + \frac{d\zeta}{dz}\cos^2\gamma\right.$$
$$\left. + u\cos\beta\cos\gamma + v\cos a\cos\gamma + w\cos a\cos\beta\right),$$

where
$$u = \frac{d\eta}{dz} + \frac{d\zeta}{dy},$$
$$v = \frac{d\zeta}{dx} + \frac{d\xi}{dz},$$
$$w = \frac{d\xi}{dy} + \frac{d\eta}{dx};$$

and also
$$\rho = \sqrt{(a^2 + b^2 + c^2)}.$$

Hence finally we obtain.

$$\rho' = \rho\left\{\frac{d\xi}{dx}\cos^2 a + \frac{d\eta}{dy}\cos^2\beta + \frac{d\zeta}{dz}\cos^2\gamma + \left(\frac{d\eta}{dz} + \frac{d\zeta}{dy}\right)\cos\beta\cos\gamma\right.$$
$$\left. + \left(\frac{d\zeta}{dx} + \frac{d\xi}{dz}\right)\cos a\cos\gamma + \left(\frac{d\xi}{dy} + \frac{d\eta}{dx}\right)\cos a\cos\beta\right\}.$$

This is the value of ρ', to be substituted in V, which will then consist of two parts, V_0 and V_1, depending upon F_0 and F_1, V_0 being homogeneous and linear with respect to

$$\frac{d\xi}{dx},\ \frac{d\eta}{dy},\ \frac{d\zeta}{dz},\ u,\ v,\ w;$$

and V_1 homogeneous, and of the second order with respect to the same six quantities.

Returning now to the distinction drawn between solids and fluids, it will appear from that distinction that in fluids the function V will be $V_0 + V_1$, while for solids $V_0 = 0$: for if in (2) we suppose $\rho' = 0$, we shall have $F_0 = f$; hence the definition of a solid requires that the forces F_0 which correspond to the case of *no external forces* acting should equilibrate.

I shall now determine the equations of equilibrium arising from V_0, which does not vanish for fluids, and then examine the case of solids; for which $V_0 = 0$, and which depend only on V_1.

By formula (3), we have

$$V_0 = \int_0^\infty\int_0^\pi\int_0^{2\pi} F_0\rho'.\rho^2 \sin\theta\, d\rho\, d\theta\, d\phi.$$

Hence

$$V_0 = \int_0^\infty \int_0^\pi \int_0^{2\pi} F_0 \left(\frac{d\xi}{dx} \cos^2 a + \frac{d\eta}{dy} \cos^2 \beta + \frac{d\zeta}{dz} \cos^2 \gamma \right.$$
$$\left. + u \cos\beta \cos\gamma + v \cos a \cos\gamma + w \cos a \cos\beta \right) \rho^3 \sin\theta \, d\rho \, d\theta \, d\phi \ldots (4).$$

In this equation F_0 is a function of ρ only, since the medium is equally elastic in all directions, and

$$\cos a = \cos\phi \sin\theta,$$
$$\cos\beta = \sin\phi \sin\theta,$$
$$\cos\gamma = \cos\theta.$$

The coefficients of $\frac{d\xi}{dx}, \frac{d\eta}{dy}, \frac{d\zeta}{dz}, u, v, w$, are six triple integrals; and in the present case it is easily seen that the coefficients of u, v, w, are zero, and those of $\frac{d\xi}{dx}, \frac{d\eta}{dy}, \frac{d\zeta}{dz}$ equal to each other. Their common value is

$$p = \frac{4\pi}{3} \int_0^\infty F_0 . \rho^3 d\rho,$$

which is obtained by integrating twice, with respect to θ and ϕ.

Hence finally we obtain

$$V_0 = p \left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} \right) \ldots\ldots\ldots\ldots (5).$$

By the general equation of equilibrium, we have

$$\iiint (X\delta\xi + Y\delta\eta + Z\delta\zeta)\, dm = \iiint \delta V_0 \, dx\, dy\, dz.$$

Substituting the value of V_0, and proceeding according to the principles of the Calculus of Variations, we obtain

$$\iiint (X\delta\xi + Y\delta\eta + Z\delta\zeta)\, dm = \iint p\, \delta\xi\, dy\, dz + \iint p\, \delta\eta\, dx\, dz + \iint p\, \delta\zeta\, dx\, dy,$$
$$- \iiint \left(\frac{dp}{dx} \delta\xi + \frac{dp}{dy} \delta\eta + \frac{dp}{dz} \delta\zeta \right) dx\, dy\, dz \ldots\ldots (6),$$

the triple integrals giving the equations of equilibrium, and the double integrals giving the conditions at the limits.

Hence, if the density be expressed by ε, we shall have the expression $dm = \varepsilon dx dy dz$, and the equations of equilibrium will be

$$\left. \begin{aligned} -\varepsilon X &= \frac{dp}{dx} \\ -\varepsilon Y &= \frac{dp}{dy} \\ -\varepsilon Z &= \frac{dp}{dz} \end{aligned} \right\} \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

which are the well-known equations of Hydrostatics. Hence, if the forces (X, Y, Z) be zero at all points of the fluid, the quantity p must be constant; and vice versâ, if p be constant for all points of the fluid, the forces (X, Y, Z) must be zero—(this includes the case of homogeneous fluids): in such a case the function V_0 will give only the condition at the limits expressed by the double integrals (6); which is identical with that found by Lagrange, and expresses that there must be a normal pressure at every point of the bounding surface, constant and equal to p, in order that the fluid should remain in equilibrium. This condition at the limits is not necessary in solids, since for them V_0, and therefore p, is equal to zero.

It follows from the views I have adopted in this paper, that the ordinary equations of Hydrostatics and Hydrodynamics are only a first approximation to the whole equations, and that in some cases, particularly in Hydrodynamics, this approximation may be insufficient: in such cases we should add to the equations (7) the terms arising from V_1, which are common to solids and fluids.

In general $$V = V_0 + V_1 + V_2 + \&c.,$$

and the terms V_0, V_1, &c. will give rise, in the equations of equilibrium, to differential coefficients of the first, second, &c. order.

I shall now take up the general discussion of the equations of equilibrium and motion arising from the function (V_1).

From equation (3) we see that

$$V_1 = \int_0^\infty \int_0^\pi \int_0^{2\pi} F_1 \cdot (\rho')^2 \cdot \rho^2 \sin\theta d\rho d\theta d\phi.$$

Hence, making $d\omega = \rho^2 \sin\theta \, d\rho \, d\theta \, d\phi$, we shall have

$$V_1 = \int_0^\infty \int_0^\pi \int_0^{2\pi} F_1 \left(\frac{d\xi}{dx} \cos^2\alpha + \frac{d\eta}{dy} \cos^2\beta + \frac{d\zeta}{dz} \cos^2\gamma \right.$$

$$\left. + u \cos\beta \cos\gamma + v \cos\alpha \cos\gamma + w \cos\alpha \cos\beta \right)^2 \rho^2 d\omega. \ldots (8).$$

Therefore

$$2V_1 = \left\{ A \left(\frac{d\xi}{dx}\right)^2 + B \left(\frac{d\eta}{dy}\right)^2 + C \left(\frac{d\zeta}{dz}\right)^2 \right\} + \{Lu^2 + Mv^2 + Nw^2\}$$

$$+ 2 \left(L \frac{d\eta}{dy} \cdot \frac{d\zeta}{dz} + M \frac{d\xi}{dx} \cdot \frac{d\zeta}{dz} + N \frac{d\xi}{dx} \cdot \frac{d\eta}{dy} \right)$$

$$+ 2 (\alpha_1 vw + \beta_2 uw + \gamma_3 uv)$$

$$+ 2\left\{u\left(a_1 \frac{d\xi}{dx} + \beta_1 \frac{d\eta}{dy} + \gamma_1 \frac{d\zeta}{dz}\right) + v\left(a_2 \frac{d\xi}{dx} + \beta_2 \frac{d\eta}{dy} + \gamma_2 \frac{d\zeta}{dz}\right)\right.$$
$$\left. + w\left(a_3 \frac{d\xi}{dx} + \beta_3 \frac{d\eta}{dy} + \gamma_3 \frac{d\zeta}{dz}\right)\right\} \ldots (9).$$

In this value of V_1 there are 21 terms, but only 15 constants, since the coefficients of six of the terms are expressed by the same integrals as some other terms of the function.

The values of the 15 coefficients are given by the following expressions:

$$A = 2\iiint F_1 \cos^4 a.\rho^2 d\omega, \qquad B = 2\iiint F_1 \cos^4 \beta.\rho^2 d\omega,$$
$$C = 2\iiint F_1 \cos^4 \gamma.\rho^2 d\omega,$$
$$L = 2\iiint F_1 \cos^2 \beta \cos^2\gamma.\rho^2 d\omega, \quad M = 2\iiint F_1 \cos^2 a \cos^2 \gamma.\rho^2 d\omega,$$
$$N = 2\iiint F_1 \cos^2 a \cos^2 \beta.\rho^2 d\omega.$$
$$a_1 = 2\iiint F_1 \cos^2 a \cos\beta \cos\gamma.\rho^2 d\omega, \quad \beta_1 = 2\iiint F_1 \cos^3 \beta \cos \gamma.\rho^2 d\omega,$$
$$\gamma_1 = 2\iiint F_1 \cos^3 \gamma \cos \beta.\rho^2 d\omega,$$
$$a_2 = 2\iiint F_1 \cos^3 a \cos\gamma.\rho^2 d\omega, \quad \beta_2 = 2\iiint F_1 \cos^2\beta \cos a \cos\gamma.\rho^2 d\omega,$$
$$\gamma_2 = 2\iiint F_1 \cos^3 \gamma \cos a.\rho^2 d\omega,$$
$$a_3 = 2\iiint F_1 \cos^3 a \cos \beta.\rho^2 d\omega, \qquad \beta_3 = 2\iiint F_1 \cos^3 \beta \cos a.\rho^2 d\omega,$$
$$\gamma_3 = 2\iiint F_1 \cos^2 \gamma \cos a \cos \beta.\rho^2 d\omega.$$

This value of V_1 must now be substituted in the equation

$$\iiint(X\delta\xi + Y\delta\eta + Z\delta\zeta)\, dm = \iiint \delta V_1 dxdydz,$$

which must then be treated by the rules of the Calculus of Variations.

$$(a)\ \tfrac{1}{2}\,\delta\iiint\left\{A\left(\frac{d\xi}{dx}\right)^2 + B\left(\frac{d\eta}{dy}\right)^2 + C\left(\frac{d\zeta}{dz}\right)^2\right\} dxdydz$$
$$= \iint A \frac{d\xi}{dx}\,\delta\xi dydz$$
$$+ \iint B \frac{d\eta}{dy}\,\delta\eta dxdz$$
$$+ \iint C \frac{d\zeta}{dz}\,\delta\zeta dxdy$$
$$- \iiint\left(A \frac{d^2\xi}{dx^2}\,\delta\xi + B \frac{d^2\eta}{dy^2}\,\delta\eta + C \frac{d^2\zeta}{dz^2}\,\delta\zeta\right) dxdydz.$$

$$(b)\ \tfrac{1}{2}\,\delta\iiint(Lu^2 + Mv^2 + Nw^2)\, dxdydz = \iint(Mv\delta\zeta + Nw\delta\eta)\, dydz$$
$$+ \iint(Nw\delta\xi + Lu\delta\zeta)\, dxdz$$
$$+ \iint(Lu\delta\eta + Mv\delta\xi)\, dxdy$$

$$-\iiint\left\{\left(M\frac{dv}{dz}+N\frac{dw}{dy}\right)\delta\xi+\left(N\frac{dw}{dx}+L\frac{du}{dz}\right)\delta\eta\right.$$
$$\left.+\left(L\frac{du}{dy}+M\frac{dv}{dx}\right)\delta\zeta\right\}dxdydz.$$

(c) $$\tfrac{1}{2}\delta\iiint 2\left(L\frac{d\eta}{dy}\cdot\frac{d\zeta}{dz}+M\frac{d\xi}{dx}\cdot\frac{d\zeta}{dz}+N\frac{d\xi}{dx}\cdot\frac{d\eta}{dy}\right)dxdydz$$
$$=\iint\left(M\frac{d\zeta}{dz}+N\frac{d\eta}{dy}\right)\delta\xi dydz$$
$$+\iint\left(N\frac{d\xi}{dx}+L\frac{d\zeta}{dz}\right)\delta\eta dxdz$$
$$+\iint\left(L\frac{d\eta}{dy}+M\frac{d\xi}{dx}\right)\delta\zeta dxdy$$
$$-\iiint\left\{\left(M\frac{d^2\zeta}{dxdz}+N\frac{d^2\eta}{dxdy}\right)\delta\xi+\left(N\frac{d^2\xi}{dxdy}+L\frac{d^2\zeta}{dydz}\right)\delta\eta\right.$$
$$\left.+\left(L\frac{d^2\eta}{dydz}+M\frac{d^2\xi}{dxdz}\right)\delta\zeta\right\}dxdydz.$$

(d) $$\tfrac{1}{2}\delta\iiint 2(\alpha_1 vw+\beta_2 uw+\gamma_3 uv)\,dxdydz$$
$$=\iint\{\alpha_1(v\delta\eta+w\delta\zeta)+u(\beta_2\delta\eta+\gamma_3\delta\zeta)\}\,dydz$$
$$+\iint\{\beta_2(w\delta\zeta+u\delta\xi)+v(\gamma_3\delta\zeta+\alpha_1\delta\xi)\}\,dxdz$$
$$+\iint\{\gamma_3(u\delta\xi+v\delta\eta)+w(\alpha_1\delta\xi+\beta_2\delta\eta)\}\,dxdy.$$
$$-\iiint\left\{\alpha_1\left(\frac{dv}{dy}+\frac{dw}{dz}\right)+\beta_2\frac{du}{dy}+\gamma_3\frac{du}{dz}\right\}\delta\xi\,dxdydz$$
$$-\iiint\left\{\beta_2\left(\frac{dw}{dz}+\frac{du}{dx}\right)+\gamma_3\frac{dv}{dz}+\alpha_1\frac{dv}{dx}\right\}\delta\eta\,dxdydz$$
$$-\iiint\left\{\gamma_3\left(\frac{du}{dx}+\frac{dv}{dy}\right)+\alpha_1\frac{dw}{dx}+\beta_2\frac{dw}{dy}\right\}\delta\zeta dxdydz.$$

If we assume

$$\aleph=\alpha_1\frac{d\xi}{dx}+\beta_1\frac{d\eta}{dy}+\gamma_1\frac{d\zeta}{dz},$$
$$\beth=\alpha_2\frac{d\xi}{dx}+\beta_2\frac{d\eta}{dy}+\gamma_2\frac{d\zeta}{dz},$$
$$\gimel=\alpha_3\frac{d\xi}{dx}+\beta_3\frac{d\eta}{dy}+\gamma_3\frac{d\zeta}{dz},$$

we shall have

$$(e)\ \tfrac{1}{2}\,\delta\iiint 2\,(u\aleph + v\beth + w\gimel)\,dxdydz = \iint(\beth\delta\zeta + \gimel\delta\eta)\,dydz$$
$$+ \iint(\gimel\delta\xi + \aleph\delta\zeta)\,dxdz$$
$$+ \iint(\aleph\delta\eta + \delta\xi)\,\beth dxdy$$
$$+ \iint(\alpha_1 u + \alpha_2 v + \alpha_3 w)\,\delta\xi\,dydz$$
$$+ \iint(\beta_1 u + \beta_2 v + \beta_3 w)\,\delta\eta\,dxdz$$
$$+ \iint(\gamma_1 u + \gamma_2 v + \gamma_3 w)\,\delta\zeta\,dxdy$$
$$-\iiint\left\{\left(\frac{d\beth}{dz} + \frac{d\gimel}{dy}\right)\delta\xi + \left(\frac{d\gimel}{dx} + \frac{d\aleph}{dz}\right)\delta\eta + \left(\frac{d\aleph}{dy} + \frac{d\beth}{dx}\right)\delta\zeta\right\}dxdydz$$
$$-\iiint\left\{\left(\alpha_1\frac{du}{dx} + \alpha_2\frac{dv}{dx} + \alpha_3\frac{dw}{dx}\right)\delta\xi\right.$$
$$+\left(\beta_1\frac{du}{dy} + \beta_2\frac{dv}{dy} + \beta_3\frac{dw}{dy}\right)\delta\eta$$
$$\left.+\left(\gamma_1\frac{du}{dz} + \gamma_2\frac{dv}{dz} + \gamma_3\frac{dw}{dz}\right)\delta\zeta\right\}dxdydz.$$

The sum of these five quantities, added together, is the expression for the variation $\iiint\delta V_1 dxdydz$. Hence we shall have

$$\iiint\delta V_1 dxdydz = \Delta - \iiint(P\delta\xi + Q\delta\eta + R\delta\zeta)\,dxdydz\ldots(10),$$

where Δ represents the sum of all the double integrals, and will belong essentially to the limits, and the quantities P, Q, R give the differential equations of equilibrium and motion, and have the following values:

$$P = A\frac{d^2\xi}{dx^2} + N\frac{d^2\xi}{dy^2} + M\frac{d^2\xi}{dz^2} + 2\left(\alpha_1\frac{d^2\xi}{dydz} + \alpha_2\frac{d^2\xi}{dxdz} + \alpha_3\frac{d^2\xi}{dxdy}\right)$$
$$+\alpha_3\frac{d^2\eta}{dx^2} + \beta_3\frac{d^2\eta}{dy^2} + \gamma_3\frac{d^2\eta}{dz^2} + 2\left(\alpha_1\frac{d^2\eta}{dxdz} + N\frac{d^2\eta}{dxdy} + \beta_2\frac{d^2\eta}{dydz}\right)$$
$$+\alpha_2\frac{d^2\zeta}{dx^2} + \beta_2\frac{d^2\zeta}{dy^2} + \gamma_2\frac{d^2\zeta}{dz^2} + 2\left(\alpha_1\frac{d^2\zeta}{dxdy} + M\frac{d^2\zeta}{dxdz} + \gamma_3\frac{d^2\zeta}{dydz}\right).$$

$$Q = B\frac{d^2\eta}{dy^2} + L\frac{d^2\eta}{dz^2} + N\frac{d^2\eta}{dx^2} + 2\left(\beta_2\frac{d^2\eta}{dxdz} + \beta_3\frac{d^2\eta}{dxdy} + \beta_1\frac{d^2\eta}{dydz}\right)$$
$$+\alpha_1\frac{d^2\zeta}{dx^2} + \beta_1\frac{d^2\zeta}{dy^2} + \gamma_1\frac{d^2\zeta}{dz^2} + 2\left(\beta_2\frac{d^2\zeta}{dxdy} + L\frac{d^2\zeta}{dydz} + \gamma_3\frac{d^2\zeta}{dxdz}\right)$$
$$+\alpha_3\frac{d^2\xi}{dx^2} + \beta_3\frac{d^2\xi}{dy^2} + \gamma_3\frac{d^2\xi}{dz^2} + 2\left(\beta_2\frac{d^2\xi}{dydz} + N\frac{d^2\xi}{dxdy} + \alpha_1\frac{d^2\xi}{dxdz}\right).$$

$$R = C\frac{d^2\zeta}{dz^2} + M\frac{d^2\zeta}{dx^2} + L\frac{d^2\zeta}{dy^2} + 2\left(\gamma_3\frac{d^2\zeta}{dxdy} + \gamma_1\frac{d^2\zeta}{dydz} + \gamma_2\frac{d^2\zeta}{dxdz}\right)$$

$$+ \alpha_2\frac{d^2\xi}{dx^2} + \beta_2\frac{d^2\xi}{dy^2} + \gamma_2\frac{d^2\xi}{dz^2} + 2\left(\gamma_3\frac{d^2\xi}{dydz} + M\frac{d^2\xi}{dxdz} + \alpha_1\frac{d^2\xi}{dxdy}\right)$$

$$+ \alpha_1\frac{d^2\eta}{dx^2} + \beta_1\frac{d^2\eta}{dy^2} + \gamma_1\frac{d^2\eta}{dz^2} + 2\left(\gamma_3\frac{d^2\eta}{dxdz} + L\frac{d^2\eta}{dydz} + \beta_2\frac{d^2\eta}{dxdy}\right).$$

The general equation of equilibrium and motion, corresponding to the function V_1, will be

$$\iiint \varepsilon(X\delta\xi + Y\delta\eta + Z\delta\zeta)\, dxdydz$$
$$= \Delta - \iiint(P\delta\xi + Q\delta\eta + R\delta\zeta)\, dxdydz,$$

where $dm = \varepsilon dxdydz$; therefore the equations of equilibrium are

$$-\varepsilon X = P, \quad -\varepsilon Y = Q, \quad -\varepsilon Z = R \ldots\ldots (11).$$

These equations are the equations of equilibrium of a solid body expressed in their most general form, without making any supposition as to the arrangement of the molecules in the body, and supposing the force in action between any two molecules to be a function, as well of the *direction* of the line joining them, as of the *length* of that line; which is the most general conception of a crystalline structure.

Note. As the continuation of this paper will not appear until the next No. of the *Cambridge and Dublin Mathematical Journal,* I shall here mention some of the results at which I have arrived. Having simplified the function V_1, I have integrated the equations of motion by means of a particular integral, which is general enough to give, by means of the relations among the constants, many geometrical properties of the motion of elastic solids. In examining the propagation of waves through the medium, I have used the *surface of wave-slowness,* which is of the sixth degree, and possesses nodes in its principal planes, which give rise to a theory of conical refraction of the vibrations of solids, somewhat analogous to the corresponding case of light. In the case of homogeneous, uncrystalline bodies, the whole theory becomes exceedingly simple.

Trinity College, Dublin, March, 1846.

(*To be continued.*)

ON A FORMULA FOR DETERMINING THE OPTICAL CONSTANTS OF DOUBLY REFRACTING CRYSTALS.

By G. G. Stokes, M.A., Fellow of Pembroke College.

In order to explain the object of this formula, it will be necessary to allude to the common method of determining the optical constants. Two plane faces of the crystal are selected, which are parallel to one of the axes of elasticity; or if such do not present themselves, they are obtained artificially by grinding. A pencil of light is transmitted across these faces in a plane perpendicular to them both, as in the case of an ordinary prism. This pencil is by refraction separated into two, of which one is polarized in the plane of incidence, and follows the ordinary law of refraction, while the other is polarized in a plane perpendicular to the plane of incidence, and follows a different law. It will be convenient to call these pencils respectively the *ordinary* and the *extraordinary*, in the case of biaxal, as well as uniaxal crystals. The minimum deviation of the ordinary pencil is then observed, and one of the optical constants, namely that which relates to the axis of elasticity parallel to the refracting edge, is thus determined by the same formula which applies to ordinary media. This formula will also give one of the other constants, by means of the observation of the minimum deviation of the extraordinary pencil, in the particular case in which one of the principal planes of the crystal bisects the angle between the refracting planes: but if this condition be not fulfilled it will be necessary to employ either two or three prisms, according as the crystal is uniaxal or biaxal, to determine all the constants. The extraordinary pencil, however, need not in any case be rejected, provided only a formula be obtained connecting the minimum deviation observed with the optical constants. It will thus be possible to determine all the constants with a smaller number of prisms; the necessity of using artificial faces may often be obviated; or if two faces are cut as nearly as may be equally inclined to one of the axes of elasticity lying in the plane of incidence, or one cut face is used with a natural face, the errors of cutting may be allowed for.

Let AEB be a section of the prism by the plane of refraction, (the reader will have no difficulty in drawing a figure,) E being the refracting edge; let i be the refracting angle; OA, OB, OC the directions of the axes of elasticity, O being any point within the prism, the two former of these lines

[redacted]

being in, and the latter perpendicular to, the plane of refraction; a, b, c the optical constants referring to them, that is, according to Fresnel's theory, the velocities of propagation of waves in which the vibrations are parallel to the three axes respectively. Everything being symmetrical with respect to the plane of incidence, we need only consider what takes place in that plane. This plane will cut the wave surface in a circle of radius c, and an ellipse whose semiaxes are a along oB and b along oA. We have only got to consider the ellipse, since it is it that determines the direction of the extraordinary ray. The form of the crystal will very often make known the directions of the axes of elasticity. Supposing these directions known, let α, β denote the inclinations of oA, oB to the produced parts of EA, EB respectively; α, β and i being of course connected by the equation $\alpha + \beta = \frac{\pi}{2} + i$.

Let ϕ, ψ be the angles of incidence and emergence, the light being supposed incident on the face EA; ϕ' the inclination of the refracted wave to EA, ψ' its inclination to EB, D the deviation, v the velocity of the wave within the crystal, u its velocity in the outer medium, which may be supposed to be either air, or a liquid of known refractive power. Then we have

$$D = \phi + \psi - i\,^* \ldots\ldots\ldots\ldots\ldots(1),$$

$$\phi' + \psi' = i \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$v \sin\phi = u \sin\phi' \ldots\ldots\ldots\ldots\ldots(3),$$

$$v \sin\psi = u \sin\psi' \ldots\ldots\ldots\ldots\ldots(4),$$

$$v^2 = a^2 \cos^2(\alpha - \phi') + b^2 \sin^2(\alpha - \phi') \ldots\ldots(5).$$

From (2), (3), (4),

$$u \sin\psi' = v \sin\psi = u \sin(i - \phi') = u \sin i \cos\phi' - v \cos i \sin\phi;$$

$$\therefore \quad \cos\phi' = \frac{v}{u \sin i}(\sin\psi + \cos i \sin\phi);$$

and
$$\sin\phi' = \frac{v}{u \sin i} \sin i \sin\phi:$$

substituting in (5),

$$u^2 \sin^2 i = a^2 \{\cos\alpha(\sin\psi + \cos i \sin\phi) + \sin\alpha \sin i \sin\phi\}^2$$
$$+ b^2 \{\sin\alpha(\sin\psi + \cos i \sin\phi) - \cos\alpha \sin i \sin\phi\}^2,$$

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* I am indebted to the Rev. P. Frost for the suggestion of employing equations (1) .... (4), rather than making use of the ellipse in which the wave surface is cut by the plane of incidence.

or $\quad u^2 \sin^2 i = a^2 (\cos \alpha \sin \psi + \sin \beta \sin \phi)^2$
$$+ b^2 (\sin \alpha \sin \psi + \cos \beta \sin \phi)^2 \ldots . (6),$$
the relation between ϕ and ψ. Putting $\psi - \phi = \theta$, and taking account of (1), (6) becomes

$$2u^2 \sin^2 i = \{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha\} \{1 - \cos (D + i + \theta)\}$$
$$+ \{a^2 \sin^2 \beta + b^2 \cos^2 \beta\} \{1 - \cos (D + i - \theta)\}$$
$$+ 2 (a^2 \cos \alpha \sin \beta + b^2 \sin \alpha \cos \beta) \{\cos \theta - \cos (D + i)\},$$

or
$$F \cos \theta + G \sin \theta + H = 0 \ldots \ldots \ldots . (7),$$
where

$$F = a^2 \{(\cos^2 \alpha + \sin^2 \beta) \cos (D + i) - 2 \cos \alpha \sin \beta\}$$
$$+ b^2 \{(\sin^2 \alpha + \cos^2 \beta) \cos (D + i) - 2 \sin \alpha \cos \beta\},$$
$$G = (a^2 - b^2) (\sin^2 \beta - \cos^2 \alpha) \sin (D + i),$$
$$H = 2u^2 \sin^2 i - a^2 (\cos^2 \alpha + \sin^2 \beta - 2 \cos \alpha \sin \beta \cos (D + i)\}$$
$$- b^2 \{\sin^2 \alpha + \cos^2 \beta - 2 \sin \alpha \cos \beta \cos (D + i)\}.$$

Now when D, regarded as a function of θ, is a maximum or minimum $\dfrac{dD}{d\theta} = 0$, whence from (7)
$$- F \sin \theta + G \cos \theta = 0;$$
and eliminating θ from this equation and (7), we have
$$F^2 + G^2 = H^2.$$
Putting for F, G and H their values, and reducing, this equation becomes

$$\sin^2 (D + i) a^2 b^2 - \{\cos^2 \alpha + \sin^2 \beta - 2 \cos (D + i) \cos \alpha \sin \beta\} u^2 a^2$$
$$- \{\sin^2 \alpha + \cos^2 \beta - 2 \cos (D + i) \sin \alpha \cos \beta\} u^2 b^2 + \sin^2 i . u^4 = 0 . . (8).$$

This equation will be rendered more convenient for numerical calculation by replacing products and powers of sines and cosines by sums and differences. Treated in this manner, the equation becomes

$$\text{versin } 2 (D + i) a^2 b^2 - (A + B) u^2 a^2 - (A - B) u^2 b^2$$
$$+ \text{versin } 2i . u^4 = 0 . \ldots (9),$$

where $\quad A = \text{versin } D + \text{versin } (D + 2i),$
$$B = \cos 2\alpha - \cos 2\beta - \cos (D + 2\alpha) + \cos (D + 2\beta).$$

If the principal plane AOC of the crystal bisects the angle between the refracting faces, we have $\alpha = \dfrac{i}{2}$, $\beta = \dfrac{\pi}{2} + \dfrac{i}{2}$, whence from (8), putting $D + i = \Delta$,
$$\left(a^2 \sin^2 \frac{\Delta}{2} - u^2 \sin^2 \frac{i}{2}\right) \left(b^2 \cos^2 \frac{\Delta}{2} - u^2 \cos^2 \frac{i}{2}\right) = 0.$$

The former of these factors is evidently that which corresponds to the problem; the latter corresponds to refraction through a prism having its faces parallel to those of the actual prism, and having its refracting angle supplemental to i. We have therefore

$$a = u\,\frac{\sin\dfrac{i}{2}}{\sin\dfrac{\Delta}{2}};$$

so that the constant a is given by the same formula that applies to ordinary media, as it should.

If the refracting faces are perpendicular to the axes of elasticity which lie in the plane of incidence, the formula (8) or (9) takes a very simple form. In this case we have $\alpha = \beta = i = \dfrac{\pi}{2}$, and therefore

$$\cos^2 D.a^2b^2 - u^2a^2 - u^2b^2 + u^4 = 0.$$

Mathematically speaking, one prism would be sufficient for determining the three constants a, b, c. For c would be determined by means of the ordinary pencil; and by observing the extraordinary pencil with the crystal in air, and again with the crystal in some liquid, we should have two equations of the form (8), by combining which we should obtain a^2 and b^2 by the solution of a quadratic equation. But since a is usually nearly equal to b, it is evident that the course of the extraordinary ray within the crystal would be nearly the same in the two observations, being in each case inclined at nearly equal angles to the refracting faces, and consequently the errors of observation would be greatly multiplied in the result. Even if a differed greatly from b, only one of these constants could be accurately determined in this manner if the refracting angle were nearly bisected by a principal plane. But two prisms properly chosen appear amply sufficient for determining accurately the three constants by the method of minimum deviations, even should neither prism have its angle exactly bisected by a principal plane of the crystal.

It is not necessary to observe the deviation when it is a minimum, as Professor Miller has remarked to me, since the angle of incidence may be measured very accurately by moving the telescope employed till the luminous slit, seen directly, appears on the cross wires, and then turning it till the slit, seen by reflection at the first face of the prism, again appears on the cross wires, the prism meanwhile remaining fixed.

The angle through which the telescope has been turned is evidently the supplement of twice the angle of incidence. If this method of observation be adopted, ϕ, D, and i will be known by observation, whence ψ will be got immediately from (1). Thus all the coefficients in (6) will be known quantities, and this equation furnishes a very simple relation between a and b. The coefficients may easily be calculated numerically by treating them like those in equation (8), or else by employing subsidiary angles.

SUR LA REPRÉSENTATION GÉOMÉTRIQUE DES FONCTIONS ELLIPTIQUES DE PREMIÈRE ESPÈCE.

Par J. Alfred Serret, (de Paris).

§. I. On sait que la fonction Elliptique de première espèce au module $\sqrt{\frac{1}{2}}$ est *identique* à l' arc de Lemniscate considéré comme fonction du rayon vecteur central, ou d' une fonction convenablement déterminée de ce rayon vecteur; mais malgré les travaux de Legendre cette propriété n' avait pu être généralisée jusqu' ici, et l' en ne connaissait aucune autre courbe algébrique dont l' arc considéré comme fonction d' une coordonée convenable fût identiquement representé par une fonction elliptique de première espèce.

J'ai résolu le premier ce problème, dans plusieurs mémoires presentés à l' Academie des Sciences et qui ont été publiés depuis dans le *Journal des Mathématiques* de M. Liouville. Voici en peu de mots quel a été l' esprit de mes nouvelles recherches.

Les coordonnées rectangulaires de la Lemniscate sont exprimables rationnellement en fonction de l' amplitude de la fonction elliptique qui représente l' arc; si en effet l' on pose

$$x = a\sqrt{2}\,\frac{z + z^3}{1 + z^4}, \quad y = a\sqrt{2}\,\frac{z - z^3}{1 + z^4};$$

on aura
$$\sqrt{(dx^2 + dy^2)} = 2a\,\frac{dz}{\sqrt{(1 + z^4)}},$$

et il est bien facile de s' assurer que les équations précédentes sont relatives à la Lemniscate: cette remarque m' a conduit naturellement a chercher les solutions *réelles et rationnelles* que peut admettre une équation de la forme

$$dx^2 + dy^2 = Zdz^2;$$

La discussion de cette équation se trouve dans mon premier mémoire, mais je me bornerai à rappeler les résultats auxquels je suis parvenu relativement à la représentation géométrique des fonctions elliptiques de première espèce: dans ce cas, je fais $z = \frac{C^2}{(z^2 - a^2)(z^2 - \alpha^2)}$, C étant une constante reelle, a et α deux constantes essentiellement imaginaires et conjuguées, en sorte que l' equation à resoudre devient

$$dx^2 + dy^2 = C^2 \frac{dz^2}{(z^2 - a^2)(z^2 - \alpha^2)} \dots\dots\dots (1).$$

Il est visible que l' on satisfera à cette équation différentielle indeterminée, en posant

$$\left.\begin{aligned} x + y\sqrt{(-1)} &= Ce_{\omega}^{\sqrt{-1}} \int \frac{(z-a)^m (z+a)^n}{(z-\alpha)^{m+1}(z+\alpha)^{n+1}}\, dz, \\ x - y\sqrt{(-1)} &= Ce^{-\omega\sqrt{(-1)}} \int \frac{(z-\alpha)^m (z+\alpha)^n}{(z-a)^{m+1}(z+a)^{n+1}}\, dz \end{aligned}\right\} \dots\dots (2),$$

ou m et n sont des nombres entiers et positifs, et ω un angle réel quelconque. Pour que les intégrales précédentes soient algébriques, *il faut et il suffit* que la quantité

$$\zeta = \frac{(a+\alpha)^2}{4a\alpha},$$

soit une racine de l' equation

$$\frac{d^n \zeta^m (1-\zeta)^n}{d\zeta^n} = 0 \dots\dots\dots\dots\dots\dots (3),$$

en supposant, ce qui est permis, que m ne soit pas supérieur à n. L' équation (3) a toutes ses racines réelles positives et inférieures à 1, ce qui est un théorème d' une bien grande importance, car autrement les quantités a et α ne seraient plus conjugueés, et les équations (2) ne se changeraient plus l'une en l'autre par le changement de $\sqrt{(-1)}$ en $-\sqrt{(-1)}$; au surplus il est facile de démontrer à priori que l'equation (1) n' admet aucune solution rationnelle et réelle si a et α sont réels.

n restant indeterminé, si l'on donne successivement à m les valeurs

$$1, 2, 3, \dots\dots$$

on aura une multitude de classes de courbes renfermant chacune une infinité de courbes individuelles, dont l' arc est identiquement représenté par une fonction elliptique de première espèce; les modules des fonctions dont il s' agit ont pour carrés les racines ζ de l'équation (3).

Je me suis plus spécialement occupé des courbes de la première classe pour les quelles $m = 1$; on a dans ce cas

$$x + y\sqrt{(-1)} = Ce^{\omega\sqrt{(-1)}}\frac{(z+a)^{n+1}}{(z-\alpha)(z+\alpha)^n},$$

$$x - y\sqrt{(-1)} = Ce^{-\omega\sqrt{(-1)}}\frac{(z+\alpha)^{n+1}}{(z-a)(z+a)^n},$$

d' où $$x^2 + y^2 = C^2\frac{(z+a)(z+\alpha)}{(z-a)(z-\alpha)};$$

avec la condition $$\frac{(a+\alpha)^2}{4a\alpha} = \frac{n}{n+1},$$

et l'en en déduit par la différentiation

$$dx + \sqrt{(-1)}\,dy = Ce^{\omega\sqrt{(-1)}}\{-(a+\alpha) - n(a-\alpha)\}\frac{(z-a)(z+a)^n}{(z-\alpha)^2(z+\alpha)^{n+1}}dz,$$

$$dx - \sqrt{(-1)}\,dy = Ce^{-\omega\sqrt{(-1)}}\{-(a+\alpha) + n(a-\alpha)\}\frac{(z-\alpha)(z+\alpha)^n}{(z-a)^2(z+a)^{n+1}}dz,$$

d'ou $$ds = 2C\sqrt{(na\alpha)}\frac{dz}{\sqrt{\{(z+a)(z+\alpha)(z-a)(z-\alpha)\}}}.$$

Si r désigne le rayon vecteur $\sqrt{(x^2+y^2)}$, et si l'on suppose pour simplifier qu' en prenne pour unité le paramètre C, on aura

$$ds = 2\sqrt{\{n(n+1)}\frac{dr}{\sqrt{\{-r^4 + 2(2n+1)r^2 - 1\}}} \ldots . (4),$$

ce qui donne ce theorème remarquable: l'arc des courbes de première classe est une fonction elliptique du rayon vecteur. L'équation précédente n' est autre chose que l'équation différentielle des courbes de première classe, lesquelles ont à leur tête la Lemniscate qui correspond au cas de $n = 1$.

Mais ici se présente une remarque bien importante due à M. Liouville, et qui consiste en ce que le nombre n, que la nature de notre analyse obligeait à être entier, n'a réellement besoin que d' etre commensurable, afin que nos courbes ne cessent pas d' être algébriques, et cette observation s' applique egalement aux classes suivantes, en sorte qu' on pourra à l' aide des courbes de la première classe seulement, representer toutes les fonctions elliptiques dont les modules ont pour carrés des nombres rationnels quelconques ; au surplus on verra plus loin que les courbes de l' equation (4) sont soumises à un mode uniforme de génération, que n soit entier ou fractionnaire, ou même incommensurable.

Si θ designe la seconde coordonné polaire, l'equation (4) donnera

$$d\theta = \frac{r^2 - (2n+1)}{\sqrt{\{-r^4 + 2(2n+1)r^2 - 1\}}} \frac{dr}{r} \dots\dots (5),$$

d'ou l'on déduit

$$\int \tfrac{1}{2} r^2 \, d\theta = -\tfrac{1}{4}\sqrt{\{-r^4 + 2(2n+1)r^2 - 1\}} + \text{constante}$$

équation qui montre que les courbes de la première classe sont toutes carrables.

Ou trouve pour l'intégrale de l'équation (5)

$$\theta = \theta_0 - \lambda - (2n+1)\mu,$$

θ_0 désignant une constante arbitraire, λ et μ deux angles compris entre 0 et $\frac{\pi}{2}$ et déterminés par les équations

$$r^2 = (2n+1) + 2\sqrt{\{n(n+1)\}} \cos 2\lambda,$$

$$\frac{1}{r^2} = (2n+1) + 2\sqrt{\{n(n+1)\}} \cos 2\mu,$$

et l'en conclut aisément de là, l'équation de nos courbes en coordonnées polaires.

§ II. Je viens d'analyser d'une manière succincte les différens résultats que j'ai publiés dans mes derniers mémoires; depuis, une étude plus approfondie des formules dont je n'ai pu indiquer ici qu'une partie, m'a conduit à deux propriétés géométriques remarquables communes à toutes les courbes elliptiques de la première classe, et qui fournissent pour ces courbes un mode uniforme de génération d'une extrême élégance; mais, bien que j'aie découvert ces propriétés, comme je viens de le dire, en suivant le cours naturel de mes recherches analytiques, je préfère, me placer ici à un point de vue différent, afin que les résultats qui vont suivre deviennent entièrement indépendants de ces considérations analytiques, et soient en conséquence parfaitement compris des lecteurs qui n'auraient pas connaissance de mes recherches antérieures.

Au nouveau point de vue ou je me place, je commence par démontrer ces propriétés sur la Lemniscate, et je les généralise ensuite bien aisément.

Théorème I. Soit r le rayon vecteur issu de l'un des foyers d'une Lemniscate, dont la demie distance focale est prise pour unité, et dont le demi axe sera dèslors $\sqrt{2}$; on pourra toujours construire un triangle dont les cotés seront respectivement r, 1 et $\sqrt{2}$, car le rayon vecteur reste compris entre $\sqrt{2} - 1$ et $\sqrt{2} + 1$: cela posé, si α designe l'angle de ce

triangle opposé au coté $\sqrt{2}$, et β celui qui est opposé au coté 1, l'angle polaire θ que forme le rayon vecteur de la Lemniscate avec l'axe, sera toujours donné par l'équation

$$\cos\theta = \cos(\alpha - 2\beta),$$

Remarque. Soit O l'origine, c'est à dire l'un des foyers de la Lemniscate, et OM un rayon vecteur quelconque; construissons le triangle OMP, de telle sorte que

$$OP = 1 \text{ et } MP = \sqrt{2},$$

(ce triangle peut être fait d'un coté ou de l'autre de OM, cela importe peu en ce moment), puis imaginons que le point M décrive d'un mouvement continu la Lemniscate entière, le point P qu'on peut toujours supposer se mouvoir d'un mouvement continu, décrira deux fois la circonférence tracée de l'origine comme centre avec l'unité comme rayon.

Corollaire. Du théorème précédent qu'on démontre bien aisément, on déduit la génération suivante de la Lemniscate.

Soit OMP *un triangle dont le sommet* O *est fixe dont les cotés mobiles* OP *et* MP *sont constamment égaux, l'un à* 1, *l'autre à* $\sqrt{2}$; *si l'on fait varier ce triangle de telle sorte que le cosinus de l'angle que fait le coté variable* OM *avec une droite fixe soit constamment égal au cosinus de l'angle* MOP – 2OMP, *le point* M *engendrera une Lemniscate dont* O *sera un foyer, et la droite fixe l'axe.*

Théorème II. Soit comme précédemment OM un rayon vecteur de la Lemniscate, et construisons le triangle OMP de part et d'autre du rayon OM, la tangente en M à la Lemniscate passera constamment par le centre du cercle circonscrit à l'un de ces triangles; si en outre on considère spécialement celui de ces triangles pour le quel cette propriété a lieu, et qu'en vertu du théorème I. on le fasse servir à la description de la Lemniscate par un *mouvement continu*, cette propriété se conservera pour tous les points de la courbe.

Remarque. Ce Théorème donne un moyen très simple de construire la tangente en un point de la courbe, car il suffira de construire le triangle correspondant à ce point, et de le joindre au centre du cercle circonscrit au triangle, mais il conduit aussi à un nouveau mode de génération pour la Lemniscate.

Théorème III. Soit OMP un triangle dont le sommet O est fixe, et dont les cotés OP et MP sont constamment égaux, l'un à 1 l'autre à $\sqrt{2}$, le sommet M décrira une lemniscate, si son deplacement infiniment petit MM' a constamment 1 suivant le rayon CM du cercle circonscrit au triangle ?.

Remarque. Le triangle dont nous venons de parler, joue, comme on voit, un rôle assez important dans la théorie de la lemniscate, aussi je ne crois pas inutile de mentionner une dernière propriété, qui consiste en ce que l'aire de ce triangle et l'aire du secteur de la courbe ont la même différentielle.

§. III. Soit maintenant n un nombre entier ou fractionnaire, ou même incommensurable et construisons le triangle OMP tel que

$$OP = \sqrt{n} \quad \text{et} \quad MP = \sqrt{(n+1)}.$$

Puis imaginons que le sommet O restant fixe, le triangle varie de telle sorte que le cosinus de l'angle θ formé par le seul coté variable OM avec une droite fixe, soit constamment égal au cosinus de l'angle

$$n.MOP - (n+1)\,OMP,$$

le point M engendrera une courbe (algébrique si n est commensurable) dont l'arc sera une fonction elliptique du rayon vecteur, réductible au module $\sqrt{\left(\frac{n}{n+1}\right)}$,

Soit en effet $MOP = \alpha$, $OMP = \beta$, l'équation de la courbe résultera de l'éliminations de α et β entre

$$\cos\theta = \cos\{n\alpha - (n+1)\,\beta\}$$

$$\left\{\begin{aligned}\cos\alpha &= \frac{r^2-1}{2r\sqrt{n}},\\ \cos\beta &= \frac{r^2+1}{2r\sqrt{(n+1)}},\end{aligned}\right. \quad \text{d'où} \quad \left\{\begin{aligned}\sin\alpha &= \frac{\Delta}{2r\sqrt{n}},\\ \sin\beta &= \frac{\Delta}{2r\sqrt{(n+1)}},\end{aligned}\right.$$

en faisant pour abréger

$$\Delta = \sqrt{\{-r^4 + 2(2n+1)\,r^2 - 1\}}.$$

cela posé on déduit par la différentiation

$$\pm\, d\theta = n d\alpha - (n+1)\,d\beta,$$

et
$$d\alpha = -\frac{r^2+1}{\Delta}\,\frac{dr}{r}, \quad d\beta = -\frac{r^2-1}{\Delta}\,\frac{dr}{r},$$

d'où
$$d\theta = \frac{r^2-(2n+1)}{\Delta}\,\frac{dr}{r},$$

et par suite
$$ds = 2\sqrt{\{n(n+1)}\,\frac{dr}{\Delta}.$$

Des équations précédentes on déduit encore les formules suivantes qu'il convient de remarquer;

$$ds = -\sqrt{n}\,\frac{d\alpha}{\cos\beta}, \quad ds = -\sqrt{(n+1)}\,\frac{d\beta}{\cos\alpha}:$$

on a d' ailleurs en posant $k = \sqrt{\left(\frac{n}{n+1}\right)}$,

$$\sin \beta = K \sin \alpha, \quad \text{d'ou} \quad \cos \beta = \sqrt{(1 - K^2 \sin^2 \alpha)},$$

donc
$$ds = -\sqrt{n}\, \frac{d\alpha}{\sqrt{(1 - K^2 \sin^2 \alpha)}},$$

et l' arc de courbe compté à partir du point de l'axe polaire qui correspond à $\alpha = 0$ ou $r = \sqrt{(n+1)} \pm \sqrt{n}$, sera exprimé par l'intégral elliptique au module K

$$\sqrt{n} \int_0^\alpha \frac{d\alpha}{\sqrt{(1 - K^2 \sin^2 \alpha)}},$$

ce qu'il s'agissait de démontrer.

On voit aisément que dans le cas de $n = 1$, la courbe dont nous parlons se confond avec la lemniscate de Bernouilli et l'on a ainsi la démonstration du théorème I, du paragraphe II.

Il est bon de remarquer encore que l'aire du triangle générateur OMP est $\frac{\Delta}{4}$, et l'on trouve aisément

$$\int \tfrac{1}{2} r^2 d\theta = \frac{\Delta}{4} + \text{constant},$$

d'où l'on conclut que l'aire du secteur du courbe compté à passer de l'axe polaire est toujours égale à l'aire du triangle générateur.

Je passe maintenant à l'examen de la seconde propriété de ces courbes remarquables : on a dans le triangle OMP

$$r^2 = 2n + 1 + 2\sqrt{\{n(n+1)\}} \cos(\alpha + \beta),$$

d'où
$$\cos(\alpha + \beta) = \frac{r^2 - (2n+1)}{2\sqrt{\{n(n+1)}} = \frac{rd\theta}{ds},$$

et
$$\sin(\alpha + \beta) = \frac{\Delta}{2\sqrt{\{n(n+1)}} = \frac{dr}{ds},$$

d' où l' on conclut que l' inclinaison de la normale sur le rayon vecteur est précisément egale à $\alpha + \beta$, si donc on fait au point M un angle $PMN = MOP$, MN sera la normale au point M de la courbe lequel correspond à la position OMP du triangle générateur. D'ailleurs le point O se trouve nécessairement sur le segment capable de l'angle PMN, que l'on décrivait sur MP, ce qui montre que MN est tangente au cercle circonscrit au triangle générateur, et si C est le centre du cercle circonscrit, le rayon MC sera précisément la tangente à la courbe.

De ce qui precede résulte le mode de génération suivant pour les courbes elliptiques :—*Si le triangle OMP varie de telle manière que le sommet O reste fixe, que les deux cotés OP et MP soient constamment égaux le premier à $\surd n$, le second à $\surd(n+1)$, et que de plus le déplacement, infiniment petit MM′ du point M ait lieu à chaque instant suivant la droite qui joint ce point au centre du cercle circonscrit, au triangle générateur, le point M engendrera la courbe elliptique que correspond au nombre n.*

On a ainsi en particulier la démonstration des théorèmes II. et III. du paragraphe II.; lesquels sont relatifs seulement à la lemniscate.

On obtient aisément l'expression du rayon de courbure; soit ϵ l'angle que fait la normal avec l'axe polaire, on aura

$$\epsilon = \theta - (a + \beta)$$

car $(a + \beta)$ est l'angle de la normal avec le rayon vecteur; on a, en différentiant, l'angle de contingence $d\epsilon$

$$d\epsilon = d\theta - da - d\beta = \frac{3r^2 - (2n+1)}{\Delta}\,\frac{dr}{r},$$

et pour le rayon de courbure

$$\frac{ds}{d\epsilon} = R = \frac{2r\surd\{n(n+1)\}}{3r^2 - (2n+1)},$$

§ IV. Les courbes dont je viens de parler sont celles que j'ai désignées sous le nom de courbes elliptiques de la première classe, dans une note insérée au Journal de M. Liouville (t. x. 1846); on voit qu'il s'en trouve une dont l'arc sera identique à telle fonction elliptique de première espèce, que l'on voudra. Les courbes de la troisième classe sont définies par l'équation

$$x + y\surd(-1) = ce^{\omega\surd-1}\int\frac{(z-a)^m\,(z+a)^n}{(z-a)^{m+1}\,(z+a)^{n+1}}\,dz$$

où la quantité $\dfrac{(a+a)^2}{4aa} = \zeta$, est une racine de l'équation

$$\frac{d^n\zeta^m(1-\zeta)^n}{d\zeta^n} = 0.$$

Si l'on a $n = m =$ un nombre entier impair $2\mu + 1$, cette équation aura toujours pour racine $\frac{1}{2}$, en sorte que toutes les classes de rang impair comprendront une courbe dont l'arc sera identique à l'arc de lemniscate. On a dans ce cas $a^2 = -a^2 = \surd(-1)$, et ces nouvelles courbes sont définies par l'équation

$$x + y\surd(-1) = ce^{\omega\surd(-1)}\int\frac{\{z^2 - \surd(-1)\}^{2\mu+1}}{(z^2 + \surd-1)^{2\mu+2}}\,dz,$$

Il est facile de voir *a posteriori* que l'intégrale du second membre est algébrique, car en posant

$$\frac{2z\sqrt[4]{(-1)}}{z^2+\sqrt{(-1)}}=t$$

on trouve $\quad x+y\sqrt{(-1)}=-\dfrac{ce^{\omega i}}{\sqrt[4]{(-1)}}\int(1-t^2)^{\mu}\,dt.$

Quant à l'arc de cette courbe, on aura évidemment pour sa différentielle $\quad \sqrt{(dx^2+dy^2)}=c\,\dfrac{dz}{\sqrt{(1+z^4)}},$

quel que soit le nombre entier μ.

Paris, le 28 *Mars*, 1846.

ON THE PRINCIPAL AXES OF A SOLID BODY.

By William Thomson.

(*Continued from* § 12, p. 133.)

§§ 13, 16. Second method of conducting part of the preceding investigation. § 14. Conditions for the equality of two of the principal moments of inertia. § 15. Remarks on the conditions for the existence of principal axes in more than one of the coordinate planes. § 17. Comparison of the different formulæ. § 18. Formulæ for the determination of the principal axes and principal moments of inertia relative to different points of a body. § 19. Geometrical representation of the preceding results by means of confocal surfaces of the second order. §§ 20...22. Equimomental surface and curves.

In the second method alluded to, either of the forms of the equations of condition for principal axes given above may be made use of; but, on account of the application which is to be made, it will be more convenient, in what follows, to employ the first form (equations (8), § 6).

13. Let

$$\left.\begin{array}{lll} A'=gh, & B'=hf, & C'=fg \\ F=f^2+\alpha, & G=g^2+\beta, & H=h^2+\gamma \end{array}\right\}\dots\dots(a),$$

from which we deduce

$$\left.\begin{array}{lll} f^2=\dfrac{B'C'}{A'}, & g^2=\dfrac{C'A'}{B'}, & h^2=\dfrac{A'B'}{C'} \\ \alpha=F-\dfrac{B'C'}{A'}, & \beta=G-\dfrac{C'A'}{B'}, & \gamma=H-\dfrac{A'B'}{C'} \end{array}\right\}\dots\dots(b).$$

These latter equations (since A', B', C' are positive) shew that the values of the constants f, g, h, α, β, γ, deduced from

the given constants, are necessarily real. By means of these assumptions, and by dividing the first of equations (8) by mn, the second by nl, the third by lm, we reduce them to the two following:

$$\frac{f(fl+gm+hn)+\alpha l}{l}=\frac{g(fl+gm+hn)+\beta m}{m}$$
$$=\frac{h(fl+gm+hn)+\gamma n}{n}\;\ldots\ldots(15).$$

If we put $\qquad S=fl+gm+hn\;\ldots\ldots\ldots\ldots\ldots\;(16),$
and denote each member of equations (15) by K, we find

$$l=\frac{Sf}{K-\alpha},\quad m=\frac{Sg}{K-\beta},\quad n=\frac{Sh}{K-\gamma}\;\ldots\ldots(17);$$

and therefore (except in the case of $S=0$, when some of the principal axes are indeterminate), we have, by (16),

$$\frac{f^2}{K-\alpha}+\frac{g^2}{K-\beta}+\frac{h^2}{K-\gamma}=1\;\ldots\ldots\ldots(18).$$

This equation determines three real values for K, (see First Series, vol. IV. p. 229), one of which lies between γ and β, another between β and α, and the third between α and ∞ * (α, β, γ being supposed to be in order of descending magnitude). Any one of these values, substituted for K in (15) give values of the ratios $l:m:n$, which fix the position of a principal axis. Hence there are three, and only three, principal axes through any point. These may be shewn to form a rectangular system, in the following manner, which is quite similar to a method given in the *Mathematical Journal* (First Series, vol. III. p. 291), for demonstrating the perpendicularity of two lines in space, in a corresponding case.

Let K_1, K_2 be two roots of the cubic (18), and (l_1, m_1, n_1) (l_2, m_2, n_2) the corresponding principal axes. Substituting for K the values K_1 and K_2 successively, in (18), we obtain, by subtraction,

$$(K_1-K_2)\left\{\frac{f^2}{(K_1-\alpha)(K_2-\alpha)}+\frac{g^2}{(K_1-\beta)(K_2-\beta)}+\frac{h^2}{(K_1-\gamma)(K_2-\gamma)}\right\}$$
$$=0.$$

Unless K_1 be equal to K_2 the second factor must vanish, and therefore, by (17),

$$l_1l_2+m_1m_2+n_1n_2=0;$$

which shews that the principal axes corresponding to any two different roots of the cubic are at right angles. Hence if the

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* If one of the quantities f, g, or h vanishes, there will be a root equal to the corresponding quantity α, β, or γ.

cubic has three unequal roots, the three principal axes form a rectangular system.

14. If two roots of the cubic equation be equal, only one principal axis will be determinate, and every line through the origin, perpendicular to it, will, as may be easily shewn, be also a principal axis, and in this case the body will, with respect to dynamics, be in the same condition as if it were symmetrical round an axis. Now if two roots be equal, each must be equal to one of the quantities α, β, γ, on account of the limits stated above. Hence, and by clearing the equation of fractions, we find that if two roots are equal the conditions

$$\alpha = \beta = \gamma;$$

or, in terms of the original coefficients,

$$F - \frac{B'C'}{A'} = G - \frac{C'A'}{B'} = H - \frac{A'B'}{C'}$$

must in general be satisfied.* The same conditions may be found by equating to zero the expression, in terms of the coefficients, for the product of the squares of the differences of the roots of the cubic equation, which has been given in a very remarkable form, as the sum of seven squares, by Kummer (*Crelle's Journal*, vol. XXI. p. 74). If we employ the quantities $\alpha, \beta, \gamma, f, g, h$, instead of the six coefficients F, G, H, A', B', C', and denote, by K_1, K_2, K_3, the roots of the cubic, the result which he has obtained may be written as follows:†

$$\begin{aligned}
&(K_2 - K_3)^2 (K_3 - K_1)^2 (K_1 - K_2)^2 \\
&\qquad = 15 f^2 g^2 h^2 \{f^2 (\beta - \gamma)^2 + g^2 (\gamma - \alpha)^2 + h^2 (\alpha - \beta)^2\} \\
&+ g^2 h^2 \{f^2 (2\alpha - \beta - \gamma) + 2g^2 (\gamma - \alpha) + 2h^2 (\beta - \alpha) - 2 (\gamma - \alpha)(\beta - \alpha)\}^2 \\
&+ h^2 f^2 \{g^2 (2\beta - \gamma - \alpha) + 2h^2 (\alpha - \beta) + 2f^2 (\gamma - \beta) - 2(\alpha - \beta)(\gamma - \beta)\}^2 \\
&+ f^2 g^2 \{h^2 (2\gamma - \alpha - \beta) + 2f^2 (\beta - \gamma) + 2g^2 (\alpha - \gamma) - 2(\alpha - \gamma)(\beta - \alpha)\}^2 \\
&+ \{[f^2 (f^2 + 2\alpha - \beta - \gamma) + g^2 h^2] (\beta - \gamma) \\
&+ [g^2 (g^2 + 2\beta - \gamma - \alpha) + h^2 f^2] (\gamma - \alpha) + [h^2 (h^2 + 2\gamma - \alpha - \beta) + f^2 g^2] (\alpha - \beta)\}^2.
\end{aligned}$$

* If however (two of the quantities A', B', C' beingequal to nothing) f, g, or h vanishes, the equation will have equal roots if a second root be equal to the root α, β, or γ implied by this circumstance. Thus if $B' = 0$ and C', which give $f = 0$, we have, as the additional condition for equal roots,

$$\frac{g^2}{\alpha - \beta} + \frac{h^2}{\alpha - \gamma} = 1,$$

or, which is equivalent, $A'^2 = (F - G)(F - H)$.

† It has been shewn by Borchhardt (*Crelle*, XXX. p. 38) that Kummer's result is the particular case of a property of the equation (of the n^{th} degree) which occurs in the reduction of homogeneous functions of the second order (of n variables).

Since all the quantities $\alpha, \beta, \gamma, f, g, h$ are, in the actual problem, whether of the reduction of the general equation of the second degree, or of the determination of principal axes of a solid, necessarily real, each of the seven squares must vanish, if the sum vanishes. Hence we obtain seven equations as the conditions for the equality of two of the roots of the cubic, which, the special cases of any of the quantities f, g, h vanishing being excluded, are equivalent to the two distinct equations

$$\alpha = \beta = \gamma.$$

If imaginary values of the coefficients were admissible, one condition would of course be sufficient.

15. In the first part of this paper (§ 10) it was shewn that the condition for there being a principal axis in the plane of (xy) is

$$B'(FA' - B'C') - A'(GB' - C'A') = 0,$$

(which is the same as (a) § 10, since $F - G = B - A$). Each member of this equation may be divided by $A'B'$, unless either A' or B' vanishes, in which case, to satisfy the equation, another also of the three quantities A', B', C' must vanish, and one of the axes of coordinates is a principal axis. Hence, if none of the axes of coordinates be a principal axis, the condition that there may be a principal axis in the plane of (xy) is

$$F - \frac{B'C'}{A'} = G - \frac{C'A'}{B'} \;\ldots\ldots\ldots\ldots (a).$$

Similarly the condition that there may be a principal axis in the plane of yz, is

$$G - \frac{C'A'}{B'} = H - \frac{A'B'}{C'} \;\ldots\ldots\ldots\ldots (b).$$

If there be a principal axis in each of the planes (xy) and (yz), equations (a) and (b) will hold simultaneously, and therefore the conditions are

$$F - \frac{B'C'}{A'} = G - \frac{C'A'}{B'} = H - \frac{A'B'}{C'} \;\ldots\ldots (c).$$

From the symmetry of the three members of these equations we infer that, if there be a principal axis in the plane (xy), and another in (yz), there shall also be a principal axis in the plane of (zx), and the conditions for this case are the same as those for two of the roots of the cubic being equal.

These results might also have been arrived at by the following simple considerations. It is impossible to find, in the planes of (xy) and (yz), two lines at right angles, of which neither is one of the axes of coordinates. Hence,

none of the axes of coordinates being principal axes, if there be a principal axis in xy, and another in (yz), these two cannot be at right angles, and therefore the roots of the cubic from which they are deduced must be equal. Also every line through the origin, in their plane, is a principal axis, and therefore there is a principal axis in the plane of (zx), (where it is cut by this plane of principal axes).

16. The roots K_1, K_2, K_3 of the cubic equation (18) are not the moments of inertia round the principal axes, but are quantities relative to these axes which correspond to F, G, H, round the axes of coordinates. For, if we take $x' = lx + my + nz$, we have

$$\Sigma\mu x'^2 = l^2\Sigma\mu x^2 + m^2\Sigma\mu y^2 + n^2\Sigma\mu z^2 + 2(mn\Sigma\mu yz + nl\Sigma\mu zx + lm\Sigma\mu xy)$$
$$= Fl^2 + Gm^2 + Hn^2 + 2(A'mn + B'nl + C'lm),$$

which is the value of each member of equations (15). Hence the moment of inertia round any one of the principal axes is the sum of the roots of the cubic corresponding to the other two. But the sum of the roots of the cubic (18) is $f^2 + g^2 + h^2 + \alpha + \beta + \gamma$, or $F + G + H$: hence, if P_1 be the moment of inertia round the axis corresponding to K_1, we have

$$P_1 = K_2 + K_3$$
$$= F + G + H - K_1 = \tfrac{1}{2}(A + B + C) - K_1.$$

17. From the last section it follows that, P_1, P_2, P_3 being the three roots of the cubic (13) given in the first part of this paper, and K_1, K_2, K_3 the corresponding roots of (18), we have

$$P_1 = F + G + H - K_1, \quad P_2 = F + G + H - K_2,$$
$$P_3 = F + G + H - K_3.$$

Hence if in the equation (18) we take $K = F + G + H - P$, we obtain an equation differing only in form from (13). Again, if we had treated equations (8) in the same manner as equations (9) were treated (in § 7), we should have obtained an equation similar in form to (13), and not differing from (18) but in form. Thus we have the following four equations, by means of any one of which the principal axes and moments might be determined:

$$\left\{\begin{aligned} &(A-P)(B-P)(C-P) - A'^2(A-P) - B'^2(B-P) - C'^2(C-P) \\ &\qquad\qquad - 2A'B'C' = 0 \ldots (a) \\ &\frac{f^2}{P - \left(A + \frac{B'C'}{A'}\right)} + \frac{g^2}{P - \left(B + \frac{C'A'}{B'}\right)} + \frac{h^2}{P - \left(C + \frac{A'B'}{C'}\right)} + 1 = 0 \ldots (a') \end{aligned}\right.$$

[redacted]

$$\left\{\begin{array}{l}(F-K)(G-K)(H-K)-A'^2(F-K)-B'^2(G-K)-C'^2(H-K) \\ \qquad\qquad\qquad\qquad\qquad\qquad + 2A'B'C' = 0\ldots(b)^* \\ \dfrac{f^2}{K-\left(F-\dfrac{B'C'}{A'}\right)}+\dfrac{g^2}{K-\left(G-\dfrac{C'A'}{B'}\right)}+\dfrac{h^2}{K-\left(H-\dfrac{A'B'}{C'}\right)}-1=0\ldots(b'),\end{array}\right.$$

the quantities F, G, H, f, g, h, being given by the equations

$$F = \tfrac{1}{2}(B+C-A),\quad G = \tfrac{1}{2}(C+A-B),\quad H = \tfrac{1}{2}(A+B-C),$$

$$f^2 = \frac{B'C'}{A'},\quad g^2 = \frac{C'A'}{B'},\quad h^2 = \frac{A'B'}{C'}.$$

It may be algebraically verified that (a) and (a') are identical, as also (b) and (b'); and that (b) or (b') may be deduced from (a) or (a') by assuming

$$P = F + G + H - K\ldots\ldots\ldots\ldots(c).$$

The roots of (a) or (a') are the three principal moments of inertia, and the roots of (b) or (b') substituted in (c), give the same quantities.

18. I shall conclude this paper by applying some of the formulæ given above to the solution of the following problem.

"Having given the moments of inertia of a body round the principal axes through its centre of gravity, shew how to determine the position of the principal axes through any other point, and the moments of inertia round them."—*St. Peter's College Examination Papers*, May 1845.

Let O be the centre of gravity; OX, OY, OZ principal axes; A, B, C the moments of inertia of the body round them: and, according to our previous notation, let

$$F = \Sigma\delta\mu x^2,\quad G = \Sigma\delta\mu y^2,\quad H = \Sigma\delta\mu z^2,$$

so that $\quad A = G + H,\ B = H + F,\ C = F + G\ldots.(a).$

Let P be any point (ξ, η, ζ) for which it is required to determine the principal axes and moments. The integrals (or sums) which will enter as coefficients in the equations for determining the required quantities will be

$$\Sigma\delta\mu\,(x-\xi)^2,\quad \Sigma\delta\mu\,(y-\eta)^2,\quad \Sigma\delta\mu\,(z-\zeta)^2,$$

$$\Sigma\delta\mu\,(y-\eta)(z-\zeta),\quad \Sigma\delta\mu\,(z-\zeta)(x-\xi),\quad \Sigma\delta\mu\,(x-\xi)(y-\eta).$$

Expanding these expressions, and taking into account the properties of the axes of coordinates (principal axes through the centre of gravity), we find that they are respectively equal to

$$F+\mu\xi^2,\quad G+\mu\eta^2,\quad H+\mu\zeta^2,$$

$$\mu\eta\zeta,\qquad \mu\zeta\xi,\qquad \mu\xi\eta,$$

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* This is the form of the "discriminating cubic" usually given. (See First Series, vol. I. p. 35; or *Earnshaw's Dynamics*, Art. 190.)

Hence, by equations (a) of § 13, and by (13), (17), and (18), which are directly applicable to this case, after the necessary change in notation, and a convenient modification of equations (17), we obtain

$$\frac{\mu\xi^2}{K-F}+\frac{\mu\eta^2}{K-G}+\frac{\mu\zeta^2}{K-H}=1 \;\ldots\ldots\ldots (b).$$

$$\frac{l}{\dfrac{\xi}{K-F}}=\frac{m}{\dfrac{\eta}{K-G}}=\frac{n}{\dfrac{\zeta}{K-H}} \;\ldots\ldots\ldots\ldots (c),$$

which contain the solution of the problem, as (b) determines three real values for K, any one of which substituted in (c) gives the ratios $l:m:n$, fixing the position of a principal axis, and subtracted from the sum of the roots, leaves as remainder the moment of inertia round this axis.

19. These equations lead to a very simple geometrical construction for the principal axes through P; but before stating this, it will be convenient to make a slight modification in their form, by means of the following assumptions.

Let $\mu Q = K-(F+G+H) \;\ldots\ldots\ldots\ldots (c')$,

and let $A=\mu a^2,\quad B=\mu b^2,\quad C=\mu c^2.$

The equations then become

$$\frac{\xi^2}{Q+a^2}+\frac{\eta^2}{Q+b^2}+\frac{\zeta^2}{\mu+c^2}=1 \;\ldots\ldots\ldots (d),$$

$$\frac{l}{\dfrac{\xi}{Q+a^2}}=\frac{m}{\dfrac{\eta}{Q+b^2}}=\frac{n}{\dfrac{\zeta}{Q+c^2}} \;\ldots\ldots\ldots\ldots (e).$$

If now we suppose a value of Q to be substituted in (d), which satisfies the equation when given values are assigned to ξ, η, ζ, and consider Q to remain constant, when ξ, η, ζ vary, the locus of ($\xi\,\eta\,\zeta$) will be a surface of the second order, confocal with the ellipsoid,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \;\ldots\ldots\ldots\ldots\ldots (f),$$

and passing through the given point, and equations (e) determine the direction-cosines of a normal to this surface at the point $\xi\,\eta\,\zeta$. The three roots of equation (d), considered as a cubic for determining Q, when ξ, η, ζ have any given values, correspond to the three surfaces of the second order, the ellipsoid, the hyperboloid of one sheet, and the hyperboloid of two sheets passing through the point ($\xi\,\eta\,\zeta$), and confocal

[redacted]

with the surface (f), which we shall, for distinction, call the *central ellipsoid* of the body.*

The principal axes through any point of a solid body are normals to the three surfaces of the second order confocal with the central ellipsoid, which intersect in that point.†

For determining the principal moments corresponding to the point ($\xi\,\eta\,\zeta$), since $F + G + H + \mu(\xi^2 + \eta^2 + \zeta^2)$ is the sum of the roots of the cubic (b), we have

$$P_1 = F + G + H + \mu(\xi^2 + \eta^2 + \zeta^2) - K_1,$$

where P_1 is the moment of inertia round the axis given by the root K_1 of the cubic. Hence if we take

$$K = F + G + H + \mu(\xi^2 + \eta^2 + \zeta^2) - P \ldots\ldots (g)$$

in (d), the equation thus obtained,

$$\frac{\xi^2}{\xi^2 + \eta^2 + \zeta^2 + a^2 - \dfrac{P}{\mu}} + \frac{\eta^2}{\xi^2 + \eta^2 + \zeta^2 + b^2 - \dfrac{P}{\mu}} + \frac{\zeta^2}{\xi^2 + \eta^2 + \zeta^2 + c^2 - \dfrac{P}{\mu}} = 1 \quad (h),$$

determines three real values for P, which are the moments of inertia round principal axes through ($\xi\,\eta\,\zeta$).

Glasgow, Jan. 6, 1846.

POSTSCRIPT.

20. If in equation (h) of the last section, P have a given value, Π; and if x, y, z be any values of ξ, η, ζ which satisfy the equation in this case, the locus of $x\,y\,z$ will be a surface

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* In the first part of this paper (see p. 130, § 6 and Note) two ellipsoids were mentioned, either of which, different for different points of the body, may be described round any point as centre, and affords a geometrical construction for determining the principal axes of the body through this point. The *central ellipsoid* is unique in a solid body; its principal axes coincide in direction with the principal axes of the body through the centre of gravity; and the semiaxes are equal to $\sqrt{\dfrac{A}{\mu}}, \sqrt{\dfrac{B}{\mu}}, \sqrt{\dfrac{C}{\mu}}$ (radii of gyration), and thus its form, position, and magnitude, are entirely fixed in the body. The axes of either of the other ellipsoids for any point of the body coincide with the principal axes of the body through the point, and, arbitrary in absolute magnitude, are proportional to $\dfrac{1}{\sqrt{A}}, \dfrac{1}{\sqrt{B}}, \dfrac{1}{\sqrt{C}}$, or to $\dfrac{1}{\sqrt{F}}, \dfrac{1}{\sqrt{G}}, \dfrac{1}{\sqrt{H}}$, the first being Poinsot's *momental ellipsoid*, and the second the ordinary *ellipsoid of construction*.

† This theorem has also been demonstrated by Mr. Townsend (Fellow of Trinity College, Dublin). His investigation, which is connected with the geometrical properties of confocal surfaces, and of enveloping cones, is entirely different from that given above, and will be published in an early Number of the *Journal*. I am informed by Mr. Townsend that another demonstration of the same theorem, which will also be contained in his paper, has been given (but not, so far as I am aware, published), several years ago, by Professor Maccullagh.

possessing the property that at each point one of the principal moments is equal to Π. Hence the equation of what may be called an *equimomental surface* is

$$\frac{x^2}{r^2 + a^2 - \frac{\Pi}{\mu}} + \frac{y^2}{r^2 + b^2 - \frac{\Pi}{\mu}} + \frac{z^2}{r^2 + c^2 - \frac{\Pi}{\mu}} = 1 \ldots\ldots (a).$$

By giving Π different values, we obtain different *conjugate* equimomental surfaces. The moment of inertia round any line not passing through the centre of gravity is greater than the moment round a line parallel to it through this point, and hence the smallest possible moment of inertia is that round the principal axis through the centre of gravity, of least moment. Hence by giving Π all values from μc^2 to ∞ (a, b, c are supposed to be in order of descending magnitude), we obtain all the possible equimomental surfaces for the body. These surfaces possess many remarkable properties, which derive additional interest from the circumstance that similar surfaces present themselves, with a very important signification, in the undulatory theory of light, Fresnel's *wave surface* in a biaxal crystal being the same as an *equimomental surface* in a solid body. The form of the equimomental surface is the same as that of the wave surface, when $\frac{P}{\mu}$ is greater than a^2, and is therefore well known in this case; but when $\frac{P}{\mu}$ is between a^2 and b^2, or b^2 and c^2, it will present many remarkable peculiarities, and may be an interesting subject for investigation.

21. The following theorem, suggested to me by a corresponding one in the undulatory theory, gives a construction for finding that principal axis through any point of the equimomental surface (a) round which the moment is Π.

If P be any point of an equimomental surface, and OQ a perpendicular from the centre to the tangent plane, then PQ is the principal axis round which the moment of inertia is Π.

To prove this, let l, m, n be the direction-cosines of the tangent plane at P, and let OQ, or $lx + my + nz$, $= v$. Then, denoting $\frac{\Pi}{\mu} - a^2$, $\frac{\Pi}{\mu} - b^2$, $\frac{\Pi}{\mu} - c^2$ by α, β, γ, and $\frac{x^2}{(r^2-\alpha)^2} + \frac{y^2}{(r^2-\beta)^2} + \frac{z^2}{(r^2-\gamma)^2}$ by S, we have

$$\frac{l}{\frac{x}{r^2-\alpha} - Sx} = \frac{m}{\frac{y}{r^2-\beta} - Sy} = \frac{n}{\frac{z}{r^2-\gamma} - Sz} \ldots . (b),$$

[redacted]

from which we deduce, by ordinary processes,

$$\frac{lx}{r^2-\alpha}+\frac{my}{r^2-\beta}+\frac{nz}{r^2-\gamma}=0\ldots\ldots\ldots\ldots(c),$$

and each member of equations (b) is

$$=\frac{v}{1-Sr^2},\quad\text{or}=\frac{1}{-Sv}.$$

Hence
$$S(r^2-v^2)=1\ldots\ldots\ldots\ldots\ldots\ldots(d),$$

and we have
$$\left.\begin{aligned}lv(r^2-\alpha)&=x(v^2-\alpha)\\ mv(r^2-\beta)&=y(v^2-\beta)\\ nv(r^2-\gamma)&=z(v^2-\gamma)\end{aligned}\right\}\ldots\ldots\ldots(e).^*$$

Now lv, mv, nv are the coordinates of Q, and hence if λ, μ, ν be the direction-cosines of PQ, we have

$$\left.\begin{aligned}&\frac{\lambda}{r^2-\alpha}=\frac{\mu}{r^2-\beta}=\frac{\nu}{r^2-\gamma}\\ \text{or}\quad&\frac{\lambda}{r^2+a^2-\frac{\Pi}{\mu}}=\frac{\mu}{r^2+b^2-\frac{\Pi}{\mu}}=\frac{\nu}{r^2+c^2-\frac{\Pi}{\mu}}\end{aligned}\right\}\ldots\ldots(f),$$

which prove the theorem enunciated, since, by (b), § 18, these are the equations for determining the principal axis, corresponding to a root Π of the cubic equation which determines the three principal moments at any point ($x\,y\,z$).

The locus of points on any surface

$$\frac{x^2}{a^2+\Theta}+\frac{y^2}{b^2+\Theta}+\frac{z^2}{c^2+\Theta}=1\ldots\ldots\ldots(g),$$

(confocal with the central ellipsoid) for which one of the principal moments is equal to Π, is determined by the equations (a) and (g) considered as simultaneous. By subtracting the former from the latter, we get

$$\left(r^2-\frac{\Pi}{\mu}-\Theta\right)\left\{\frac{x^2}{(a^2+\Theta)\left(r^2+a^2-\frac{\Pi}{\mu}\right)}+\frac{y^2}{(b^2+\Theta)\left(r^2+b^2-\frac{\Pi}{\mu}\right)}\right.$$
$$\left.+\frac{z^2}{(c^2+\Theta)\left(r^2+c^2-\frac{\Pi}{\mu}\right)}\right\}=0.$$

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* See *Math. Journal*, vol. I. (First Series) p. 8, or Gregory's *Examples*, p. 232, where the same formulæ occur in the investigation of the wave surface.

Hence the required locus on the surface (g) consists of two portions, given by the equations

$$r^2 = \frac{\Pi}{\mu} + \Theta \quad \ldots\ldots\ldots\ldots\ldots \quad (h),$$

$$\frac{x^2}{(a^2+\Theta)\left(r^2+a^2-\dfrac{\Pi}{\mu}\right)} + \frac{y^2}{(b^2+\Theta)\left(r^2+b^2-\dfrac{\Pi}{\mu}\right)} + \frac{z^2}{(c^2+\Theta)\left(r^2+c^2-\dfrac{\Pi}{\mu}\right)} = 1 \ldots (k),$$

taken separately. Now, by comparing (h) with equations (g) and (c') of § 19, we see that the moment of inertia round a normal at $x\,y\,z$ to the surface (g) is Π. Hence the first portion of the curve is the locus of points for which the moment of inertia round a normal to the surface (g) is constant. Also unless (h) be satisfied at any point, the moment of inertia round a normal is not Π, and hence (k) must represent the locus of points for which one of the principal moments round axes in the tangent plane is constant. Now these axes are, by what has been proved above, normals to the other two confocal surfaces through the same point, and are therefore tangents to the principal sections of the surface (g). Again, the principal axis round which the moment is Π, lies in the tangent plane to the equimomental surface through the point, corresponding to the moment Π; hence it must be the intersection of the tangent plane to this surface and the surface (g), and consequently this line of intersection must touch a principal section of (g). Hence the second portion of the curve on (g) is a line of curvature.* Thus we conclude that the equimomental surface (a) cuts the surface of the second order (g) in a "spherical conic" and a line of curvature. The moment of inertia round a tangent at any point of the latter curve and round a normal to the surface (g) at any point of the former is equal to Π. Also, since the principal axis round which the moment is Π, lies in the equimomental surface, it follows that, along the spherical conic, the equimomental surface cuts the surface of the second order at right angles, a result which may be easily verified.

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* This theorem was communicated to me by Mr. Cayley before I had convinced myself of its truth by the proof given in the text. His demonstration is given below.

[redacted]

22 As an illustration of what has been proved in the last section, let us suppose $\Theta = 0$, so that (g) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

the equation of the central ellipsoid. It may be readily shewn that the moments of inertia round normals to this surface lie between the limits c^2 and a^2 (the mass being taken as unity), and those round lines touching it, between $b^2 + c^2$ and $a^2 + b^2$. Hence if Π have all values from c^2 to a^2 given it in succession, all the equimomental surfaces which intersect the central ellipsoid in a real spherical conic will be obtained; and if all values from $b^2 + c^2$ to $a^2 + b^2$ be given, all the equimomental surfaces which intersect the central ellipsoid in real lines of curvature will be obtained. The values of Π from $b^2 + c^2$ to $c^2 + a^2$ will give equimomental surfaces which cut the ellipsoid in the lines of curvature corresponding to the confocal hyperboloids of one sheet; and the other set of lines of curvature are therefore given by values of Π between $c^2 + a^2$ and $a^2 + b^2$.

Since $b^2 + c^2 > a^2$, except in the limiting case of the body being reduced to a portion of the plane of yz, when these quantities are equal, the values of Π which make one curve of intersection real must make the other imaginary. Hence the same equimomental surface can only cut the central ellipsoid in a "spherical conic" or a line of curvature, but not in both. The same may be shewn to be true for any confocal ellipsoid (but not for any of the hyperboloids). Hence any of the equimomental surfaces which cuts one of the ellipsoids must do so along either a spherical conic or a line of curvature, but not along both.

There are many interesting subjects of investigation relative to these equimomental surfaces which present themselves; such as the properties of the three equimomental surfaces which intersect in a given point and of the curves along which they cut one another, the forms of the different classes of equimomental surfaces, and the remarkable properties of principal axes at different points (corresponding to the *conical refraction* of light), which are due to the singular points of these surfaces. These may be discussed in a future communication; but the length to which this paper has already been extended, prevents me from going farther in the subject at present.

St. Peter's College, May 8, 1846.

NOTE ON A GEOMETRICAL THEOREM CONTAINED IN THE PRECEDING PAPER.

By Arthur Cayley.

It is easily shown that if three confocal surfaces of the second order pass through a point P, then the square of the distance of this point from the origin is equal to the sum of the squares of three of the axes, no two of which are parallel or belong to the same surface (the squares of one or two of the axes of the hyperboloids being considered negative); *i.e.* if

$$\frac{x^2}{a^2+h}+\frac{y^2}{b^2+h}+\frac{z^2}{c^2+h}=1,$$

$$\frac{x^2}{a^2+k}+\frac{y^2}{b^2+k}+\frac{z^2}{c^2+k}=1,$$

$$\frac{x^2}{a^2+l}+\frac{y^2}{b^2+l}+\frac{z^2}{c^2+l}=1;$$

then $$x^2+y^2+z^2=a^2+b^2+c^2+h+k+l.$$

In fact these equations give

$$x^2=\frac{(a^2+h)(a^2+k)(a^2+l)}{(a^2-b^2)(a^2-c^2)},$$

$$y^2=\frac{(b^2+h)(b^2+k)(b^2+l)}{(b^2-a^2)(b^2-c^2)},$$

$$z^2=\frac{(c^2+h)(c^2+k)(c^2+l)}{(c^2-a^2)(c^2-b^2)}.$$

And adding these and reducing, we have the relation in question; which is also immediately obtained by forming the cubic whose roots are h, k, l.

From this property may be deduced the theorem given by Mr. Thomson in the preceding memoir. In fact, writing

$$r^2=x^2+y^2+z^2,$$

and $$k=-a^2-b^2-c^2+h,$$

we see that in consequence of these relations the equations

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

$$\frac{x^2}{r^2+a^2-h}+\frac{y^2}{r^2+b^2-h}+\frac{z^2}{r^2+c^2-h}=1,$$

$$\frac{x^2}{a^2+k}+\frac{y^2}{b^2+k}+\frac{z^2}{c^2+k}=1,$$

[redacted]

are equivalent to two independent equations, *i.e.* the third can be deduced from the two first. Now the first equation is that of an ellipsoid (or generally a surface of the second order, since a, b, c are not necessarily real). The second is that of what may be called a conjugate equimomental surface, defining the term as follows: "The conjugate equimomental surfaces of an ellipsoid (or surface of the second order) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, are the equimomental surfaces derived in the usual manner from any surface of the second order $\frac{x^2}{h - a^2} + \frac{y^2}{h - b^2} + \frac{z^2}{h - c^2} = 1$, which is confocal with the *conjugate* surface $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1$ of the given ellipsoid," viz. by measuring along any line through the centre distances equal to the axes of the section by a plane through the centre perpendicular to this line, and taking the locus of the points so determined for the equimomental surface. The third equation is that of a surface confocal with the given ellipsoid; hence the theorem, "The curves of curvature of a given ellipsoid lie upon a system of conjugate equimomental surfaces."

But since the first and second equations are evidently satisfied by the combination of the first equation with the relation $r^2 = h$, which is that of a sphere, we have also, "The curve of intersection of the ellipsoid with any one of the conjugate equimomental surfaces, is composed of the line of curvature, and of a spherical conic." And these two curves being each of them of the fourth order make up the complete curve of intersection, which should obviously be of the eighth order.

It would be an interesting question to determine the relations existing between the curve of curvature and the spherical conic, which have been thus brought into connection by means of the conjugate equimomental surfaces; *i.e.* between the two curves obtained by combining the equation of the ellipsoid with

$$\frac{x^2}{a^2 + k} + \frac{y^2}{b^2 + k} + \frac{z^2}{c^2 + k} = 1,$$

$$r^2 = a^2 + b^2 + c^2 + k,$$

respectively: but it will be sufficient at present to have suggested the problem.

ON PRINCIPAL AXES OF A BODY, THEIR MOMENTS OF INERTIA, AND DISTRIBUTION IN SPACE.

By Richard Townsend, Fellow of Trinity College, Dublin.

1. As the centre of gravity of a body or system of bodies may be defined either by its analytical or by its fundamental physical property, and as it may be readily shewn that either follows from the other, so may we define a principal axis at any point of a solid body either by its analytical or by its dynamical property with respect to that point, and these properties may be easily proved to be each a necessary consequence of the other. This is usually done as follows.

Let O be any point, and OZ any axis drawn therethrough, round which suppose the body to revolve with an angular velocity ω; then will every element dm have a centrifugal force impelling it perpendicularly out from the axis, and equal to $\omega^2 rdm$, r being the distance of dm from OZ.

Transferring all these forces to O, each by a parallel movement, we shall have a number of forces passing through O and lying all in a plane perpendicular to OZ, and a number of moments in planes passing all through OZ; the forces will compound a single force (which may = 0) at right angles to the axis, and the pairs a single pair in a plane passing therethrough.

To find the values of the resultant force and of the resultant moment, let each component force $\omega^2 rdm$ be resolved into two $\omega^2 xdm$, $\omega^2 ydm$ along any two fixed axes OX, OY at right angles to OZ, and therefore in the plane which contains the whole system of forces; and let each component moment $\omega^2 rzdm$ be resolved into two $\omega^2 xzdm$, $\omega^2 yzdm$ in the planes ZOX, ZOY: the sums of the resolved parts of all these forces along each axis, and of all the pairs in each plane, will be then equal to the resolved parts of their respective resultants along the same axes and in the same planes.

The quantities $\omega^2.\Sigma xdm$ and $\omega^2.\Sigma ydm$ are, therefore, the components of the resultant force along OX and OY, and the quantities $\omega^2.\Sigma zxdm$ and $\omega^2.\Sigma zydm$ are those of the resultant moment in the planes ZX and ZY.

Now, if any number of forces be in equilibrium, their resultant moment with respect to any point whatsoever must vanish, and if they compound a single resultant force, their resultant moment with respect to any point through which that force passes must vanish; and conversely, if at any point the resultant moment, or, which is the same thing, the re-

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solved parts of that resultant in any directions be equal to nothing; then will the forces either equilibrate or compound a single resultant passing through that point.

If, therefore, OZ be a *permanent* axis of rotation (the point O of the body alone being fixed), then must the quantities $\Sigma xzdm$ and $\Sigma yzdm$ be $= 0$ for *every* pair of coordinate axes through O at right angles to OZ, and conversely. If these analytical quantities be both $= 0$, then for the point O will the resultant moment of the centrifugal forces vanish, or the axis be permanent.

2. The forces will not equilibrate except in the particular case when the point is the centre of gravity; for, the components of their resultant along OX and OY are respectively $\omega^2\Sigma xdm$ and $\omega^2\Sigma ydm = \omega^2\bar{x}M$ and $\omega^2\bar{y}M$.

Hence, an axis which is principal for one point of a body is principal for no other point on itself, and therefore for no other point whatever, except in the particular case of the centre of gravity whose principal axes are principal for all points along them. This is evident, for when any number of forces compound a single resultant, their resultant moment with respect to an assumed point will vanish only when that point is on the resultant, except in the particular case when that resultant is nothing, that is, when the forces are in equilibrium.

Again, since in general systems of forces neither equilibrate nor compound a single resultant, it follows that an axis taken at random in a body may not be principal for any point at all; and that such is often actually the case, appears at once from the values for the components of the resultant force and moment, which for *every* point on the axis must be such that $\Sigma xzdm : \Sigma yzdm :: \Sigma xdm : \Sigma ydm$, in order that the resultants of the two systems of parallel forces, $\omega^2 xdm$ in the plane of xz and $\omega^2 ydm$ in the plane of yz, which both pass through the axis OZ, should meet it at the same point, and therefore compound a single resultant: but this proportion does not in general hold.

Hence, though the number of principal axes in a body may perhaps be infinite, it may still be small in comparison with the number of axes which are not principal.

3. If on every line diverging from any point O of a solid body we measure off a portion r, such that the square of its reciprocal multiplied by the mass of the body shall equal the moment of inertia round it; then, as is well known, will the locus of the extremity of r be an ellipsoid (the "Momental" Ellipsoid of Poinsot). This is usually shewn as follows.

Assuming arbitrarily any three rectangular axes through O, let A' B' C' be equal to the moments of inertia round them, and let A'' B'' C'' be equal to the quantities $\Sigma yzdm$, $\Sigma zxdm$, and $\Sigma xydm$ with respect to the same; then will the moment of inertia round the line whose direction-angles are $\alpha\,\beta\,\gamma$ be equal to

$$A' \cos^2 \alpha + B' \cos^2 \beta + C' \cos^2 \gamma - 2A'' \cos \beta \cos \gamma - 2B'' \cos \gamma \cos \alpha - 2C'' \cos \alpha \cos \beta = \frac{M}{r^2}.$$

Multiplying this by r^2, we get the equation of the surface in question, viz.

$$A'x^2 + B'y^2 + C'z^2 - 2A''yz - 2B''zx - 2C''xy = M \;.\;. \text{(I)},$$

a central surface of the second order, which, as moments of inertia are essentially positive, is therefore an ellipsoid referred to its centre.*

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* M. Poinsot, in his elegant tract on Rotation, has denominated the above ellipsoid at any point of a body, the "central" ellipsoid of that point, and that name had been at first used in this paper, but having been found very inconvenient, in consequence of the *centre* of gravity appearing in almost every enquiry, and, as such, having no connexion with the ellipsoid at any other point, the term "momental" (for which there is high authority) has been since adopted in preference.

If round O as centre we conceive a sphere to be described with radius = 1, and that the polar plane of every point of the momental ellipsoid be taken with respect to that sphere, then will the envelope of all these planes be another ellipsoid, which, as is evident, will be concentric and coaxal with the former, and which, as obviously possessing the property that the squares of the perpendiculars from O upon all its tangent planes multiplied each by the mass of the body are equal to their moments of inertia, may be called the *ellipsoid of inertia* of that point. These two concentric ellipsoids, *sphero-polars reciprocal* to each other, and thus intimately connected at each individual point of a body, are of constant occurrence in all enquiries respecting the equilibrium and motion of solid bodies, and should be familiarly known and always considered together, both being important and possessing their advantages each over the other. If we know either for any point of a body we of course have the other for the same point, but they both obviously vary, in magnitude, position, and figure, from one point of a body to another.

The *particular* ellipsoid of inertia round the centre of gravity (which is a very important surface in all that relates to the present subject) has been called (for an obvious reason) the *ellipsoid of gyration*, and is the ellipsoid which Mr. Thomson, Fellow of St. Peter's College, has distinguished by the shorter and more expressive name of *the central ellipsoid*.

The surface locus of feet of perpendiculars from any point O, upon all the tangent planes to its ellipsoid of inertia, that is, the surface round any point of a body the squares of whose radii multiplied each by the mass of the body are equal to their moments of inertia, may be called the *surface of inertia* of that point. We shall have occasion as we proceed to notice it also; it varies like the others from point to point of every body, is always of the fourth order, has two circular sections passing both through its mean axis, and coincident with those of the momental ellipsoid, and is the same in shape as the well-known "surface of elasticity" in the Wave Theory of Light.

For each point O, this ellipsoid is, from its very nature, fixed in magnitude and position with respect to the body, and is of course independent of the arbitrarily assumed directions of the coordinate axes; though if the latter be changed, the coefficients $A'\ B'\ C'\ A''\ B''\ C''$ in the new equation will no longer represent the *same* sums as before, but will be equal to the similar sums with respect to the new axes. Conversely, therefore, this ellipsoid once determined gives us the values of these six integrals for every rectangular system of coordinate axes assumed at pleasure through O.

Now, in *every* ellipsoid (and therefore in the above) there exist *three* lines, passing through its centre and at right angles to each other (and generally but three), to which as axes of coordinates if the surface be referred, the coefficients $A''\ B''\ C''$ will all disappear from the equation, and for each of which, whatever be the directions of the other two coordinate axes, provided they be both at right angles to the first, two of the same quantities will vanish. This, therefore, being the analytical property of a principal axis, proves that for every point of a solid body there are, at least, three principal axes at right angles to each other, the axes, viz., of its momental ellipsoid.*

4. By means of the momental ellipsoid we may also arrive at the same result from the dynamical property of a principal axis.

For, suppose the body to revolve round any diameter $2r$ of that ellipsoid, with which, as the coordinate rectangular axes are quite arbitrary, let one of them, that of z, coincide; then, in the equation of the surface (3),

$$A'x^2 + B'y^2 + C'z^2 - 2A''yz - 2B''zx - 2C''xy = M,$$

will the quantities A'' and B'', multiplied each by the square of the angular velocity, be equal to the components in the planes of zy and of zx respectively of the resultant moment of all the centrifugal forces.

A'' and B'' are, therefore, proportional to the cosines of the angles which the plane of that resultant passing through the axis of z makes with the planes of zy and of zx respectively, and their *signs* are the same as those of the *directions* in which the components tend to draw the axis of rotation.

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* This proof is due to Professor MacCullagh, and was given by him at his mathematical lectures in Trinity College, Dublin.

At the point where that axis meets the surface, let now a tangent plane be drawn, then (the coordinates of that point being $x = 0, y = 0, z = r = \frac{\sqrt{M}}{\sqrt{C'}}$) will its equation be

$$-B''r.x - A''r.y + C'r.z = M \ldots\ldots\ldots (\mathrm{II}),$$

from which we see that the perpendicular p, let fall from the centre on this tangent plane, makes with the axes of x and of y, respectively, angles whose cosines are proportional to the quantities $-B''$ and $-A''$, and therefore proportional to the components of the resultant moment with their signs changed, that is, with their directions changed into the opposite.

If, therefore, a body revolve round any radius of its momental ellipsoid at any point, the plane of the resultant centrifugal moment will be that of the radius and perpendicular, and its tendency will be to draw the axis of rotation *from* the perpendicular; theorems which are due to Poinsot and to Professor Maccullagh.

Again, from (II) we have

$$p^2.(B''^2 + A''^2 + C'^2).r^2 = M^2;$$

from which, since $C'^2 = \frac{M^2}{r^4}$, we get

$$B''^2 + A''^2 = \frac{M^2}{p^2r^2} - \frac{M^2}{r^4} = \frac{M^2}{r^2}\left(\frac{1}{p^2} - \frac{1}{r^2}\right) = M^2p'^2(r'^2 - p'^2) \ldots (\mathrm{III}),$$

by changing at every point of the ellipsoid r and p into their reciprocals p' and r', p' being supposed $= \frac{1}{r}$ and $r' = \frac{1}{p}$.

If, therefore, a body revolve with the same angular velocity round different radii of its momental ellipsoid at any point, then will the magnitude of the resultant centrifugal moment be proportional to the area of the right-angled triangle $p'r'$.*

Now, in every ellipsoid there exist *three* diameters at right angles to each other (the axes of the surface), for which p will coincide with r, and therefore p' with r', and for which,

* Translating the above properties to the *ellipsoid of inertia*, we, hence, know that—If a body revolve round a perpendicular on a tangent plane to its ellipsoid of inertia at any point, then will the plane of the resultant centrifugal moment be that of the perpendicular p' and of the corresponding radius r', its direction will be *from* r′ *towards* p′, and its magnitude (for an invariable angular velocity) will be proportional to the area of the right-angled triangle $p'r'$, being in all cases equal to the quantity $M.\omega^2$. (double that area).

consequently, the area of the triangle $p'r'$ will vanish. If, therefore, a body revolve at any point round either of these diameters of its momental ellipsoid, the resultant centrifugal moment with respect to that point will vanish; which proves that at every point of a body the axes of its momental ellipsoid are principal axes.

5. At any point O let $A\ B\ C$ be the three principal moments of inertia; then, referring the momental ellipsoid to the principal axes, its equation will be

$$Ax^2 + By^2 + Cz^2 = M \dots\dots\dots\dots (\text{IV}):$$

from which (since all lines which are axes of that surface, and none others, are principal axes at O) it appears that, to know the number of principal axes which a body admits of at a given point, we need only know the relative magnitudes of $A\ B$ and C for that point.

There are, therefore, three varieties of points in a body with respect to the number of principal axes which they admit of.

The most frequent case is, when A, B, and C are all unequal; such points admit of but three principal axes, viz. those of the ellipsoid. Secondly, two of them may be equal, in which case the ellipsoid will be of revolution and, besides the one perpendicular thereto, will have an infinite number of axes in the plane of its equator. Such points admit, therefore, of an infinite number of principal axes in the plane which contains the two of equal moment: these axes will moreover be all *equimomental*, for, their plane intersecting the ellipsoid in a circle, the radii of that surface with which they coincide, and therefore their moments of inertia will be all equal, (3). Thirdly, A, B, and C may be all equal; the ellipsoid will then be a sphere, and *all* axes for such points will be principal, and also equimomental.

(In *every* body there exists, as we shall see, a curve locus of points of the second species; but if a point of the third species exist, either it must be the centre of gravity, or else that centre must be a point of the second species where the two *equal* principal moments are each *less than* the third; in which latter case there will be two such points, both on the third central principal axis, at opposite sides of the centre of gravity, and equidistant therefrom by the interval $\pm\sqrt{\left(\frac{A-C}{M}\right)}$, A and C being the unequal principal moments, and M the mass of the body).

6. If, in the general case, the momental ellipsoid at any point O be intersected by any concentric sphere, then will all axes through O which pass through the curve of intersection be equimomental.

This is obvious, for their intercepts between the centre and the curve, being radii of a sphere, are all equal to each other; and being radii of the momental ellipsoid, are equal to the inverse square roots of their moments of inertia.

Let I be the moment round a system of such axes, then will $I.(x^2+y^2+z^2)=M$ be the equation of the sphere; subtracting this from that of the ellipsoid $Ax^2+By^2+Cz^2=M$, we get that of the cone which the axes generate, viz.

$$(A-I)\,x^2+(B-I)\,y^2+(C-I)\,z^2=0\ldots\ldots(\mathrm{V}).$$

At every point, therefore, of a body, every system of equimomental axes generates a cone of the second order.

Each point of the body has, of course, an infinite number of such cones; for I may have all values between the limits A and C: and from the above equation it appears that the cones of each system have all the same principal axes, viz. those of the body at their common vertex.

(From this it appears, that if at any point of a body we could find one of its cones of equimomental axes, we should have an obvious geometric construction by which to determine the principal axes for that point).

The cones of each system have also all the same cyclic planes, viz. those of the momental ellipsoid.

For, the quantity $\pm\sqrt{\left\{\dfrac{(A-I)-(B-I)}{(B-I)-(C-I)}\right\}}$, which expresses the tangent of the angle that the cyclic planes (which here pass all through the axis of y) make with the plane of xy, is independent of I, and $=\pm\sqrt{\left(\dfrac{A-B}{B-C}\right)}$, which last expresses the tangent of the similar angle in the momental ellipsoid.

When $I=B$, the cone (V) degenerates into two planes passing through the axis of y: all axes, therefore, at any point of a body for which the moment of inertia equals the mean principal moment, lie in two planes, equally inclined to the axis of least, and therefore of greatest, moment; viz. the cyclic planes of the momental ellipsoid.

If the ellipsoid be of revolution, so will also the whole system of equimomental cones; the cyclic planes will also coincide in one which will be the limit to the system of cones, and we get the property already noticed, that when two of

the principal moments are equal, all axes in their plane will be equimomental.

Hence, conversely, if at any point of a body one of the cones of equimomental axes be a cone of revolution, such point will admit of an infinite number of principal axes in the principal plane perpendicular to the internal axis of the cone.

If the ellipsoid be a sphere, the cones will be all indeterminate, as they ought to be.

7. In the particular case when the point O is taken off at infinity, the reasonings by which the conclusions of the preceding section were established, of course fail, the momental ellipsoid being there infinitely slender; but (as is usual in all such cases) the results are then simpler. It should be mentioned, however, lest an erroneous conclusion might be hastily drawn, that some of the latter *appear* to be at variance with others which have been obtained in the general case.

For instance, from the well-known and very important property of the centre of gravity with respect to the moments of inertia round parallel axes, it appears that every system of equimomental axes which are all parallel to each other in any direction whatever, will generate a cylinder of revolution round the parallel axis through the centre of gravity; and, by giving different values to the moment of inertia, that the whole system of equimomental cylinders will be therefore all co-axal. But these cylinders (since their sides, like every other system of parallel lines, pass all through an infinitely distant point) are cones of the second order having a common vertex at infinity, and from this we might hastily conclude that *every* point at infinity admits of an infinite number of principal axes, since for every such point the system of equimomental cones is of revolution.

But that is not the case, for (as we shall presently see) the principal axes of a body are as accurately determinable for a point at infinity as for any other point; and we shall then also see from the equation of the equimomental system of cones, which we can find for any point, why it is that in the above particular case they are of revolution, while at the same time their three axes are all fixed. For the present we will only remark, that should an ellipse, variable, in magnitude, position, and figure, according to any law, and whose axis, determinable in each case from that law, enjoy some property peculiar to themselves, pass through the particular figure of a circle, we have obviously no right to extend to *any* pair of diameters of that circle the peculiar property of

the axes, but must take *the particular pair* determined by the law which all the other axes follow. Such is actually the case in the present instance, and instances of a similar nature are of frequent occurrence, as, for example, in the determinable points of intersection of two consecutive curves which ultimately coincide. The above, therefore, presents no real difficulty, and has only been noticed to prevent the chance of an erroneous conclusion.

8. At every point of a body the sum of the moments of inertia round any three rectanglar axes is equal to the sum of the three principal moments.

For, in every ellipsoid, and therefore in the momental ellipsoid at every point of a body, the sum of the squared reciprocals of any three rectangular semidiameters is equal to the sum of the squared reciprocals of the three semiaxes.

If, therefore, round the centre of gravity of a body as centre, a sphere be described, then for all points of that sphere will the sum of the three principal moments be constant (a property of the centre of gravity which may be considered as an extension of its ordinary property respecting equidistant axes).

For that sum at every point of the sphere is equal to the sum of the moments round three parallel axes through the centre + Mr^2, M being the mass of the body and r the radius of the sphere.

If at any point of a body a number of axes be drawn all in the same plane, then will the sum of the moments round every two at right angles to each other be constant.

For that sum, together with the moment round the axis perpendicular to the plane is constant, and the latter moment is the same for all.

If the plane be fixed, the locus of the points therein, for which this sum, constant for each, will have the same value, will be a circle of which the centre will be the foot of the perpendicular dropped on the plane from the centre of gravity.

For perpendiculars to the plane erected at every point of such a circle will be all equimomental axes (7), and also the sum of the moments round every three rectangular axes, constant for each of these points, will be the same for them all, since they lie on a sphere whose centre is the centre of gravity.

If a system of equimomental axes lie all in a plane passing through the centre of gravity, they will envelope an ellipse (or hyperbola) of which that point will be the centre.

To shew this, let I' I'' be the moments of inertia round the semiaxes r' r'' of the ellipse in which the plane intersects the momental ellipsoid at the centre of gravity, ρ' ρ'' the radii of the same parallel and perpendicular to one of the equimomental axes, d the distance between that axis and ρ', and α β the angles which ρ'' or d makes with r' and r''; then, denoting by I the constant moment common to all the axes, we shall have $\dfrac{M}{\rho^2} + M.d^2 = I$, from which, since $\dfrac{1}{\rho^2} = \dfrac{1}{r'^2}\cos^2\alpha + \dfrac{1}{r''^2}\sin^2\alpha$, $\dfrac{M}{\rho^2} = I'\cos^2\alpha + I''\sin^2\alpha$, and we get

$$M.d^2 = (I - I')\cos^2\alpha + (I - I'')\sin^2\alpha \;\ldots\ldots\; (a):$$

the equimomental axes therefore all envelope a central conic, the squares of whose semi-axes are $\dfrac{I - I'}{M}$ and $\dfrac{I - I''}{M}$, and which will be therefore an ellipse or hyperbola, as the case may be.

For different systems of equimomental axes in the same plane I will vary but I' and I'', and their axes will remain the same: hence we know that the system of conics enveloped by all the systems of axes will be coaxal and confocal, their common axes being those of the section of the central ellipsoid.

The ellipse a', b', or $\sqrt{\dfrac{M}{I'}}, \sqrt{\dfrac{M}{I''}}$, with which the conics are all confocal, is obviously the ellipse cyclo-polar reciprocal* to the ellipse section of the momental ellipsoid by the plane of the axes, the radius of the reciprocating circle being 1, and the pole being the centre of that ellipse, or the centre of gravity.

9. Now, if a system of equimomental axes, lying all in any plane whatever, be projected orthographically upon a parallel plane through the centre of gravity (and therefore upon *any* parallel plane), the projections will be also a system of equimomental axes.

For the moment round each axis will exceed that round its projection by $M\delta^2$, δ being the distance between the planes.

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* The terms *cyclo* and *sphero* polar reciprocal, were first introduced by Mr. Ingram, Fellow of Trinity College, Dublin, in order to distinguish from the general class of polars with respect to *any* curve or surface of the *second* order (which class alone possesses the important property of *reciprocality*) that particular class where the reciprocating curve or surface is a *circle* or *sphere*.

Hence, from the above, we know that every system of equimomental axes which lie all in a plane will envelope a conic, an ellipse, or hyperbola, as the case may be. And, by giving different values to the moment of inertia, the whole system of conics thus enveloped will be all concentric, coaxal, and confocal; the common centre, axes, and foci, being those of the ellipse contour of the orthographic projection on that plane of the ellipsoid of inertia round the centre of gravity, that ellipsoid being the sphero-polar reciprocal of the momental ellipsoid round the same point to the concentric sphere $x^2 + y^2 + z^2 = 1$.

In the particular case, when the plane of the equimomental axes is parallel to either cyclic plane of the central ellipsoid, the conics will be obviously all concentric circles. But of this, as well as of the general case, more hereafter.

Any two of these conics in any plane being confocal, the locus of the intersection of a tangent to either with a rectangular tangent to the other, will be a circle concentric with both: hence we arrive at the property, already noticed, that the locus of points in any plane for which rectangular axes in that plane have a constant sum of moments, is a circle of which the centre is the projection on the plane of the centre of gravity.

If we have one of these conics in any plane, we may at once find the axes of maximum and of minimum moment in that plane for every point thereon; for we have only to draw from the point two tangents to the conic, and then will the two rectangular lines which bisect the angles between them be the axes required: or, which comes to the same, we may describe through the point the ellipse and hyperbola confocal to the conic, and then will their normals at the point be the axes sought.

Since the tangents drawn from any point in the plane of a conic make equal angles with the tangents drawn from the same point to any confocal conic, a still easier construction than either of the above would be to connect the point with the foci of the given conic, and to bisect the angles between the connecting lines; the bisecting lines would then be the axes sought.

10. Now the axes so found at any point in a given plane coincide obviously with the geometric axes of the section in which the plane intersects the momental ellipsoid of that point; the above therefore affords an easy means of finding the axes of every central section of that ellipsoid at any point of a body.

It also enables us to find the locus of points for which these axes in a given plane enjoy a given property: for instance, let it be required to find the locus of points in the plane for which they remain parallel to each other.

Connecting all the points with the two foci of the system of conics in that plane, we shall have a series of triangles having all the same base, viz. the line joining the foci, and of which the bisectors of the internal and external vertical angles are all parallel to two given rectangular lines: the base, therefore, and the difference of the base angles being the same for all these triangles, the locus of their vertices (that is, of the points in question) will be an equilateral hyperbola, passing through the two foci, and having for centre the middle point of the line joining the same; that is, the projection of the centre of gravity on the given plane, and for asymptotes, lines parallel to the given bisectors.

In every plane, therefore, taken at random in a body, there exists a series of equilateral hyperbolas, such that for all the points of each the momental ellipsoid will intersect the plane in ellipses whose axes will be all parallel to each other, and these hyperbolas will have all the same centre, viz. the projection on the plane of the centre of gravity, they will all pass through two fixed points, viz. the foci of the system of confocal conics which envelope the systems of equimomental axes in that plane, and the locus of their vertices will be the lemniscata of the principal hyperbola, that, viz., whose axis is the line joining the two foci.

Since perpendiculars to the plane erected at the centre and foci of its system of conics pass through the centres and foci of the systems of conics in every parallel plane (9), it appears that if we move the above plane parallel to itself, each of the hyperbolas will describe a cylinder of the second order. Hence we know that the locus of those points in a body, for which two rectangular radii of their momental ellipsoids drawn in directions parallel to two given lines are the axes of the central sections whose planes they determine, will be a surface of the second order, a cylinder whose base will be an equilateral hyperbola in the plane parallel to the two given directions, whose axis perpendicular to that plane will pass through the centre of gravity, and whose asymptotic planes passing both through that axis will be parallel each to one of the given directions.

If in any plane we take the particular conic of its confocal system of equimomental envelopes which passes through the two foci, that is, the infinitely flat ellipse or hyperbola which

constitutes the transition state from one of these species to the other, then will all axes in the plane which pass through either of these foci be tangents to that particular conic, and be therefore equimomental. Hence we know that in every plane taken at random in a body there exist two points (the foci of a certain ellipse (9),) for which all axes in that plane will be equimomental, and for every system of parallel planes the loci of the points will be two parallel right lines perpendicular to the system of planes, and equidistant from, and in a plane passing through, the centre of gravity.

Now if at any point of a body a system of equimomental axes lie all in a plane, that plane must be a cyclic plane of the momental ellipsoid at that point (6). Hence, from the above, we know that in every plane assumed at pleasure in a body, there exist two points for which the momental ellipsoid will intersect it in a circle. These points are easily found, being the foci of an ellipse contour of the projection on the plane of the ellipsoid of inertia round the centre of gravity (9), from which we see that their loci for every system of parallel planes will be two parallel right lines, viz. the focal lines of the projecting cylinder.

In the particular case when the plane is parallel to a cyclic plane of the momental ellipsoid at the centre of gravity, there will exist but one such point, viz. the projection of that centre: this is evident, for the projecting cylinder will then be of revolution, and its axis passing through the centre of gravity will intersect the plane in a single point, and will be the locus of the similar points for the system of parallel planes.

The centre of gravity, in this case, is itself the point in the particular plane passing therethrough; that plane intersects the cylinder of revolution (which, as we shall see, is an important surface) in a circle, the squares of whose radii, multiplied each by the mass of the body, are equal to their moments of inertia; and if at all the other points we describe in their own planes, circles possessing the same property, the whole system of circles thus described will obviously generate a surface of revolution round the axis of the cylinder: this it is easy to see will be an hyperboloid of one sheet touching the cylinder along its principal section through the centre of gravity.

For, let r be the radius of the cylinder, x the distance of one of the planes from the centre of gravity, and y the radius of the circle in that plane, then have we $My^2 = Mr^2 + Mx^2$, and therefore $y^2 - x^2 = r^2$, an equilateral hyperbola.

More generally, if at every point of any axis through the centre of gravity perpendiculars be erected all round the axis, and that portions be taken on each whose squares, multiplied each by the mass of the body, shall equal their moments of inertia; then, for the same reason as above, will the locus of the extremities of these portions be a surface all whose sections by planes through the axis will be equilateral hyperbolas.

(It is, perhaps, needless to say that this surface will not be an hyperboloid, for, except in the particular case just noticed, its sections perpendicular to the axis will not be curves of the second order. See Note Art. 3.)

If at all points of any plane drawn at will through the centre of gravity of a body, we erect perpendiculars whose squares multiplied each by the mass shall equal their moments of inertia, the locus of their extremities will be an hyperboloid of two sheets of revolution round the perpendicular through the centre of gravity:

For, drawing through that perpendicular any two rectangular planes, let the coordinates referred thereto of the extremity of one of the variable perpendiculars z be xy, then have we $Mz^2 = Mz_1^2 + M(x^2 + y^2)$, and therefore $x^2 + y^2 - z^2 = z_1^2$; the meridians therefore are equilateral hyperbolas, and the surface is of revolution round their transverse axis.

(The plane obviously need not pass through the centre of gravity, and the only difference will be that the centre of the hyperboloid will not be that centre, but its projection on the plane.)

For every plane through the centre of gravity we have a different hyperboloid; these are all connected by the property that the squares of their axes represent their moments of inertia. If, therefore, at the vertices of each we draw tangent planes, the whole system of planes thus produced will envelope an ellipsoid (of which we have spoken and shall frequently use again), the ellipsoid of gyration, sphero-polar reciprocal with respect to the sphere $x^2 + y^2 + z^2 = 1$ of the momental ellipsoid at the centre of gravity. This is evident, for the coincident radii of the latter surface are inversely as the axes of the hyperboloids, shewing that the extremities of these radii are the poles with respect to that sphere of the tangent planes in question (Note Art. 3), and that, conversely, the locus of the vertices of the polar reciprocal system of hyperboloids is the momental ellipsoid at the centre of gravity.

Since every plane parallel to the axis of any one of these hyperboloids intersects it in an equilateral hyperbola, of which the centre is the projection on the plane of the centre of gravity, it appears that, if in any plane taken at will we draw any line whatever, and at all its points erect perpendiculars in the plane whose squares multiplied by M shall equal their moments of inertia, the locus of the extremities of those ordinates will be an equilateral hyperbola of which the line will be the imaginary axis, and whose centre will be the projection on that line of the centre of gravity.

If in place of drawing the ordinates perpendicularly to the assumed line, we draw them all obliquely to the same, the locus will be still an hyperbola, but its centre will no longer be the same as before, being in all cases the foot of the particular ordinate drawn through the projection on the plane of the centre of gravity, and that ordinate with the assumed line will be always conjugate diameters of the hyperbola.

11. Throughout the preceding section we had frequent opportunities of observing the symmetrical arrangement on every arbitrary plane round the projection thereon of the centre of gravity, of every system of axes whose moments of inertia follow all the same law, that point not only being a centre round which every thing is the same in opposite directions, but also the projected axes of the parallel central section of the momental ellipsoid at the centre of gravity dividing the plane into four regions, in which the axes of equal moment are arranged symmetrically and similarly round that centre, and the same being, of course, true for the centre of gravity itself with respect to every plane passing therethrough. This, however, we might have easily anticipated; for, from its property respecting the moments of inertia of parallel axes, we readily see that, more generally, the centre of gravity of every body possesses the same properties in *space*, everything being the same with respect to the moments of inertia of all axes similarly situated in opposite directions from that point, and from the additional circumstance that the surface round the same centre whose radii represent their moments of inertia is symmetrical and equal in the eight regions of space determined by the three central principal planes: we see also that everything is the same respecting the moments of inertia of axes and of systems of axes which are symmetrically situated in these eight different regions with respect to the three principal axes through the centre of gravity.

With respect to *principal* axes the same important properties tell us—

(1) That at every two points of a body equidistant, and in opposite directions from the centre of gravity, the principal axes are all mutually parallel to each other and respectively equimomental.

For, the moments of inertia being equal round every pair of parallel lines through any two such points, their momental ellipsoids will be equal, similar, and similarly placed; the axes, therefore, of these ellipsoids (with which the principal axes of the body at the same points coincide) must be parallel, and their moments of inertia respectively equal.

(2) That at every eight points of a body equidistant from, and similarly and symmetrically situated with respect to the three principal axes at the centre of gravity, the principal axes will be also all similarly and symmetrically situated with respect to the same three axes; which is evident from the symmetry of the momental ellipsoid round that centre.

Whatever, therefore, be the geometric distribution in space of the principal axes of a body fixed in position, we see that the centre of gravity is a point round which they are similarly and symmetrically situated in the eight regions of space determined by the three principal planes of that centre; a circumstance which will, of course, be verified when we ascertain the laws (whatever they be) which determine their directions at different points, but which, when we consider the dynamical property of these axes and the utter irregularity of almost all bodies with respect to figure, density, and structure, is by no means evident *à priori.**

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* The symmetry here noticed holds also in many other cases in which it would be still more difficult, or rather impossible, to perceive, *à priori*, its necessary existence. For instance, in the highly elastic and absolutely incompressible ether from the vibrations of which arise the phenomena of light, the discoveries of Professor MacCullagh have shewn that at every point there exist three axes at right angles to each other with respect to which everything is the same in every eight directions of symmetry. Also, in the altogether different case of solid elastic bodies, the recent researches of Mr. Haughton, Fellow of Trinity College, Dublin, respecting their equilibrium and motion, have indicated that, whatever be their internal structure, there exist also at every point three rectangular axes round which a symmetry of a remarkable and unexpected nature is always found. It is a circumstance not a little surprising, that strict mathematical reasoning, setting out from the simple elementary principles which characterise each case, should be able to prove, in any case, the existence of this symmetry, which is a physical fact not apparently at all connected with the principles from which we set out. It was to call attention to this circumstance that we have made the above remark, lest in more difficult cases it might be conceived to be impossible to prove such existences.

12. Equimomental axes, in a body considered merely as such with no other restriction, are, of course, infinite in number, being so for each individual point: they are all, however, confined within very narrow limits round the centre of gravity, passing all through the space between two spheres which may be readily found.

For, let I be their common moment, and A and C the greatest and least central moments; then describing round the centre two spheres whose radii are $\sqrt{\left(\frac{I-A}{M}\right)}$ and $\sqrt{\left(\frac{I-C}{M}\right)}$, they must obviously pass all without the former and within the latter.

The greater the value of I, the narrower therefore will be the limits, for the rectangle under the sum and difference of the radii of the spheres is constant; so that the greater their sum, the less their difference.

In the particular class of bodies for which the three central principal moments are equal, they will be still more restricted, for then obviously must each system be equidistant from, and therefore envelope a sphere round, the centre of gravity; the different spheres for different values of the common moment being all concentric, that point being their common centre.

In such bodies, therefore, the different systems of equimomental axes in every plane assumed at pleasure will envelope a system of concentric circles round the projection on each plane of the centre of gravity: this is also evident from (10), where it appeared that in *every* body there exist two systems of parallel planes which possess the same property, viz. those parallel to the cyclic planes of the momental ellipsoid at the centre of gravity.

The most interesting class of bodies, with respect to the arrangement of their equimomental axes, are those for which two of the three principal central moments are equal, the third not having the same value. For in these, the momental ellipsoid at the centre of gravity, and therefore the ellipsoid of inertia its reciprocal to the concentric sphere $r^2 = 1$, will be of revolution, and so will also the whole central system of equimomental cones, the axis of revolution common to all these surfaces being that of the unequal moment.

Now, for every plane passing through that axis every system of equimomental axes (10) will envelope a conic confocal with the meridian of the ellipsoid of inertia, and which will be an ellipse or hyperbola, according as the common moment exceeds the greater of the two principal central moments, or is intermediate to the greater and less, the

asymptotes of the hyperbola in the latter case being the particular pair of axes which pass through the centre.

Hence every system of equimomental axes which pass all through the axis of unequal moment will envelope a central surface of the second order of revolution round that axis, and which will be an ellipsoid or hyperboloid, as the case may be. And, for different values of the common moment, the whole system of envelopes will be confocal with the ellipsoid of inertia at the centre of gravity, and the central system of equimomental cones will be obviously the asymptotic cones of the system of hyperboloids.

(It hence immediately appears, that if upon any plane assumed at will we project orthographically this system of confocal surfaces, then will the confocal system of conics, contours of the several projections, be the system of envelopes of equimomental systems of axes in that plane (10). But this, as well as the theorem itself, are particular cases of one still more general, which shall be subsequently noticed).

Of bodies of this class there are two varieties (quite distinct in many of their properties), according as the unequal moment is greater or less than those that are equal: in the former case the system of confocal ellipsoids will be all *prolate* spheroids, and the hyperboloids will be all of *two sheets;* in the latter the ellipsoids will be all *oblate* spheroids and the hyperboloids all of *one sheet:* this is obvious, for the system of confocal conics which have been supposed to generate these surfaces, in the former case revolve round their primary axis, and in the latter, round their secondary.

The rectilinear generatrices of each hyperboloid, in the latter case, are also a system of equimomental axes: this is evident, for they are in pairs, parallel to, and equidistant from, the sides of their asymptotic cones, which for each hyperboloid are a system of equimomental axes through the centre of gravity.

Bodies of the former variety possess the important property, that in them there exist two points for which *all* axes are principal, (5). These points are the two foci of the central spheroid of inertia and of its prolate system of confocal surfaces; that is, of the ellipsoid of revolution reciprocal to the momental spheroid at the centre of gravity.

For *all* axes through either of these foci pass through the axis of unequal moment, and are tangents to the infinitely slender surfaces of revolution which, terminated by the two foci, form the transition state between the ellipsoids and the hyperboloids of the confocal system: these axes are therefore *all* equimomental, and therefore (5) all principal.

Bodies of the second variety admit of no point for which *all* axes are principal, but all points on the focal circle of their oblate system of confocal surfaces of revolution admit of an *infinite number* of principal axes (5), which, for each point, lie all in the plane containing that point and the axis of revolution, that is, in the normal plane to the circle at the same point.

For all axes through any of these points which lie in the normal plane to the circle intersect the axis of unequal moment, and are tangents to the infinitely flat surfaces of revolution which, bounded by that circle, form the transition state between the ellipsoids and the hyperboloids: these axes are therefore all equimomental, and, as lying in the equators of the momental ellipsoids at each point, are therefore all principal.

For both varieties of body, that is, for all bodies of the class we are now considering, the axis of revolution is obviously principal for all points along it, which points also admit all of an infinite number of principal axes in planes perpendicular to that axis: but these, being particular cases of a more general property common to all bodies, which will be fully discussed in a subsequent article, we shall notice no further at present.

(*To be continued.*)

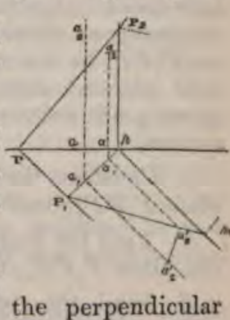
Trinity College, Dublin, May 1846.

MISCELLANEOUS NOTES ON DESCRIPTIVE GEOMETRY. NO. I.

By T. S. Davies, F.R.S.L. & E., F.S.A.

The following constructions are, I think, simpler than any which have been given by the French writers.

Prop. I. *To construct the perpendicular from a given point to a given plane.*



Let P_1PP_2 be the given plane exhibited by means of its traces, and a_1, a_2 the projections of the given point.

Through a_1 draw the perpendicular pP_1 to PP_1, and from p draw pP_2 perpendicular to the axis Pp. Make pp_2 at right angles to pP_1 and equal to pP_2, and join p_2P_1.

Again, draw a_1a_2' perpendicular to pP_1 and equal to aa_2; and from a_2' draw $a_2'a_2'$ perpendicular to P_1p_2. Then $a_2'a_2'$ is the perpendicular required.

Also, for the projections of the point of intersection, draw $a_2'a_1$ perpendicular to pP_1, and a_1a_2 perpendicular Pp, taking aa_2 equal to a_1a_2'. Then a_1, a_2 are the projections of the point in which the perpendicular meets the plane.

It is clear that the construction might have been similarly made on the vertical plane.

The truth of this construction will be so obvious to those who are familiar with the subject, as to render a formal demonstration superfluous.

PROP. II. *A point is situated in a given plane, and one of its projections is given to construct the* rabattement *of the plane and of the point in it, upon either of the coordinate planes.*

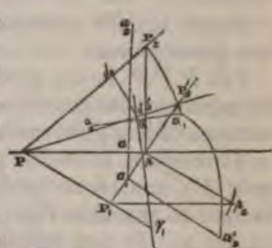
Find the vertical projection a_2 of the given point in the usual manner as indicated by the dotted lines.

Through a_1 draw the perpendicular P_1P_2', and make PP_2' equal to PP_2: then PP_2' is the *rabattement* of PP_2. In $a_1\beta_1$ make a_1a_2' equal to aa_2, and then P_1a' equal to P_1a_2'. Then a' is the rabattement of the point.

If a line in the plane P_1PP_2 is to be *rabatted,* let its traces be γ_1, δ_2. Make $P\delta_2'$ equal to $P\delta_2$, and join $\gamma_1\delta_2'$. This is the position of the line required.

The angle P_1PP_2' is that formed by the traces in their natural position: for finding which *directly,* this construction is well adapted.

PROP. III. *Given the projections of a point upon two coordinate planes, to find its projections upon any other two, both systems being rectangular.*



Let $\gamma_1\delta_2$ be the intersection of the new coordinate planes referred to the original ones whose axis is Pp; and let P_1PP_2 be one of the new coordinate planes. Also, let a_1a_2 be the given point referred to the original system.

1. Construct the *rabattement* of the plane P_1PP_2 about PP_1, (by Prop. II.)

2. Find the length a_2a' of the perpendicular from a_1a_2 to the plane P_1PP_2, (by Prop. I.)

3. Find the *rabattement* $\gamma_1\delta_2'$ of the line $\gamma_1\delta_2$, and the rabattement a_1 of the point a_1a_2, (by Prop. II.)

4. Draw a_1a_2 perpendicular to $\gamma_1\delta_2'$, so that aa_2 shall be equal to $a_2'a'$.

Then a_1a_2 is the point referred to the new system.

Scholium.

The use of oblique coordinate planes has not been introduced into Descriptive Geometry, except in one or two simple cases. No general system of construction by means of them has ever been suggested; nor indeed does there seem to exist an impression that their introduction would be attended with any advantage in actual construction.

There is no doubt that the problems of the most usual practical occurrence do, *in general*, admit of more elegant construction by means of rectangular coordinate planes than by means of oblique: still my own experience convinces me that this rule is very far from being universal. At any rate, as a matter of mathematical interest, it seems advisable to give a development of such a system of construction, in order that we might have a choice of means (analogous to those we possess in the ordinary geometry of coordinates) of selecting the methods best adapted to any given problem.

Such a system, with the Editor's permission, I propose to give hereafter through the medium of this Journal.

College for Civil Engineers, Putney, March 20, 1846.

ANALYTICAL INVESTIGATIONS OF TWO OF DR. STEWART'S GENERAL THEOREMS.

By T. S. Davies, F.R.S.L. & E., F.S.A.

About two years ago I sent the Royal Society of Edinburgh "An Analytical Investigation of Dr. Matthew Stewart's General Theorems," which paper was printed in the *Transactions* of the Society. Those mathematicians who have looked into these theorems do not need to be informed that not the slightest clue to a general mode of investigation can be deduced from the solutions which Dr. Stewart himself gave of the first eight of them; or, at any rate, that no one has ever succeeded, by following his steps or imitating his processes, in proving the truth of the remoter and most general ones.

There are two of those theorems (the *seventh* and *eighth*) which Dr. Stewart demonstrated by geometrical consider-

ations, of which it did not fall in my plan in drawing up that paper to take special notice. I had, however, subjected these to the same process which I had found to be effective in treating those which had a certain analogy (though not a very close one) to the theorems which had been merely enunciated; and possibly the solutions so obtained might not be without interest to those geometers who have devoted any attention to these subjects.

I have given the steps with adequate detail to render the reading of the solutions more facile than they would be if much abbreviated: and I would remark that by the use of rectangular coordinates the expressions may be rendered more brief to the *eye*, although it does not appear likely that the reasoning itself could be materially abbreviated by such means.

PROP. VII. GENERAL THEOREMS.

"*Let there be any circle whose centre is* A, *and let* BCD *be a segment of the circle, and* BD *the chord of the segment; about the segment let there be any equilateral figure circumscribed touching the circle in the points* E, F, G, *etc., and let the two sides of the figure next* BD *meet* BD *in* H, K; *bisect the segment* BCD *in* F, *and join* AF; *in* AF *take the point* L *on the same side of the centre* A *with the point* F, *and let the sum of the sides of the figure circumscribed about the segment into* HK *as the semidiameter to* AL; *draw* ML *perpendicular to* AL *meeting the circle in* M. *If from the points* E, F, G, *etc., the points of contact of the circumscribed figure, and the point* L, *there be drawn right lines to any point* N, *the sum of the squares of* EN, FN, GN, *etc., will be equal to the multiple of the sum of the squares of* LM, LN *by the number of the sides of the figure.*"

Dr. Stewart divides his demonstration into two cases, according as N is in the circumference of the given circle or not, the former being a "case of case" to the latter. In the method here employed it will be more convenient to consider the two cases dependent on the number of sides of the polygon being even or odd.

1. *The polygon having* $2n$ *sides.*

Let the given segment be $\dfrac{2n\pi}{m}$, and ρ the radius of the given circle. Take the line through the centre of the circle, viz. AF, as angular origin, which will also pass through an

angular point of the figure. Then it will be obvious from the figure itself that

$$\text{perimeter} = 4n\rho \tan \frac{\pi}{2m},$$

$$HK = 2\rho \sec \frac{\pi}{2m} \sin \frac{n\pi}{m},$$

$$AL = \frac{\rho \sin \frac{n\pi}{m}}{2n \sin \frac{\pi}{2m}},$$

and

$$LM^2 = \rho^2 \left\{1 - \frac{\sin^2 \frac{n\pi}{m}}{4n^2 \sin^2 \frac{\pi}{2m}}\right\} \ldots\ldots\ldots\ldots (1).$$

Now the coordinates of the points of contact referred to the centre A and axis AF will be

$$\left(\rho, \frac{\pi}{2m}\right), \left(\rho, \frac{3\pi}{2m}\right) \ldots \left\{\rho, \frac{(2n-1)\pi}{2m}\right\};$$

and if $r\theta$ denote the arbitrary point N, and the expressions for the several squares enunciated be formed, we shall have

$$NL^2 = r^2 - 2r.AL \cos\theta + AL^2$$

$$= r^2 - 2r\rho . \frac{\sin \frac{n\pi}{m} \cos\theta}{2n \sin \frac{\pi}{2m}} + \rho^2 . \frac{\sin^2 \frac{n\pi}{m}}{4n^2 \sin^2 \frac{\pi}{2m}} \ldots (2),$$

and from (1, 2), we get at once

$$NL^2 + LM^2 = r^2 - r\rho \frac{\sin \frac{n\pi}{m}}{n \sin \frac{\pi}{2m}} \cos\theta + \rho^2 \ldots (3).$$

Also taking the squares of the lines drawn from N to the points of contact equidistant from the circular origin in pairs, we shall have them represented by

$$\left.\begin{array}{l} r^2 - 2r\rho \cos\left(\theta - \dfrac{\pi}{2m}\right) + \rho^2 \\ r^2 - 2r\rho \cos\left(\theta + \dfrac{\pi}{2m}\right) + \rho^2 \end{array}\right\},$$

$$\left.\begin{array}{l} r^2 - 2r\rho\cos\left(\theta - \dfrac{3\pi}{2m}\right) + \rho^2 \\ r^2 - 2r\rho\cos\left(\theta + \dfrac{3\pi}{2m}\right) + \rho^2 \end{array}\right\},$$

. . . .

$$\left.\begin{array}{l} r^2 - 2r\rho\cos\left\{\theta - \dfrac{(2n-1)\pi}{m}\right\} + \rho^2 \\ r^2 - 2r\rho\cos\left\{\theta + \dfrac{(2n-1)\pi}{m}\right\} + \rho^2 \end{array}\right\}.$$

Expanding the cosines and adding, and putting S^2 for the sum of the squares of the lines, we have

$$S^2 = 2n(\rho^2 + \gamma^2) - 4r\rho\cos\theta\left(\cos\frac{\pi}{2m} + \cos\frac{3\pi}{2m} + \ldots\right).$$

But $$\cos\frac{\pi}{2m} + \cos\frac{3\pi}{2m} + \ldots + \cos\frac{(2n-1)\pi}{2m} = \frac{\sin\dfrac{n\pi}{m}}{2\sin\dfrac{\pi}{2m}},$$

and hence $$S^2 = 2(\rho^2 + r^2) - 2r\rho\frac{\sin\dfrac{n\pi}{m}}{\sin\dfrac{\pi}{2m}}\cos\theta \ldots\ldots (4).$$

Wherefore, multiplying (3) by $2n$, we have (4), which shews that in this case the proposition is true; viz. that

$$S^2 = 2n(NL^2 + LM^2).$$

2. *Let the polygon have* $2n + 1$ *sides.*

Take origin and axis as before, the axis now passing through one of the points of contact of the figure. Then we shall have, the segment being denoted by $\dfrac{(2n+1)\pi}{m}$,

$$\text{perimeter} = 2(2n+1)\rho\tan\frac{\pi}{2m},$$

$$HK = 2\rho\sec\frac{\pi}{2m}\sin\frac{(2n+1)\pi}{2m},$$

$$AL = \frac{\rho\sin\dfrac{(2n+1)\pi}{2m}}{(2n+1)\sin\dfrac{\pi}{2m}},$$

whence $$ML^2 = \rho^2 \left[1 - \frac{\sin^2 \frac{(2n+1)\pi}{2m}}{(2n+1)^2 \sin^2 \frac{\pi}{2m}}\right] \ldots\ldots (1),$$

and $$NL^2 + LM^2 = r^2 - 2r\rho \frac{\sin \frac{(2n+1)\pi}{2m} \cos\theta}{(2n+1)\sin\frac{\pi}{2m}} + \rho^2 \ldots (2).$$

Again, for the lines drawn to the points of contact, that drawn to the circular origin being taken alone, and those to the points of contact equidistant from it in pairs, we shall have

$$r^2 - 2r\rho\cos\theta + \rho^2,$$

$$\left.\begin{array}{l} r^2 - 2r\rho\cos\left(\theta - \frac{\pi}{m}\right) + \rho^2 \\ r^2 - 2r\rho\cos\left(\theta + \frac{\pi}{m}\right) + \rho^2 \end{array}\right\},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\left.\begin{array}{l} r^2 - 2r\rho\cos\left(\theta - \frac{n\pi}{m}\right) + \rho^2 \\ r^2 - 2r\rho\cos\left(\theta + \frac{n\pi}{m}\right) + \rho^2 \end{array}\right\}.$$

Adding all these, except the first, we get the sum of all the squares except that one:

$$= 2n(r^2 + \rho^2) - 4r\rho\cos\theta\left\{\cos\frac{\pi}{m} + \cos\frac{2\pi}{m} + \ldots \cos\frac{n\pi}{m}\right\},$$

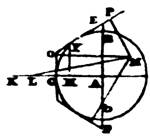
$$= 2n(r^2 + \rho^2) - 2r\rho\cos\theta\left[\frac{\sin\frac{(2n+1)\pi}{2m}}{\sin\frac{\pi}{2m}} - 1\right].$$

Add the first line to this: then there results

$$S^2 = (2n+1)(r^2 + \rho^2) - 2r\rho \frac{\sin\frac{(2n+1)\pi}{2m}}{\sin\frac{\pi}{2m}} \cos\theta \ldots (3).$$

Multiply (2) by $2n + 1$, and we have the result equal to (3); which therefore also proves the theorem in this case.

PROP. VIII. GENERAL THEOREMS.



"*Let there be any circle whose centre is* A, *and let* BCD *be a semicircle, and* BD *the diameter of the circle; about the semicircle let there be any regular figure described, and let the sides of the figure next to* BD *meet* BD *in* E, F; *bisect the semicircle in* G *and join* AG; *and in* AG *take the point* H *on the same side of the centre* A *with the point* G, *and let* AG *be to* AH *as the sum of the sides of the figure to* EF; *and let the rectangle* HAK *be equal to the square of the semidiameter, and* HL *be equal to* AH: *if from any point* M *there be drawn* MN, MO, MP, *etc., perpendicular to the sides of the figure circumscribed about the semicircle, and likewise there be drawn* ML *to the point* L; *twice the sum of the squares of the perpendiculars* MN, MO, MP, *etc., will be equal to the multiple of the square of* ML *by the number of the sides of the figure together with the multiple of the rectangle* KLA *by the same number.*"

1. *Let the figure have* $2n$ *sides.*

Taking the line AG as angular origin, the angular axis will pass through an angular point of the circumscribed figure. Denote the radius by ρ: then the polar equations of the sides of the figure whose points of contact lie respectively above and below the circular origin will be (Hutton's *Course*, vol. II. p. 264, *twelfth edition*),

$$\rho = r\cos\left(\theta - \frac{\pi}{4n}\right),\quad \rho = r\cos\left(\theta - \frac{3\pi}{4n}\right),\ldots$$

$$\rho = r\cos\left\{\theta - \frac{(2n-1)\pi}{4n}\right\},$$

$$\rho = r\cos\left(\theta + \frac{\pi}{4n}\right),\quad \rho = r\cos\left(\theta + \frac{3\pi}{4n}\right),\ldots$$

$$\rho = r\cos\left\{\theta + \frac{(2n-1)\pi}{4n}\right\},$$

and the perpendiculars from the arbitrary point $r\theta\,(M)$ upon these will be (*ib.*)

$$\rho - r\cos\left(\theta - \frac{\pi}{4n}\right),\quad \rho - r\cos\left(\theta - \frac{3\pi}{4n}\right),\ldots$$

$$\rho - r\cos\left\{\theta - \frac{(2n-1)\pi}{4n}\right\},$$

$$\rho - r\cos\left(\theta + \frac{\pi}{4n}\right),\quad \rho - r\cos\left(\theta + \frac{3\pi}{4n}\right), \ldots$$

$$\rho - r\cos\left\{\theta + \frac{(2n-1)\pi}{4n}\right\}.$$

Taking the sums of the squares of these in pairs as they stand beneath each other, and expressing the results in multiple cosines, we get

$$2\rho^2 + r^2 - 4r\rho\cos\theta\cos\frac{\pi}{4n} + r^2\cos 2\theta\cos\frac{\pi}{n},$$

$$2\rho^2 + r^2 - 4r\rho\cos\theta\cos\frac{3\pi}{4n} + r^2\cos 2\theta\cos\frac{3\pi}{n},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$2\rho^2 + r^2 - 4r\rho\cos\theta\cos\frac{(2n-1)\pi}{4n} + r^2\cos 2\theta\cos\frac{(2n-1)\pi}{n}.$$

Now the sum of these being taken, the column in $\cos 2\theta$ vanishes, its coefficient being $\dfrac{\sin\pi\cos\pi}{\sin\frac{\pi}{n}} = 0$; and the column in $\cos\theta$ has for its coefficient the value $\dfrac{\sin\frac{\pi}{4}\cos\frac{\pi}{4}}{\sin\frac{\pi}{4n}} = \dfrac{1}{2\sin\frac{\pi}{4n}}$:

and hence twice the sum of the squares, $2S^2$, of the $2n$ perpendiculars is

$$2S^2 = 4n\rho^2 + 2nr^2 - 4r\rho\frac{\cos\theta}{\sin\frac{\pi}{4n}} \quad \ldots\ldots\ldots (1).$$

Again, from the obvious properties of the figure,

$$\text{perimeter} = 4n\rho\tan\frac{\pi}{4n},$$

$$AH = \frac{\rho}{2n\sin\frac{\pi}{4n}},$$

$$AK = 2n\rho\sin\frac{\pi}{4n},$$

$$AL = \frac{\rho}{n\sin\frac{\pi}{4n}},$$

[redacted]

$$KL = \rho \,.\, \frac{2n^2 \sin^2 \frac{\pi}{4n} - 1}{n \sin \frac{\pi}{4n}},$$

$$KL.LA = 2\rho^2 - \frac{\rho^2}{n^2 \sin^2 \frac{\pi}{4n}},$$

$$ML^2 = LA^2 - 2LA.AM \cos LAM + AM^2$$

$$= \frac{\rho^2}{n^2 \sin^2 \frac{\pi}{4n}} - \frac{2r\rho \cos \theta}{n \sin \frac{\pi}{4n}} + r^2;$$

and hence $ML^2 + KL.LA = 2\rho^2 + r^2 - \dfrac{2r\rho \cos \theta}{n \sin \frac{\pi}{4n}}$ .... (2).

Whence multiplying this by $2n$, we have the same result as in (1); and this identity of value is that enunciated in the proposition, viz.

$$2S^2 = 2n\,(ML^2 + KL.LA).$$

2. *Let the figure have* $2n + 1$ *sides.*

In this case the angular axis will pass through the middle point of contact; and the perpendiculars from $r\theta$ upon the sides of the figure, whose points of contact are respectively above and below the circular origin, will be (omitting that upon the side whose point of contact is the circular origin)

$$\rho - r \cos\left(\theta - \frac{\pi}{2n+1}\right),\quad \rho - r \cos\left(\theta - \frac{2\pi}{2n+1}\right) \ldots.$$

$$\rho - r \cos\left(\theta - \frac{n\pi}{2n+1}\right),$$

$$\rho - r \cos\left(\theta + \frac{\pi}{2n+1}\right),\quad \rho - r \cos\left(\theta + \frac{2\pi}{2n+1}\right) \ldots.$$

$$\rho - r \cos\left(\theta + \frac{n\pi}{2n+1}\right).$$

Twice the sum of the squares of these being taken, and the results expressed in multiple cosines, we get

$$2\left\{2\rho^2 + r^2 - 4r\rho \cos \theta \cos \frac{\pi}{2n+1} + r^2 \cos 2\theta \cos \frac{2\pi}{2n+1}\right\},$$

$$2\left\{2\rho^2 + r^2 - 4r\rho\cos\theta\cos\frac{2\pi}{2n+1} + r^2\cos 2\theta\cos\frac{4\pi}{2n+1}\right\},$$

. . . .

$$2\left\{2\rho^2 + r^2 - 4r\rho\cos\theta\cos\frac{n\pi}{2n+1} + r^2\cos 2\theta\cos\frac{2n\pi}{2n+1}\right\},$$

the sum of which is

$$2n(2\rho^2 + r^2) - 4r\rho\cos\theta\left\{\frac{1}{\sin\frac{\pi}{2n+1}} - 1\right\} - r^2\cos 2\theta \quad \ldots(1).$$

Also double the square of the omitted perpendicular is

$$2\rho^2 + r^2 - 4r\rho\cos\theta + r^2\cos 2\theta,$$

which, added to (1), gives

$$2S^2 = (2n+1)(2\rho^2 + r^2) - 4r\rho \cdot \frac{\cos\theta}{\sin\frac{\pi}{2n+1}} \quad \ldots\ldots(2).$$

Again, we have, as in the preceding case,

$$\text{perimeter} = 2(2n+1)\rho\tan\frac{\pi}{2(2n+1)},$$

$$AH = \frac{\rho}{(2n+1)\sin\frac{\pi}{2(2n+1)}},$$

$$AK = (2n+1)\rho\sin\frac{\pi}{2(2n+1)},$$

$$AL = \frac{\pi}{(2n+1)\sin\frac{2\rho}{2(2n+1)}},$$

$$KL = \rho \cdot \frac{(2n+1)^2\sin^2\frac{\pi}{2(2n+1)} - 2}{(2n+1)\sin\frac{\pi}{2(2n+1)}},$$

$$KL.LA = 2\rho^2 - \frac{4\rho^2}{(2n+1)^2\sin^2\frac{\pi}{2(2n+1)}},$$

$$LM^2 = \frac{4\rho^2}{(2n+1)^2\sin^2\frac{\pi}{2(2n+1)}} - \frac{4r\rho\cos\theta}{(2n+1)\sin\frac{\pi}{2(2n+1)}} + r^2;$$

whence, adding, we get

$$KL.LA + LM^2 = 2\rho^2 + r^2 - \frac{4r\rho\cos\theta}{(2n+1)\sin\dfrac{\pi}{2(2n+1)}},$$

$$\text{or } (2n+1)(KL\,LA+LM^2)=(2n+1)(2\rho^2+r^2)-\frac{4r\rho\cos\theta}{\sin\dfrac{\pi}{2(2n+1)}} \;..(3).$$

The identity of (1) and (3) proves the truth of the theorem, also, when the number of sides is odd.

Royal Military Academy, Woolwich, March 25, 1846.

ON ARBOGAST'S FORMULÆ OF EXPANSION.

By Augustus De Morgan,
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§ 1. *General Theory of Derivatives.*

The theory of Arbogast has received so little attention in this country, that no excuse is necessary for an attempt to exhibit its rules in a short and comparatively easy manner. Arbogast himself was more occupied in proving the ease with which his method could be applied to very complicated cases, than in illustrating the connexion of its principles with those of other parts of analysis.

The first attempt of which I know, to write on this subject in English, is contained in the posthumous work* of Mr. West, which is a very complete attempt, as far as series of one variable are concerned. The next is that which I made myself in my work on the Differential Calculus, (at which time I did not know of Mr. West's work): and I am not aware of any other. I think that many mathematicians are under the impression that Argobast's method belongs to the *combinatorial analysis* of Hindenburg and his followers. This, however, any one who carefully examines both will find is not the case.

When Arbogast developes $\phi(a + bx + cx^2 + \ldots)$, he presumes

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* 'Mathematical Treatises, containing, 1. the Theory of Analytical Functions .... By the Rev. John West .... edited .... from his MSS. .... by the late Sir John Leslie ....' Edinburgh, 1838. 8vo.—Mr. West died in Jamaica in 1817, aged 61.

that $a+bx+cx^2+\ldots$ is some particular case of $F(a+x)$: so that the development is

$$\phi Fa + (\phi F)'a \,.\, x + (\phi F)''a \,.\, \frac{x^2}{2} + \ldots$$

Now $(\phi F)'a$ is $\phi' Fa \,.\, F'a$, or $\phi' a \,.\, b$, &c., whence he derives his fundamental rule, namely, that the coefficients of x, x^2, &c. are obtained from ϕa by successive differentiations (followed by divisions by 1, 2, 3, &c.) of ϕa with respect to an imaginary variable; on the supposition that a, b, c, &c. have the differential coefficients b, $2c$, $3d$, &c. From this method, partly by inspection, he obtains the subordinate rules which distinguish his treatise from others on the same subject. This is enough to shew that Arbogast's method begins by the use of the usual methods of analysis.

I shall first explain the general form which contains these rules, and then proceed to establish them, and to connect them with a problem of combinations, from which such simplifications as they admit of may be easily deduced.

There is a difficulty about notation, particularly in the case of series of two variables. Some will think it best to proceed by letters, a, b, c, e, &c.; others by superfixes or suffixes, as a, a', a'', &c., a_0, a_1, a_2, &c. Arbogast proceeds by letters with respect to x, and by superfixed accents with respect to y. I shall (at least where two variables are concerned) use both methods, denoting a series of one variable by

$$a + b'x + c''x^2 + e'''x^3 + f^{\text{IV}}x^4 + \ldots,$$

and one of two variables by

$$\begin{array}{l} a + b'x + c''x^2 + e'''x^3 \\ \quad + b_{\prime}y + c_{\prime}'xy + e_{\prime}''x^2y \\ \qquad + c_{\prime\prime}y^2 + e_{\prime\prime}'xy^2 \\ \qquad\quad + e_{\prime\prime\prime}y^3 \end{array} + \&\text{c}.$$

For this I have a twofold reason. First, those who prefer either plan alone may drop that portion of my distinctions which is to them superfluous. Secondly, I have myself found this double system of notation a very useful check in the algebraical operations which this subject contains.

I now proceed to the definitions and rules of the system.

Let there be a function of any number of letters, say a, b, c, e, &c.; and let there be a number of *convertible and distributive* operations, $\alpha, \beta, \gamma, \epsilon$, &c., of which each acts only on one letter, α on a, β on b, &c., so that $\gamma\phi(a, b)$, for instance, is $= 0$.

Derivation, the process by which so much of algebraical development is performed, is the performance of the operations $a\beta^{-1}$, $\beta\gamma^{-1}$, $\gamma\varepsilon^{-1}$, &c. These give *partial derivatives,* and the complete derivative is obtained from the performance of $a\beta^{-1}+\beta\gamma^{-1}+\gamma\varepsilon^{-1}+\ldots$; *but only on this condition, that no term which is produced by any one of the partial derivatives shall be allowed to appear again as a result of any other.* So that, in fact, complete derivation means—the result of $a\beta^{-1}$ + so much of that of $\beta\gamma^{-1}$ as is not given by $a\beta^{-1}$ + so much of that of $\gamma\varepsilon^{-1}$ as is not given by $a\beta^{-1}+\beta\gamma^{-1}+$ &c. It is obvious that the order of the terms is convertible in the expression of this condition.

If the quantity operated upon be a function only of a, then it is useless to perform β or any subsequent operation except upon terms which have undergone the inverse operation. The inverse operation must produce its letter, or the process would be unmeaning: thus, if $\gamma^{-1}\phi(a, b)$ were not a function of c, $\gamma\gamma^{-1}\phi(a, b)$, which should be $\phi(a, b)$, would be 0. Thus again, beginning with a function of b only, the first derivative has the operation $\beta\gamma^{-1}$ only; the second has only $(\beta\gamma^{-1}+\gamma\varepsilon^{-1})\,\beta\gamma^{-1}$, or $\beta^2\gamma^{-2}+\beta\varepsilon^{-1}$. The third has only $\beta^3\gamma^{-3}+\beta^2\gamma^{-1}\varepsilon^{-1}+(\beta^2\gamma^{-1}\varepsilon^{-1}$ rejected as having already occurred) $+\beta\zeta^{-1}$, and so on.

It is clear that any derivative formed from a function of b, by means of $\beta\gamma^{-1}+\ldots$, contains direct forms of β only, and inverse ones of all the others. The reason is that, (λ, μ, ν being consecutive operations) $\lambda\mu^{-1}$ must have been performed in any term, to introduce m, before anything is produced on which $\mu\nu^{-1}$ is effective: and μ is destroyed by μ^{-1}. If we write down a few of the above derivations from a function of b, we have, calling D the symbol of derivation,

$$D = \beta\gamma^{-1},$$
$$D^2 = \beta\epsilon^{-1}+\beta^2\gamma^{-2},$$
$$D^3 = \beta\zeta^{-1}+\beta^2\gamma^{-1}\epsilon^{-1}+\beta^3\gamma^{-3},$$
$$D^4 = \beta\eta^{-1}+\beta^2\gamma^{-1}\zeta^{-1}+\beta^2\epsilon^{-2}+\beta^3\gamma^{-2}\epsilon^{-1}+\beta^4\gamma^{-4},$$

and so on. Here, by reason of the function containing only b, no term can contain any one letter, as e, except those in which the inverse operation of ϵ occurs to introduce it.

And it is plain from the mode of formation, that any direct operation, as ϵ, performed on any one derivative, gives the same result as the next, ζ, performed on the next derivative. For if any term of the first be $\beta^m\gamma^{-n}\epsilon^{-p}\zeta^{-q}\ldots$, which can come only once, there will come in the next the term

$\beta^m\gamma^{-n}\epsilon^{-p+1}\zeta^{-q-1}\ldots$; and ζ performed on the second yields the same as ϵ performed on the first. This, and the vanishing of all terms in the two, when ϵ and ζ are performed on terms which do not contain ϵ^{-1} and ζ^{-1}, proves the theorem alleged. This theorem is the immediate cause of the sufficiency of Arbogast's derivatives for their purpose. Its converse is as easily shewn, namely, that no other functions except these derivatives can have this property.

The formation of the derivatives in this very simple case, namely, in which the function operated upon is only a function of the letter belonging to the first operation, may be facilitated by the following method of selecting the terms to be retained. Operate only on the last operation of each term, except when the last but one in the term is also the next before the last in the series; in which case operate also upon the last but one. Thus, λ, μ, ν, ξ being consecutive, the term $\ldots\lambda^{-l}\nu^{-n}$ yields only $\ldots\lambda^{-l}\nu^{-n+1}\xi^{-1}$ to the next derivative, but $\ldots\lambda^{-l}\mu^{-m}\nu^{-n}$ yields

$$\ldots\lambda^{-l}\mu^{-m}\nu^{-n+1}\xi^{-1} + \ldots\lambda^{-l}\mu^{-m+1}\nu^{-n-1}.$$

This result is established by Arbogast upon a process of observation, but it admits of a very easy proof by combinations, as follows.

Let there be a box B, containing an unlimited number of counters, followed in succession by other boxes C, E, F, G, &c., all empty. Let it be allowable to remove a counter out of one box into the next *after* it: but let no other single operation be allowed. When n such operations have been performed, let the distribution in the boxes be called an n^{th} state. Let β, γ, ϵ, ζ, &c. be interpreted as directions to take a counter out of B, C, &c.; let γ^{-1}, ϵ^{-1}, ζ^{-1}, be interpreted as directions to put a counter into C, E, &c. Then it is clear that each step of transference is either $\beta\gamma^{-1}$, or $\gamma\epsilon^{-1}$, or $\epsilon\zeta^{-1}$, &c.; and also that what is written opposite to D above is the one possible first state, and opposite to D^2 we have all possible second states, and so on. For it is manifest that the way of deriving every $(n+1)^{\text{th}}$ state is to take every possible n^{th} state, throw a counter out of every box which has one into the next, and strike out every state which is thus brought about more than once, so often as it appears after the first time.

Now it is plain that the simplest way of converting the distribution $\infty\,0000$, &c. into $\infty\,p, q, r, s$, &c., is to take $p+q+r+$ &c. counters from B, and throw them into C at $p+q+r+$ &c. steps. Then take $q+r+$ &c. out of C and throw them into E, at $q+r+$ &c. steps. Then take $r+$ &c.

out of E and throw them into F, and so on. The same distribution may be gained by many permutations of the order of the steps, but this is enough. Now in the preceding process there is no one step but what may be described as follows: either one is taken out of the last occupied box, and thrown into the first which has till then been empty; or one is taken out of the last occupied box but one, and thrown into the last which is occupied. And since every $(n+1)^{\text{th}}$ state must be an n^{th} state with one step more, and there is no state of which the final process need be anything but that of the *last or last but one* just described, it follows that if this process be applied to *every* n^{th} state, it will give *every* $(n+1)^{\text{th}}$ state.

Again, if in every n^{th} state in which there are one or more counters in, say C, we throw one counter into E, we shall have every $(n+1)^{\text{th}}$ state in which there are one or more counters in E: and the same for E and F, &c. This proves the rule for the successive operations upon successive derivatives, otherwise established above.

All the β-operations included in $\beta^{\pm m}$ are convertible with derivations, when the function is one of b only. A look at the process will establish this; which is moreover no more than saying, that if m balls be added to or taken from the first box, it matters nothing whether this be done before or after the establishment of any state. Accordingly, the derivatives of $\beta^{-m}(\phi b)$ contain the following operations:

$$D^0 \text{ contains } \beta^{-m}, \qquad D \text{ contains } \beta^{-(m-1)}\gamma^{-1},$$
$$D^2 \quad \ldots\ldots \quad \beta^{-(m-1)}\varepsilon^{-1} + \beta^{-(m-2)}\gamma^{-2},$$

and so on.

Let us now suppose that every such operation, as ϵ for instance, either introduces or takes away multipliers which are functions of its letter, as e; so that

$$\epsilon . P \text{ must mean } P \times \text{some function of } e.$$

The consequence is, that derivation is altogether a convertible operation, when the conversion is made with letters which the rule of the last or last but one renders inoperative. Thus, though $D(\beta\gamma^{-1})$ is not $\beta D\gamma^{-1}$ or $\beta\varepsilon^{-1}$, but $\beta^2\gamma^{-2} + \beta\varepsilon^{-1}$, yet $D\beta\gamma^{-1}\varepsilon^{-1}$ is the same thing as $\beta D(\gamma^{-1}\varepsilon^{-1})$. And thus we see that $D\{\Sigma\beta^m\gamma^{-n}\varepsilon^{-p}..\}$ is made up of $\Sigma\{\beta^m D.\gamma^{-n}\varepsilon^{-p}..\}$ together with a term from every term in which β comes last but one and γ last.

From this we easily prove the following theorem, for any function of any letter, the first one, a, for example,

$$D^m = aD^{m-1}\beta^{-1} + a^2D^{m-2}\beta^{-2} + \ldots\ldots + a^m\beta^{-m}.$$

Let this be true for any one case; say that

$$D^{m-1} = aD^{m-2}\beta^{-1} + a^2D^{m-3}\beta^{-2} + \ldots + a^{m-2}D\beta^{-(m-2)} + a^{m-1}\beta^{-(m-1)}.$$

Perform D on both sides,

$$D^m = D(aD^{m-2}\beta^{-1}) + D(a^2D^{m-3}\beta^{-2}) + \ldots.$$

Now in every term except the last, a cannot in any case be the last letter but one, and in the last term

$$D(a^{m-1}\beta^{-(m-1)}) \text{ yields } a^{m-1}D\beta^{-(m-1)} + a^m\beta^{-m}.$$

Hence, inverting the order of D and a in all but the last, and substituting for the last as just found, we see that the theorem is true for D^m, if true for D^{m-1}; but

$$D = a\beta^{-1}, \quad D^2 = aD\beta^{-1} + a^2\beta^{-2},$$

so that it is true in all cases.

In the problem of combinations, this theorem is the immediate consequence of the following. If we wish to ascertain all the m^{th} states which can arise from an unlimited number of counters in A, none in B, C, &c., we must collect all the $(m-1)^{\text{th}}$ states which arise from 1, 0, 0, 0, &c. in B, C, &c., all the $(m-2)^{\text{th}}$ states from 2, 0, 0, 0, in B, C, &c., all the $(m-3)^{\text{th}}$ states from 3, 0, 0, 0 in B, C, &c., and so on, ending with the state of m, 0, 0, &c. in B, C, &c. And it is evident that the law of rejection is here inoperative: it is impossible that any state consequent upon p, 0, 0, &c. and q, 0, 0, &c. can be identical, if p and q be different numbers.

§. 2. *Application to Functions of one Variable.*

For use, undoubtedly development by means of derivation is an *application* of the differential calculus: but in theory it is an *extension*. And this, although the last operations must be those of the differential calculus: just as the last operations of the differential calculus itself must be those of algebra.

There is nothing that so much tends to destroy a proper perception of the full extent of a method, and of its separate meaning, as previous knowledge of equivalent processes.

In the function $\phi(a + bx + cx^2 + ex^3 + \ldots)$ which we might expand by Maclaurin's theorem, let us ask how we can avoid the change of x into $x + \xi$, by processes performed upon a, b, c, &c. The answer is, that for a we must write $a + b\xi + c\xi^2 + e\xi^3 + \text{\&c.}$; for b we must write $b + 2c\xi + 3c\xi^2 + \ldots$; for c we must write $c + 3e\xi + 6f\xi^2 + 10g\xi^3 + \ldots$; for e we must write $e + 4f\xi + 10g\xi^2 + 20h\xi^3 + \ldots$; and so on. The function

ϕ, thus altered, being ϕ_1, the limit of $\phi_1 - \phi$ divided by ξ is one set of *derivative coefficients*, complete. The partial *derivatives* are got by making the single letters vary as above.

It is obvious that the complete derivative amounts to the result obtained by differentiating with respect to a, b, c, &c., considered as functions of some one imaginary variable, and then writing b, $2c$, $3e$, &c. as the differential coefficients of a, b, c, &c. This was Arbogast's first process, and by itself is a great saving of trouble. It is the process for the development of $\phi\,\{a + b\,(x + \xi) + c\,(x + \xi)^2 + ..\}$ and by making $x = 0$, *before the operations*, or working upon ϕa only, gives the process required in the development of $\phi\,(a + bx + cx^2 + ..)$.

I have called the preceding a derivation because Arbogast did so. But this derivation is *differentiation*, and nothing more. He soon drops it for his *divided derivation*, which is the real distinctive feature of his system, and which consists in *differentiation accompanied by integration*, to which, and to which only, I shall in future apply the term derivation.

If we look at $\phi\,(a + b)$, we see in the development the following theorem. The operation which furnishes $\phi\,(a + b)$ is of the form

$$1 + \alpha\beta^{-1} + \alpha^2\beta^{-2} + \alpha^3\beta^{-3} + \ldots,$$

giving, if we return to the problem of combinations, every first, second, third, &c. state of which ∞ 0 0 0, &c. is susceptible, on the condition that no advance shall be made beyond the second box. Here α means differentiation with respect to a, and β integration (from $b = 0$) with respect to b. The change of a into $a + c$ in $\phi\,(a+b)$ is equivalent to the addition of $\alpha\gamma^{-1} + \alpha\gamma^{-2} + ..$, and further to the change of α into $\alpha + \alpha\gamma^{-1} + \alpha\gamma^{-2} + ..$ of α^2 into $\alpha^2 + \alpha^2\gamma^{-1} + \alpha^2\gamma^{-2} + ..$ &c. on the second side.

Suppose P_n denotes the sum of terms like the preceding, answering to every possible final state (the initial state included) of which the initial state is ∞ 0 0 0 0, &c., the n^{th} being the last box which has ever received a counter. It is plain that P_{n+1} can be formed from P_n by combining with every case in P_n, separately, 0 in the $\overline{n + 1}^{\text{th}}$ box, 1 in the $(n + 1)^{\text{th}}$ box, 2 in the same, &c., which, as all the counters must be originally brought from the first box, gives the performance of

$$1 + \alpha\nu^{-1} + \alpha^2\nu^{-2} + ..$$

upon every term of P_n, ν denoting the operation which comes n^{th} after α. Now this is precisely the process by which $\phi\,(a_0 + a_1 + .. + a_{n-1})$ is changed into $\phi\,(a_0 + .. + a_n)$, by writing $a_0 + a_n$ for a_0. So that the development of $\phi(a + b + c + e + ..)$

is the result of the sum of the operations arising from every possible final state in which as many of the successive boxes are or have been occupied as there are letters in a, b, c, e, &c. Whence $\phi(a+b+c+..) = \phi a + D\phi a + D^2\phi a + ...$, D being the operation described in the first section, on the supposition that α, β, γ, &c. mean differentiations with respect to a, b, c, &c., and that β^{-1}, γ^{-1}, &c. mean the corresponding $\int_0$ integrations.

If for b we write bx, for c, cx^2, and so on, the dimensions easily shew that

$$\phi(a+bx+cx^2+..) = \phi a + D\phi a.x + D^2\phi a.x^2 + ..;$$

but this will be better seen as follows. The general equation $\phi(a+b+..) = \phi a + D\phi a + ..$ obliges us to interpret $a+b+..$ as $a + Da + ..$, whence b, c, e, &c. are the successive derivatives of a: as, indeed, appears from the rule. This derivation preserving its notation, let the operations of the original theorem, β^v, γ^v, &c. include multiplication by x, x^2, &c.: the development of the series of powers of x will easily follow.

It is hardly necessary to shew that the special definitions of α, β, &c. above given satisfy the conditions. Indeed, so restrictive are the conditions, that a person accustomed to the calculus of operations will wonder what, except differential coefficients, they could mean. But it is true that other meanings might be given: the difficulty would lie in finding operations in which those meanings could be made useful.

By the rule of derivation established (not attending to the restriction of the last or the last but one), we have

$$D\phi a = \phi' a.b,$$

$$D^2\phi a = \phi' a.c + \phi'' a\,\frac{b^2}{2},$$

$$D^3\phi a = \phi' a.e + \phi'' a.bc + \begin{pmatrix}\phi'' a.b.c\\ \text{rejected}\end{pmatrix} + \phi''' a\,\frac{b^3}{2.3},$$

and so on. But the rule of the last or last but one, or a theorem already proved to which it led, shews that

$$D^n\phi a = \phi' a D^{n-1}b + \phi'' a D^{n-2}\frac{b^2}{2} + \phi''' a D^{n-3}\frac{b^3}{2.3} + .. + \phi^{(n)} a\,\frac{b^n}{2.3..n},$$

more conveniently written

$$D^n\phi a = \phi' a D^{n-1}b + \frac{\phi'' a}{2}D^{n-2}b^2 + \frac{\phi''' a}{2.3}D^{n-3}b^3 + .. + \frac{\phi^{(n)} a}{2.3..n}b^n.$$

The values of $D^m b^n$ are formed with very little practice almost as fast as they can be written. The only thing worth men-

tioning is the convenience of making the division which is to arise from the integration before the multiplication, or in abatement of the multiplication, which arises from the differentiation. Thus the derivatives of b^7 are

$$7b^6c,\ 7b^6e + 21b^5c^2,$$
$$7b^6f + 42b^5ce + 35b^4c^3,$$
$$7b^6g + 42b^5cf + 21b^5e^2 + 105b^4c^2e + 35b^3c^4,$$

and so on.

That fractional coefficients cannot enter, may be inferred from their general form, which may be obtained as follows.

If the initial state of the boxes B, C, E, F, &c. be $n, 0, 0, 0$, &c., the m^{th} derivative of b^n contains a term arising from every m^{th} state which can be thence produced. Now it is evident that in every removal from one box to the next, the process gives a multiplier, the number in the losing box before the loss, and a divisor, the number in the gaining box after the gain. Consequently the coefficient depends only on the numbers in the several boxes after the final step: for each divisor which comes in at any accession afterwards to be removed, is compensated by the multiplier introduced at the removal. By the time then that $n, 0, 0$, &c. has become p, q, r, &c the only effective multipliers are $n, n-1, \ldots$ $n-p+1$, and the divisors are $1, 2, 3..q$, $1, 2, 3..r$, &c. Hence the coefficient which multiplies $b^pc^qe^r$, &c. in any derivative of b^n, is evidently

$$\frac{1.2.3..n}{1.2.3..p \times 1.2.3..q \times 1.2.3..r \times \&c.}.$$

This form also shews how it happens that the coefficients must be the same in $b^qc^pe^r$.. or in $b^rc^qe^p$.. as in $b^pc^qe^r$..

With respect to the power of this method, none can judge but those who have tried both it and the substitutes for it. There is no producing conviction of the superiority of any process by description. But if any one will write down for himself even as much as

$$(a + bx + cx^2)^5 = a^5 + Da^5.x + D^2a^5.x^2 + \&c.$$
$$= a^5 + 5a^4bx + (5a^4c + 10a^3b^2)\,x^2$$
$$+ (20a^3bc + 10a^2b^3)\,x^3 + (10a^3c^2 + 30a^2b^2c + 5ab^4)\,x^4$$
$$+ (30a^2bc^2 + 20ab^3c + b^5)\,x^5 + (10a^2c^3 + 30ab^2c^2 + 5b^4c)\,x^6$$
$$+ (20abc^3 + 10b^3c^2)\,x^7 + (5ac^4 + 10b^2c^3)\,x^8$$
$$+ 5bc^4x^9 + c^5x^{10},$$

forming the derivatives by the method of the last or last but one,

and remembering $e = 0$, $f = 0$, &c. whenever they arise, he may then try the same by the binomial theorem, and decide the question for himself. Nor is this by any means a case peculiarly favourable to the method. Take the reversion of the series $y = a_1x + a_2x^2 +..$ into $x = A_1y + A_2y^2 +..$ and determine A_1, A_2, &c. up to A_{20} *first* by Arbogast's method (which will be the safest plan), and then by any other, until he is satisfied as to the relative ease of the two methods; and this will be a proper trial, with justice to the former method.

The point of view in which the calculus of operations places the main result is remarkable. It is

$$\phi\left\{(1 - xD)^{-1}a\right\} = (1 - xD)^{-1}\phi a,$$

shewing the convertibility of the ordinary operation, ϕ, of algebra, with the operation $(1 - xD)^{-1}$; and we have

$$\frac{1}{1 - xD}\,\phi a = \varepsilon^{\left(\frac{xD}{1-xD}\,a\right)\frac{d}{da}}\,\phi a.$$

But we must not use the calculus of operations in obtaining results. For that calculus assumes a definite and permanent subject of operation, as a: while derivation introduces new subjects of operation at every step.

§. 3. *Application to Functions of two Variables.*

We have now obtained the equation

$$\phi(a + Da.x + D^2a.x^2 + \ldots.) = \phi a + D\phi a.x + D^2\phi a.x^2 + \ldots.$$

in which we may, if we please, make the operation D include, besides its hitherto expressed meaning, multiplication by x, and thus write the preceding as $\phi(a + Da + \ldots) = \phi a + D\phi a + \ldots$ This I shall not do, but I shall write the symbol D_x instead of D, as a distinction from other derivations presently to be noticed, and I shall write a, b', c'', e''', &c. for a, b, c, e, &c., or a, Da, D^2a, &c.

If we now take a derivation of another kind, denoted in full by yD_y, in which the D_y derivatives of a are $b_{,}$ $c_{,,}$ $e_{,,,}$ &c.; those of b' are $c'_{,}$ $e'_{,,}$ $f'_{,,,}$ &c.; those of c'' are $e''_{,}$ $f''_{,,}$ $g''_{,,,}$ &c.; we shall find that $1 + D_y.y + D^2_y.y^2 + \ldots$ performed upon

$$a + b'x + c''x^2 + \ldots.$$

is
$$a + b'x + b_{,}y + c''x^2 + c'_{,}xy + c_{,,}y^2 + \ldots.$$

and by a repetition of the above theorem, we find that the development of $\phi(a + b'x + b_{,}y + \ldots)$ is

$$\phi a + D_x\phi a.x + D_y\phi a.y + D_x^2\phi a.x^2 + D_yD_x\phi a.xy + D_y^2\phi a.y^2 + \ldots$$

in which the coefficient of $x^m y^n$ is $D_x^m D_y^n \phi a$. The convertibility of D_x and D_y is proved either from the nature of the operations, or from an inversion of the preceding process. And the convertibility of D_x or D_y with $\frac{d}{da}$ is proved by differentiating $\phi(a + Da.x + \ldots) = \phi a + \ldots$ with respect to a, and comparing the result with $\phi'(a + \ldots)$ obtained from the theorem.

The above rule is not difficult to use, though not the easiest which the subject admits of. Let it be required to exhibit the value of $D_y^2 D_x^2 \phi a$. We have

$$D_x^2 \phi a = \phi' a.c'' + \frac{\phi'' a}{2} b'^2,$$

$$D_y D_x^2 \phi a = \phi' a.e_{,}'' + \frac{\phi'' a}{2} \cdot 2b_{,}c'' + \frac{\phi'' a}{2} 2b'c_{,}' + \frac{\phi''' a}{2.3} 3b_{,}b'^2,$$

$$D_y^2 D_x^2 \phi a = \phi' a.f_{,,}'' + \frac{\phi'' a}{2} 2b_{,}e_{,}'' + \frac{\phi'' a}{2}(2b'e_{,,}' + c_{,}'^2 + 2b_{,}e_{,}'' + 2c_{,,}c'')$$

$$+ \frac{\phi''' a}{2.3}(6b_{,}b'c_{,}' + 3b_{,}^2 c'') + \frac{\phi''' a}{2.3}(6b_{,}b'c_{,}' + 3c_{,,}b'^2) + \frac{\phi^{iv} a}{2.3.4} 6b_{,}^2 b'^2,$$

in which the terms occurring twice are to be rejected. It must be remembered that the rule of the last or last but one does not apply to any thing but derivatives of a function of one quantity only.

If we take $b'x + b_{,}y + \ldots$ as an increment of a, use Taylor's theorem, and also the preceding theorem applied to the powers of the increment, we have the following theorem:

$$D_x^m D_y^n \phi a = \phi' a D_x^m D_y^n a + \frac{\phi'' a}{2} D_x^m D_y^n a^2 + \ldots$$

$$+ \frac{\phi^{(m+n)} a}{2.3\ldots m+n} D_x^m D_y^n a^{m+n},$$

in which a is made to vanish in the derivatives. Now it is obvious that $D_x^m D_y^n a^{m+n+k}$, k being a positive integer, must vanish with a, for no derivation lowers a by more than one dimension. Hence, we have the following,

$$D_x^m D_y^n \phi a = D_x^m D_y^n \{\phi(a + z) - \phi a\},$$

on condition that all the derivatives of a be transferred to z, and z be made finally to vanish.

We have now two modes of derivation, D_x and D_y, in which the symbols x and y might have implied that multiplication by x or by y is a part of the operation. Let

there be two systems of derivation in which multiplication by $\frac{x}{y}$ or by $\frac{y}{x}$ might have been a part of the operation, and let their symbols be $D_{x:y}$ and $D_{y:x}$: when we begin with D_x derivations, we shall always want to combine $D_{y:x}$ derivations with them; accordingly, in $D^m_{y:x}D_x{}^n\phi a$, $y:x$ is superfluous; the colon alone will do, or we may write $D_:^m D_x{}^n\phi a$. In $D_{y:x}$ derivation, let the derivatives be as follows:

of a	0	0	0	0	0	&c.
b'	$b_,$	0	0	0	0	&c.
c''	$c_,'$	$c_{,,}$	0	0	0	&c.
e'''	$e_,''$	$e_{,,}'$	$e_{,,,}$	0	0	&c.,

and so on.

By interchanging superfixes and suffixes, we get the $D_{x:y}$ derivations of $a, b_,, c_{,,}, e_{,,,}$, &c. The separate letters a, b, c, e, &c. uniformly represent coefficients of terms of the same order. Thus f is the index of the fourth order, and must always appear with four accents, either as f^{iv}, $f_,'''$, $f_{,,}''$, $f_{,,,}'$, or f_{iv}. As I before observed, those who think this superfluous, may use only one letter; but I should recommend them to do as is here done.

Take $\phi(a)$ and perform the operations $1 + D_x . x + D_x^2 . x^2 + . .$ and $1 + D_: . x^{-1}y + D_:^2 . x^{-2}y^2 + . .$ both internally and externally. We have then

$$\phi(a + b'x + b_,y + . .) = \phi a + D_x\phi a . x + D_:D_x\phi a . y$$
$$+ D_x^2\phi a . x^2 + D_:D_x^2\phi a . xy + D_:^2D_x^2\phi a . y^2$$
$$+ D_x^3\phi a . x^3 + D_:D_x^3\phi a . x^2y + D_:^2D_x^3\phi a . xy^2 + D_:^3D_x^3\phi a . y^3 + . .$$

At the same time it is obvious that if the beginning had been made with $\phi(a + b_,y + . .)$ followed by $D_{x:y}$ derivations, and if $D_:$, written before D_y, had signified $D_{x:y}$, we should have had

$$\phi(a + b'x + b_,y + ...) = \phi a + D_:D_y\phi a . x + D_y\phi a . y + ...$$

from which we obtain, by equating the coefficients of $x^m y^n$,

$$D^n_{y:x}D_x{}^{m+n}\phi a = D^m_{x:y}D_y{}^{m+n}\phi a,$$

for all positive values of m and n from zero inclusive. Thus we have

$$D^n_{y:x}D_x{}^n\phi a = D_y{}^n\phi a, \quad D^m_{x:y}D_y{}^m\phi a = D_x{}^m\phi a.$$

To look at these results by the method of combinations,

let us suppose the following series of boxes, of which A has an unlimited number of counters and the rest none,

$$\begin{array}{ccccc}
 & & & E''' & \\
 & & C'' & & \\
 & B' & & E_{,}'' & \\
A & & C_{,}' & & \&c. \\
 & B_{,} & & E_{,,}' & \\
 & & C_{,,} & & \\
 & & & E_{,,,} &
\end{array}$$

In $D_x^m\phi a$ we have, as has been seen, a term descriptive of every m^{th} state made by taking counters from A into B', C'', &c. at m steps. Now, since there are but m steps out, there are but m steps back again: so that if we changed the direction of transference, there is no result from m back-steps, except finally bringing all the counters back into A, and leaving B', C'', &c. empty. But there are no more steps from B' to A than from B' to B; no more from C'' to A than from C'' to $C_{,,}$, and so on: hence if, after establishing any m^{th} state along A, B', C'', &c., we make m vertical transferences, we must end by transposing all that were in B into $B_{,}$, all there were in C'' into $C_{,,}$, and so on, in neither more nor less than m steps. Hence, from the connexion previously established between the m^{th} states and m^{th} derivations, we have $D_{:}^m D_x^m \phi a = D_y^m \phi a$. And if we were, having established an m^{th} state in A, B', C'', &c., to make only $m-1$ vertical steps downwards, it follows that we produce neither more nor less than all the states which would have been obtained by establishing a corresponding m^{th} state in A, $B_{,}$, $C_{,,}$, &c. and making one vertical step upwards; and so on.

This establishes

$$D_{y:x}^{m-1} D_x^m \phi a = D_{x:y} D_y^m \phi a,$$

and similarly for the rest.

The reader will perhaps have noticed, that $D_{y:x}$ derivation is done by the operation $D_y D_x^{-1}$, for the passage from C'' to $C_{,}'$, for instance, may be made by $C''E_{,}''C_{,}'$, or by $C''B'C_{,}'$. If we write $D_y D_x^{-1}$ for $D_{y:x}$, then $D^n_{y:x} D_x^{m+n}$ becomes $D_y^n D_x^n$, and the two modes of developing $\phi(a + b'x + b_{,}y + \&c.)$ are identical. But it is to be remembered that, without an extension, we cannot affirm the convertibility of D_x^m and D_y^n for all values, positive and negative, of the exponents of operation. If indeed we complete our system thus,

[redacted]

$$
\begin{array}{cccccccc}
a_{-,,,}''' & & c_{-,,}^{iv} & & f_{-,}^{v} & & h^{vi} & \\
& b_{-,,}''' & & e_{-,}^{iv} & & g^{v} & & \\
a_{-,,}'' & & c_{-,}''' & & f^{iv} & & h_{,}^{v} & \\
& b_{-,}'' & & e''' & & g_{,}^{iv} & & \\
a_{-,}' & & c'' & & f_{,}''' & & h_{,,}^{iv} & \\
& b' & & e_{,}'' & & g_{,,}''' & & \\
a & & c_{,}' & & f_{,,}'' & & h_{,,,}''' & \&c. \\
& b_{,} & & e_{,,}' & & g_{,,,}'' & & \\
a_{,}^{-\prime} & & c_{,,} & & f_{,,,}' & & h_{iv}'' & \\
& b_{,,}^{-\prime} & & e_{,,,} & & g_{iv}' & & \\
a_{,,}^{-\prime\prime} & & c_{,,,}^{-\prime} & & f_{iv} & & h_{v}' & \\
& b_{,,,}^{-\prime\prime} & & e_{iv}^{-\prime} & & g_{v} & & \\
a_{,,,}^{-\prime\prime\prime} & & c_{iv}^{-\prime\prime} & & f_{v}^{-\prime} & & h_{vi} &
\end{array}
$$

with the understanding that all the negatively accented letters are made to vanish at the end of the process, we may then declare the four modes of derivation to be entirely convertible.

Leaving these extensions, however, we proceed to the details of development of $\phi(a + \&c.)$ by means of D_{s} derivations

and $D_{y:x}$ ones performed upon them. The coefficient of $x^m y^n$ is $D^n_{y:x} D_x^{m+n} \phi a$, or $D_{:}$ is to be performed n times upon

$$\phi' a \,.\, D^{m+n-1} b' + \frac{\phi'' a}{2} D^{m+n-2} b'^2 + \ldots + \frac{\phi^{(m+n)} a}{2 \ldots m+n} b^2 b'^{m+n}.$$

Now, because all the $D_{:}$ derivatives of a vanish, the process is wholly inoperative upon $\phi' a$, $\phi'' a$, &c.; and it is clear that by the difference of dimensions (derivation never producing change of dimension) no term of $D_{:}^p D_x^q b'^r$ can ever be identical with any sum of $D_{:}^s D_x^t b'^u$ if r and u be different. Hence the coefficient of $x^m y^n$ is

$$\phi' a D_{:}^n D_x^{m+n-1} b' + \frac{\phi'' a}{2} D_{:}^n D_x^{m+n-2} b'^2 + \ldots + \frac{\phi^{(m+n)}}{2 \ldots m+n} D_{:}^n b'^{m+n}.$$

It may be worth while to give an instance of the truth of the equation $D_{:}^m D_x^m b'^n = D_y^m b'^n$. Let us construct $D_{:}^3 D_x^3 b'^4$,

$$D_x b'^4 = 4b'^3 c'',$$

$$D_x^2 b'^4 = 4b'^3 e''' + 6b'^2 c''^2,$$

$$D_x^3 b'^4 = 4b'^3 f^{iv} + 12b'^2 c'' e''' + 4b' c''^3.$$

We cannot now use the rule of the last or last but one, but must proceed with every letter, dropping terms already obtained. I begin from the last letters in each term:

$$D_{:} D_x^3 b'^4 = 4b'^3 f_{,}''' + 12b'^2 b_{,} f^{iv} + 12b'^2 c'' e_{,}'' + 12b'^2 c_{,}' e''' + 24b' b_{,} c'' e''' + 12b' c''^2 c_{,}' + 4b_{,} c''^3$$

$$\begin{aligned} D_{:}^2 D_x^3 b'^4 = {} & 4b'^3 f_{,,}'' + 12b'^2 b_{,} f_{,}''' + 12b' b_{,}^2 f^{iv} + 12b'^2 c'' e_{,,}' \\ & + 12b'^2 c_{,}' e_{,}'' + 24b' b_{,} c'' e_{,}'' + 12b'^2 c_{,,} e''' \\ & + 24b' b_{,} c_{,}' e''' + 12b_{,}^2 c'' e''' + 12b' c''^2 c_{,,} \\ & + 12b' c'' c_{,}'^2 + 12b_{,} c''^2 c_{,}' \end{aligned}$$

$$\begin{aligned} D_{:}^3 D_x^3 b'^4 = {} & 4b'^3 f'_{,,,} + 12b'^2 b_{,} f_{,,}'' + 12b' b_{,}^2 f_{,}''' + 4b_{,}^3 f^{iv} \\ & + 12b'^2 c'' e_{,,,} + 12b'^2 c_{,}' e_{,,}' + 24b' b_{,} c'' e_{,,}' \\ & + 12b'^2 c_{,,} e_{,}'' + 24b' b_{,} c_{,}' e_{,}'' + 12b_{,}^2 c'' e_{,}'' + 24b' b_{,} c_{,,} e''' \\ & + 12b_{,}^2 c_{,}' e''' + 12b_{,}^2 c'' e_{,}'' + 12b_{,}^2 c_{,}' e''' + 24b' c'' c_{,}' c_{,,} \\ & + 12b_{,} c'^2 c_{,,} + 4b' c_{,}'^3 + 12b_{,} c'' c_{,}'^2. \end{aligned}$$

Now this term occurs in $D_{:}^3 D_x^7 \phi a$, the coefficient of $x^4 y^3$, in which it is the coefficient of $\phi^{iv} a \div 2.3.4$.

But $$D^3_{y:x} D_x^7 \phi a = D^4_{x:y} D_y^7 \phi a;$$

in which last the coefficient of $\phi^{iv} a \div 2.3.4$ is $D_{:}^4 D_y^3 b_{,}^4$, which is therefore $= D_{:}^3 D_x^3 b'^4$. And thus we have, generally,

$$D^m_{y:x} D_x^n b'^p = D^{n+p-m}_{x:y} D_y^n b_{,}^p.$$

Now $D_y^3 b_{,}^4$ is only $D_x^3 b'^4$ with superfixes and suffixes interchanged; and $D_{:}^4 D_y^3 b_{,}^4$ will only be $D_{:}^4 D_x^3 b'^4$ with similar

interchanges. From this we gather, that if one more $D_{y:x}$ derivation be made in the preceding result, superfixes and suffixes will be interchanged, other things remaining the same. And this will be found on trial to be the case.

As another instance, we try $D_:^4 D_x^2 b'^2$, which ought to give $D_y^2 b_,^2$. We have

$$D_x^2 b'^2 = 2b'e''' + c''^2,$$
$$D_: D_x^2 b'^2 = 2b'e_,'' + 2b_,e''' + 2c''c_,',$$
$$D_:^2 D_x^2 b'^2 = 2b'e_{,,}' + 2b_,e_,'' + 2c''c_{,,} + c_,'^2,$$
$$D_:^3 D_x^2 b'^2 = 2b'e_{,,,} + 2b_,e_{,,}' + 2c_,'c_{,,},$$
$$D_:^4 D_x^2 b'^2 = 2b_,e_{,,,} + c_{,,}^2 = D_y^2 b_,^2.$$

The easiest mode of preparing for the development of a function is to write down the coefficients of $\phi'a$, $\frac{1}{2}\phi''a$, &c. in the coefficients of the simple powers of x, and then to perform as many $D_{y:x}$ derivations as there are in the exponent of the power, the test of correctness being the ultimate production of the D_y derivations in the coefficients of the simple powers of y. The following is as much of an instance as the page will allow of; it is for the terms of the third order.

	$\phi'a$	$\dfrac{\phi''a}{2}$	$\dfrac{\phi'''a}{2.3}$
x^3	$D_x^2 b' = e'''$	$D_x b'^2 = 2b'c''$	b'^3
x^2y	$e_,''$	$2b'c_,' + 2b_,c''$	$3b'^2b_,$
xy^2	$e_{,,}'$	$2b'c_{,,} + 2b_,c_,'$	$3b'b_,^2$
y^3	$e_{,,,}$	$2b_,c_{,,}$	$b_,^3$

and the complete exhibition of the terms of the third order is

$$\left(\phi'a.e''' + \frac{\phi''a}{2}\,2b'c'' + \frac{\phi'''a}{2.3}\,b'^3\right)x^3$$
$$+\left(\phi'a.e_,'' + \frac{\phi''a}{2}\,\{2b'c_,' + 2b_,c''\} + \frac{\phi'''a}{2.3}\,3b'^2b_,\right)x^2y$$
$$+\left(\phi'a.e_{,,}' + \frac{\phi''a}{2}\,\{2b'c_{,,} + 2b_,c_,'\} + \frac{\phi'''a}{2.3}\,3b'b_,^2\right)xy^2$$
$$+\left(\phi'a.e_{,,,} + \frac{\phi''a}{2}\,.\,2b_,c_{,,} \qquad + \frac{\phi'''a}{2.3}\,b_,^3\right)y^3.$$

To enable the reader to exercise himself further, I insert the terms of the fourth and fifth order, making ϕ_n stand for $\phi^{(n)}a \div 1.2.3..n$. The coefficients of

[redacted]

x^4, x^3y, x^2y^2, xy^3, y^4, are

$f^{\mathrm{iv}}\phi_1 + (2b'e''' + c''^2)\,\phi_2 + 3b'^2c''\phi_3 + b'^4\phi_4$

$f_{,}'''\phi_1 + (2b'e_{,}'' + 2b_{,}e''' + 2c''c_{,}')\,\phi_2 + (3b'^2c_{,}' + 6b'b_{,}c'')\,\phi_3 + 4b'^3b_{,}\phi_4$

$f_{,,}''\phi_1 + (2b'e_{,,}' + 2b_{,}e_{,}'' + 2c''c_{,,} + c_{,}'^2)\,\phi_2 + (3b'^2c_{,,} + 6b'b_{,}c_{,}' + 3b_{,}^2c'')\,\phi_3 + 6b'^2b_{,}^2\phi_4$

$f_{,,,}'\phi_1 + (2b'e_{,,,} + 2b_{,}e_{,,}' + 2c_{,}'c_{,,})\,\phi_2 + (6b'b_{,}c_{,,} + 3b_{,}^2c_{,}')\,\phi_3 + 4b'b_{,}^3\phi_4$

$f_{\mathrm{iv}}\phi_1 + (2b_{,}e_{,,,} + c_{,,}^2)\,\phi_2 + 3b_{,}^2c_{,,}\phi_3 + b_{,}^4\phi_4.$

The coefficients of x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5, are

$g^{\mathrm{v}}\phi_1 + (2b'f^{\mathrm{iv}} + 2c''e''')\,\phi_2 + (3b'^2e''' + 3b'c''^2)\,\phi_3 + 4b'^3c''\phi_4 + b'^5\phi_5$

$g_{,}^{\mathrm{iv}}\phi_1 + (2b'f_{,}''' + 2b_{,}f^{\mathrm{iv}} + 2c''e_{,}'' + 2c_{,}'e''')\,\phi_2 + (3b'^2e_{,}'' + 6b'b_{,}e''' + 6b'c''c_{,}' + 3b_{,}c''^2)\,\phi$

$\quad + (4b'^3c_{,}' + 12b'^2b_{,}c'')\,\phi_4 + 5b'^4b_{,}\phi_5$

$g_{,,}'''\phi_1 + (2b'f_{,,}'' + 2b_{,}f_{,}''' + 2c''e_{,,}' + 2c_{,}'e_{,}'' + 2c_{,,}e''')\,\phi_2$

$\quad + (3b'^2e_{,,}' + 6b'b_{,}e_{,}'' + 3b_{,}^2e''' + 6b'c''c_{,,} + 3b'c_{,}'^2 + 6b_{,}c''c_{,}')\,\phi_3$

$\quad + (4b'^3c_{,,} + 12b'^2b_{,}c_{,}' + 12b'b_{,}^2c'')\,\phi_4 + 10b'^3b_{,}^2\phi_5$

$g_{,,,}''\phi_1 + (2b'f'_{,,,} + 2b_{,}f_{,}'' + 2c''e_{,,,} + 2c'_{,}e_{,,}' + 2c_{,,}e_{,}'')\,\phi_2$

$\quad + (3b'^2e_{,,,} + 6b'b_{,}e_{,,}' + 3b_{,}^2e_{,}'' + 6b'c_{,}'c_{,,} + 6b_{,}c''c_{,,} + 3b_{,}c_{,}'^2)\,\phi_3$

$\quad + (12b'^2b_{,}c_{,,} + 12b'b_{,}^2c_{,}' + 4b_{,}^3c'')\,\phi_4 + 10b'^2b_{,}^3\phi_5$

$g'_{\mathrm{iv}}\phi_1 + (2b'f_{\mathrm{iv}} + 2b_{,}f'_{,,,} + 2c_{,}'e_{,,,} + 2c_{,,}e'_{,,})\,\phi_2 + (6b'b_{,}e_{,,,} + 3b_{,}^2e'_{,,} + 3b'c_{,,}^2 + 6b_{,}c_{,}'c_{,,})\,\phi_3$

$\qquad + (12b'b_{,}^2c_{,,} + 4b_{,}^3c_{,}')\,\phi_4 + 5b'b_{,}^4\phi_5$

$g_{\mathrm{v}}\phi_1 + (2b_{,}f_{\mathrm{iv}} + 2c_{,,}e_{,,,})\,\phi_2 + (3b_{,}^2e_{,,,} + 3b_{,}c_{,,}^2)\,\phi_3 + 4b_{,}^3c_{,,}\phi_4 + b_{,}^5\phi_5.$

§ 4. *On Functions of two series of one Variable.*

Having $V = \phi(a, A)$, it is proposed to develope

$$\phi(a + bx + cx^2 + \ldots,\quad A + Bx + Cx^2 + \ldots).$$

If we write this with two different variables, as

$$\phi(a + bx + \ldots,\quad A + By + \ldots),$$

and if we first write $a + bx + \ldots$ for a in $\phi(a, A)$ or V, the development is $V + D_xV.x + D_x^2V.x^2 + \ldots$ If in each of the functions V, D_xV, &c. we write $A + By + \ldots$ for A, we find for the development required,

$$V + D_xV.x + D_yV.y + D_x^2V.x^2 + D_xD_yV.xy + D_y^2V.y^2 + \ldots$$

Change y into x, and the coefficient of x^n is

$$D_x^nV + D_x^{n-1}D_yV + \ldots + D_xD_y^{n-1}V + D_y^nV.$$

If the sum of the partial derivatives with respect to a, b, &c. and A, B, &c. be called the total derivative, and denoted by $D_{,,}$, we have

$$\phi(a, A) + D_{,,}\phi(a, A)\,x + D_{,,}^2\phi(a, A).x^2 + \ldots$$

for the development. To expand this form, observe that in

$$D_x^m\,\phi(a, A) = \phi'\,D^{m-1}\ldots \text{[illegible]} \ldots \frac{\phi^{(m)}}{1 \ldots m}\,b'^m,$$

if we perform the operation D_y, it can affect nothing except the factors ϕ', ϕ'', &c. And if we adopt the notation $V_{m,n}$ to signify

$$\frac{d^{m+n}V}{da^m\,dA^n} \times \frac{1}{2.3\ldots m} \times \frac{1}{2.3\ldots n},$$

we have $$D_x^m D_y^n V = V_{1,1} D^{m-1}b\, D^{n-1}B + \ldots$$
$$= \Sigma V_{p,q}\, D_x^{n-p} b^p \,.\, D_y^{n-q} B^q,$$

for every pair of values of p and q, in which the former lies between 1 and m, the latter between 1 and n, both inclusive. Now $D_x^k b^p \,.\, D_y^l B^q$ is not distinguishable from $D_x^k D^l (b^p B^q)$. If we collect the expressions for $D_x^n V$, $D_x^{n-1} D_y V$, &c. up to $D_y^n V$, we find that $V_{p,q}$ occurs in every form in which $p+q$ does not exceed n, and that the coefficient of $V_{p,q}$ is

$$D_{,,}^{n-p-q}(b^p B^q), \text{ or } b^p D_y^{n-p-q} B^q + D_x b^p D_y^{n-p-q-1} B^q + \ldots$$

So that the coefficient of x^n is

$$\begin{aligned} &V_{1,0} D_{,,}^{n-1} b + V_{0,1} D_{,,}^{n-1} B \\ &+ V_{2,0} D_{,,}^{n-2} b^2 + V_{1,1} D_{,,}^{n-2} . bB + V_{0,2} D^{n-2} B^2 \\ &+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ &+ V_{n,0} b^n + V_{n-1,1} b^{n-1} B + \ldots + V_{1,n-1} b B^{n-1} + V_{0,n} B^n, \end{aligned}$$

in which the $D_{,,}$ derivations require further development.

This paper has extended to such a length, that I will not make any further remark except the following. Arbogast's methods in general must be valued more than in proportion to the extent of development which they are used to obtain. For terms of the first and second orders they save no trouble, and very little for terms of the third order. From thence upwards they continue saving a larger and larger fraction of the time and labour which the common methods require. Independently of any value which the extension of the principle of differentiation in development may be found to have in high analysis, the insight which these methods give into the structure of algebraical expressions, and the power which they add to operation, render them deserving of much more attention than they have received.

University College, London, Feb. 16, 1846.

ON SYMBOLICAL GEOMETRY.

By Sir William Hamilton.

[*Continued.*]

On the Distributive Character of the Operation of Multiplication, as performed generally on Geometrical Fractions.

14. We are now prepared to extend the formulæ (76), (77), respecting the multiplication of sums of geometrical fractions; and to shew that similar results hold good, even when the condition of colinearity, assumed in those two formulæ, is no longer supposed to be satisfied. That is, the two equations

$$\left(\frac{\mathrm{h}}{\mathrm{g}}+\frac{\mathrm{f}}{\mathrm{e}}\right)\times\frac{\mathrm{k}}{\mathrm{i}}=\left(\frac{\mathrm{h}}{\mathrm{g}}\times\frac{\mathrm{k}}{\mathrm{i}}\right)+\left(\frac{\mathrm{f}}{\mathrm{e}}\times\frac{\mathrm{k}}{\mathrm{i}}\right)\ldots\ldots(104),$$

$$\frac{\mathrm{k}}{\mathrm{i}}\times\left(\frac{\mathrm{h}}{\mathrm{g}}+\frac{\mathrm{f}}{\mathrm{e}}\right)=\left(\frac{\mathrm{k}}{\mathrm{i}}\times\frac{\mathrm{h}}{\mathrm{g}}\right)+\left(\frac{\mathrm{k}}{\mathrm{i}}\times\frac{\mathrm{f}}{\mathrm{e}}\right)\ldots\ldots(105),$$

can both be shown to be true, whatever may be the lengths and directions of the six lines e, f, g, h, i, k; although, by the general non-commutativeness of geometrical fractions as factors, which was pointed out in the last article, the expressions contained in these two equations are not to be confounded with each other.

Making for this purpose

$$\left.\begin{array}{l}\dfrac{\mathrm{f}}{\mathrm{e}}=\beta_1+b_1,\quad \dfrac{\mathrm{h}}{\mathrm{g}}=\beta_2+b_2,\quad \dfrac{\mathrm{k}}{\mathrm{i}}=a+a,\\ \mathrm{I}\beta_1'\parallel\mathrm{I}a,\quad \mathrm{I}\beta_1''\perp\mathrm{I}a,\quad \mathrm{I}\beta_1''+\mathrm{I}\beta_1'=\mathrm{I}\beta_1,\\ \mathrm{I}\beta_2'\parallel\mathrm{I}a,\quad \mathrm{I}\beta_2''\perp\mathrm{I}a,\quad \mathrm{I}\beta_2''+\mathrm{I}\beta_2'=\mathrm{I}\beta_2,\\ \beta_2'+\beta_1'=\beta',\ \beta_2''+\beta_1''=\beta'',\ \beta_2+\beta_1=\beta,\ b_2+b_1=b,\end{array}\right\}\ldots(106),$$

the conditions (83) will be satisfied; and if we still assign to γ and c the meanings (87), the equation (88) will hold good, and $\gamma+c$ will be an expression for the first member of (104). Making also, in imitation of (87),

$$\left.\begin{array}{l}c_1=\beta_1'a+b_1a,\quad \gamma_1=\beta_1''a+\beta_1 a+b_1 a,\\ c_2=\beta_2'a+b_2a,\quad \gamma_2=\beta_2''a+\beta_2 a+b_2 a,\end{array}\right\}\ldots(107),$$

the second member of the same equation (104) becomes, by the principles of the 11th article, $(\gamma_2+c_2)+(\gamma_1+c_1)$; and the equation resolves itself into the two following,

$$c=c_2+c_1,\quad \gamma=\gamma_2+\gamma_1\ldots\ldots\ldots(108);$$

which are easily seen to reduce themselves to these two,

$$(\beta_2'+\beta_1')\,a=\beta_2'a+\beta_1'a;\quad (\beta_2''+\beta_1'')\,a=\beta_2''a+\beta_1''a\ldots(109);$$

the one being an equation between scalars, and the other between vectors. In like manner the equation (105) may be shown to depend on the two following equations, less general than itself, but of the same form,

$$a(\beta_2' + \beta_1') = a\beta_2' + a\beta_1'\,;\quad a(\beta_2'' + \beta_1'') = a\beta_2'' + a\beta_1'' \,\ldots (110).$$

And since, by (101), the three scalar products in the equations (110) are respectively equal, and the three vector products are respectively opposite (in their signs) to the corresponding products in the equations (109), it is sufficient to prove either of these two pairs of equations; for example, the pair (110). Now the first equation of this pair is true, because the scalars denoted by the three products $a\beta_1'$, $a\beta_2'$, $a(\beta_2' + \beta_1')$, are proportional, both in their magnitudes and in their signs, to the indices of the three parallel vectors β_1', β_2', $\beta_2' + \beta_1'$; and the second equation of the same pair is true, because the indices of the vectors denoted by the three other products $a\beta_1''$, $a\beta_2''$, $a(\beta_2'' + \beta_1'')$ are formed from the indices of the three coplanar vectors β_1'', β_2'', $\beta_2'' + \beta_1''$, by causing the three latter indices to revolve together, as one system, in their common plane, round the index Ia, their lengths being at the same time changed (if at all) in one common ratio, namely, in that of $\bar{a}$ to 1. The formulæ (104) (105) are therefore proved to be true; and the same reasoning shows, that in any multiplication of two geometrical fractions, either of the factors may be *distributed* into *any number* of parts, and that the sum of the partial products will be equal to the total product: so that we may write, generally,

$$\left(\Sigma \frac{\mathrm{k}}{\mathrm{i}}\right) \times \left(\Sigma \frac{\mathrm{f}}{\mathrm{e}}\right) = \Sigma \left(\frac{\mathrm{k}}{\mathrm{i}} \times \frac{\mathrm{f}}{\mathrm{e}}\right) \ldots\ldots (111).$$

The *multiplication of geometrical fractions* is therefore *a distributive operation;* although it has been shown to be *not*, in general, a *commutative* one.

On the Associative Property of the Multiplication of Geometrical Fractions.

15. Proceeding now, with the help of the distributive property established in the last article, and of the principle that a product is multiplied by a scalar when any one of its factors is multiplied thereby, to prove that the multiplication of geometrical fractions is generally an *associative* operation, or that the formula

$$\frac{k}{i} \times \left(\frac{h}{g} \times \frac{f}{e}\right) = \left(\frac{k}{i} \times \frac{h}{g}\right) \times \frac{f}{e} \ldots\ldots (112),$$

holds good for *any three fractions* (with other formulæ of the same sort for more fractional factors than three), it will be sufficient to prove that the formula is true for *any three vectors;* or that we may write generally

$$\gamma \times \beta\alpha = \gamma\beta \times \alpha \ldots\ldots\ldots\ldots (113),$$

the vector γ being not here obliged to satisfy the equation (87); we may even content ourselves with proving that the equation (113) is true in the two following cases, namely first, when any two of the three vectors are parallel; and secondly, when all three are rectangular to each other. The first case may be expressed by the three following equations as its types—

$$\beta \times \beta\alpha = \beta\beta \times \alpha \ldots\ldots\ldots\ldots (114),$$

$$\beta \times \alpha\beta = \beta\alpha \times \beta \ldots\ldots\ldots\ldots (115),$$

$$\alpha \times \beta\beta = \alpha\beta \times \beta \ldots\ldots\ldots\ldots (116);$$

and the second case may be expressed by the equation

$$\alpha\beta \times \beta\alpha = (\alpha\beta \times \beta) \times \alpha, \quad \text{when } \beta \perp \alpha \ldots (117);$$

because, under this last condition, $\alpha\beta$ is, by Art. 13, a vector, rectangular to both α and β. Under the same condition we may, by (99), change $\alpha\beta$ to $-\beta\alpha$; therefore the first member of the equation (117) may be equated to $-(\beta\alpha)^2$, and consequently, by (96), to $(-\beta^2) \times (-\alpha^2) = \beta^2 \times \alpha^2 = \beta^2\alpha \times \alpha = (\alpha \times \beta\beta) \times \alpha$, because β^2 or $\beta\beta$ is, by Art. 12, a scalar; thus we may make (117) depend on (116), which again depends on (114), and on the following equation,

$$\beta \times \beta\alpha = \alpha\beta \times \beta \ldots\ldots\ldots\ldots (118).$$

Equations (118) and (115) may both be proved by observing that, by Art. 13, whatever two vectors may be denoted by α and β, we have the expressions

$$\left.\begin{aligned}\beta\alpha &= \mathrm{S}.\beta\alpha + \mathrm{V}.\beta\alpha,\\ \alpha\beta &= \mathrm{S}.\beta\alpha - \mathrm{V}.\beta\alpha,\end{aligned}\right\} \ldots\ldots\ldots (119),$$

with the relations

$$\left.\begin{aligned}\beta \times \mathrm{S}.\beta\alpha - \mathrm{S}.\beta\alpha \times \beta = 0,\\ \beta \times \mathrm{V}.\beta\alpha + \mathrm{V}.\beta\alpha \times \beta = 0,\end{aligned}\right\} \ldots\ldots (120).$$

It remains then to prove the equation (114); and it is sufficient to prove this for the case where α and β are two rectangular vectors. But, in this case, $\beta\alpha$ is a vector formed from α by causing its index Iα to r[illegible] [illegible]ledly through a right angle round the [illegible] is

perpendicular, changing at the same time in general the length of this revolving index from $\bar{a}$ to $\bar{\beta} \times \bar{a}$; and the repetition of this process, directed by the symbol $\beta \times \beta a$, conducts to a new vector, of which the index is in direction opposite to the original direction of Ia, and in length equal to $\bar{\beta}^2 \times \bar{a}$: this new vector may therefore be otherwise denoted by $-\bar{\beta}^2 \times a$, or by $\beta^2 \times a$, and the equation (114) is true. The equations (113) and (112) are therefore also true; and since the latter formula may easily be extended to any number of fractional factors, we are now entitled to conclude what it was at the beginning of the present article proposed to prove; namely, that the *multiplication of geometrical fractions is always an associative operation:* as the addition of fractions, and the addition of lines, have in former articles been shown to be. In other words, any number of successive fractional factors may be *associated* or grouped together by multiplication (without altering their order) into a single product, and this product substituted as a single factor in their stead; a result which constitutes a new agreement (the more valuable on account of the absence of identity in some other important respects), between the *rules of operation* of ordinary algebra, and those of the present Symbolical Geometry.

Other forms of the Associative Principle of Multiplication.

16. By the principles already established respecting the transformation of geometrical fractions, any three such fractions, $\frac{\text{f}}{\text{e}}$, $\frac{\text{h}}{\text{g}}$, $\frac{\text{k}}{\text{i}}$, may be so prepared that the numerator of the first shall be in the plane of the second, and that the numerator of the second shall coincide with the denominator of the third; we may, therefore, without diminishing the generality of the theorem expressed by the formula (112), suppose that the line i is equal to h, and that the fourth proportional to g, h, f, is a new line l; and with this preparation the associative principle of multiplication, established in the foregoing article, may be put under the following form, in which the mark of multiplication between two fractional factors is omitted for the sake of conciseness:

$$\text{if } \frac{\text{h}}{\text{g}} = \frac{\text{l}}{\text{f}}, \quad \text{then } \frac{\text{k}}{\text{h}}\,\frac{\text{l}}{\text{e}} = \frac{\text{k}}{\text{g}}\,\frac{\text{f}}{\text{e}} \;\ldots\ldots (121);$$

that is to say, *the product of any two geometrical fractions will remain unaltered* in value, or will still continue to repre-

sent the same third fraction, *if the denominator of the multiplier and the numerator of the multiplicand be changed to any two new lines to which they are proportional,* or with which they form a *symbolic analogy,* including a relation between *directions* as well as a proportion of lengths, of the kind considered in Mr. Warren's work, (and earlier by Argand and Français,) and in the seventh article of this paper. Reciprocally, by the associative principle, the former of the two equations (121) is in general a consequence of the latter; that is, if the product of two geometrical fractions be equal to the product of two other fractions of the same sort, and if the multipliers have a common numerator, and the multiplicands a common denominator, then the numerators of the two multiplicands and the denominators of the two multipliers are the antecedents and consequents of a symbolical proportion or analogy, of the kind considered in the seventh article: for we may write

$$\frac{\mathrm{h}}{\mathrm{g}} = \frac{\mathrm{h}}{\mathrm{k}}\left(\frac{\mathrm{k}}{\mathrm{g}}\ \frac{\mathrm{f}}{\mathrm{e}}\right)\frac{\mathrm{e}}{\mathrm{f}}, \quad \frac{\mathrm{h}}{\mathrm{k}}\left(\frac{\mathrm{k}}{\mathrm{h}}\ \frac{\mathrm{l}}{\mathrm{e}}\right)\frac{\mathrm{e}}{\mathrm{f}} = \frac{\mathrm{l}}{\mathrm{f}};$$

so that the first equation (121) may be obtained from the second, by suitably grouping or associating the factors.

Again, the same associative principle shows that

$$\text{if } \frac{\mathrm{c}}{\mathrm{c}'} = \frac{\mathrm{b}'}{\mathrm{b}}\ \frac{\mathrm{a}'}{\mathrm{a}}, \quad \text{then } \frac{\mathrm{c}}{\mathrm{b}'} = \frac{\mathrm{c}'}{\mathrm{a}}\ \frac{\mathrm{a}'}{\mathrm{b}} \ldots\ldots (122);$$

for the first equation (122) may be replaced by the system of the three following equations,

$$\frac{\mathrm{a}'}{\mathrm{a}} = \frac{\mathrm{b}''}{\mathrm{a}''}, \quad \frac{\mathrm{b}'}{\mathrm{b}} = \frac{\mathrm{c}''}{\mathrm{b}''}, \quad \frac{\mathrm{c}'}{\mathrm{c}} = \frac{\mathrm{a}''}{\mathrm{c}''} \ldots\ldots (123);$$

of which the two last give, for the first member of the second equation (122), the expression

$$\frac{\mathrm{c}}{\mathrm{b}'} = \frac{\mathrm{c}'}{\mathrm{a}''}\ \frac{\mathrm{b}''}{\mathrm{b}},$$

which is equal to the second member of the same second equation (122), by the first of the three equations (123), and by the theorem (121): whenever, therefore, we meet an equation between one geometrical fraction and the product of two others, we are at liberty to *interchange the denominator of the product and the numerator of the multiplier,* provided that we at the same time *interchange the denominators of the two factors;* no change being made in the numerators of the product and the multiplicand. Conversely, this assertion respecting the liberty to make t[illegible], and the

formula (122), to which the assertion corresponds, are modes of expressing the associative principle of multiplication; for by introducing the equations (123) we find that the theorem (122) conducts to the following relation, or *identity between the two ternary products of three fractions,* associated in two different ways, but with one common order of arrangement,

$$\frac{c'}{a''}\left(\frac{a''}{a}\ \frac{a'}{b}\right)=\left(\frac{c'}{a''}\ \frac{a''}{a}\right)\frac{a'}{b} \;\ldots\ldots\ldots (124);$$

in which last form, as in (112), the three factors multiplied together may represent any three geometrical fractions. We may also present the same principle under the form of the following theorem—

$$\text{if } \frac{c'}{c}\ \frac{b'}{b}\ \frac{a'}{a}=1, \quad \text{then } \frac{c'}{a}\ \frac{a'}{b}\ \frac{b'}{c}=1 \;\ldots (125);$$

and may derive from it, with the help of (123), the following value of a certain product of six fractional factors,

$$\frac{a''}{c''}\ \frac{c}{a}\ \frac{b''}{a''}\ \frac{a}{b}\ \frac{c''}{b''}\ \frac{b}{c}=1 \;\ldots\ldots\ldots (126):$$

which must hold good whenever the three lines a, b, c are respectively coplanar with the three pairs $a''b''$, $b''c''$, $c''a''$. Finally, it may be stated here, as a theorem essentially equivalent to the associative principle of multiplication, although not expressly involving any product of two or more fractions, that *in the system of the six equations* of which those marked (123) are three, and of which the others are the three following analogous equations,

$$\frac{a}{c'}=\frac{a'''}{c'''},\quad \frac{b}{a'}=\frac{b'''}{a'''},\quad \frac{c}{b'}=\frac{c'''}{b'''} \;\ldots\ldots (127);$$

any five equations of the system include the sixth.

Geometrical Interpretation of the Associative Principle: Symbolic Equations between Arcs upon a Sphere: Theorem of the two Spherical Hexagons.

17. If we attended only to the *lengths* of the various lines compared, the associative principle of multiplication, under all the foregoing forms, would be nothing more than an easy and known consequence of a few elementary theorems respecting compositions of ratios of magnitudes. On the other hand it is permitted, in the present symbolical geometry, to assume at pleasure the *situations* of straight lines denoted by small roman letters, provided that the lengths and directions are preserved. The general theorem or property of

multiplication, which has been expressed in various ways in the two foregoing articles, may therefore be regarded as being essentially a *relation, or system of relations, between the directions of certain lines in space.*

In this view of the subject no essential loss of generality (or at least none which cannot easily be supplied by known and elementary principles) will be sustained by supposing all the straight lines abc, a′b′c′, a″b″c″, a‴b‴c‴, e f g h i k l, of the two last articles to be *radii of one sphere,* setting out from one *common origin* or centre O, and terminating in points upon one *common spheric surface,* which may be denoted respectively by the symbols ABC, A′B′C′, A″B″C″, A‴B‴C‴, EFGHIKL. In order more conveniently to study and express relations between points so situated, we may agree to say that two *arcs upon one sphere,* such as those from G to H and from F to L, are *symbolically equal,* when they are *equally long and similarly directed portions of the circumference of one great circle;* and may denote this *symbolical equality between arcs,* so called for the sake of suggesting that (like the symbolical equality between straight lines considered in the second article) it involves a relation of *identity of directions,* as well as a relation of equality of lengths, by writing any one of the three formulæ,

$$\left.\begin{array}{r}\frown \mathrm{LF} = \frown \mathrm{HG},\\ \frown \mathrm{FL} = \frown \mathrm{GH},\\ \frown \mathrm{LH} = \frown \mathrm{FG},\end{array}\right\}\dots\dots\dots\dots(128);$$

of which the second may be called the *inverse,* and the third the *alternate* of the first. Any one of these three formulæ (128) will thus express the *same relation between the directions of four coplanar radii,* namely, the four lines f g h l, as that expressed by the first equation (121), or by its inverse, or its alternate equation; that is, by any one of the three following equations between geometrical fractions,

$$\frac{\mathrm{l}}{\mathrm{f}} = \frac{\mathrm{h}}{\mathrm{g}}, \quad \frac{\mathrm{f}}{\mathrm{l}} = \frac{\mathrm{g}}{\mathrm{h}}, \quad \frac{\mathrm{l}}{\mathrm{h}} = \frac{\mathrm{f}}{\mathrm{g}} \dots\dots\dots(129).$$

The formulæ (128) express also the same relation between the same four directions, as that which would be expressed in a notation of a former article, by any one of the three following *symbolic analogies* between the same four lines,

$$\mathrm{l} : \mathrm{f} :: \mathrm{h} : \mathrm{g}, \quad \mathrm{f} : \mathrm{l} :: \mathrm{g} : \mathrm{h}, \quad \mathrm{l} : \mathrm{h} :: \mathrm{f} : \mathrm{g} \dots(130);$$

although it must not be forgotten that any one of the six latter formulæ, (129) and (130), expresses at the same time a

proportion between the lengths of four straight lines, not generally equal to each other, which is not expressed by any one of the three former symbolical equations (128), between pairs of arcs upon a sphere. In this notation (128), the last form of the associative principle of multiplication which was assigned in the foregoing article, so far as it relates to directions only, may be expressed by saying that *any one of the six following symbolical equations between arcs is a consequence of the other five,*

$$\left.\begin{array}{l} \frown A'A = \frown B''A'', \\ \frown B'B = \frown C''B'', \\ \frown C'C = \frown A''C'', \end{array}\right\} \ldots\ldots\ldots (131);$$

$$\left.\begin{array}{l} \frown BA' = \frown B'''A''', \\ \frown CB' = \frown C'''B''', \\ \frown AC' = \frown A'''C''', \end{array}\right\} \ldots\ldots\ldots (132).$$

Regarding *any six points* upon a spheric surface, in *any one order* of succession, as the *six corners of a spherical hexagon* (which may have re-entrant angles, and of which two or more sides may cross each other without being prolonged), we may speak of the arcs joining *successive corners* as the *sides;* those joining *alternate* corners, as the *diagonals;* and those joining *opposite* corners, as the *diameters* of this hexagon: the first side, first diagonal, and first diameter, respectively, being those three arcs which are drawn from the first corner to the second, third, and fourth corners of the figure. With this phraseology, the form just now obtained for the result of the two foregoing articles may be expressed as a relation between two spherical hexagons, AA'BB'CC', A''A'''B'''B''C''C''', and may be enunciated in words as follows: *If five successive sides of one spherical hexagon be respectively and symbolically equal to five successive diagonals of another spherical hexagon, the sixth side of the first hexagon will be symbolically equal to the sixth diagonal of the second hexagon.* This theorem of spherical geometry, which may be called, for the sake of reference, the *theorem of the two hexagons,* is therefore a consequence, and may be regarded as an interpretation of the associative principle of multiplication: and conversely, in all applications to spherical geometry, and generally in all investigations respecting relations between the directions of straight lines in space, the associative principle of multiplication may be replaced by the theorem of the two spherical hexagons.

[*To be continued.*]

[redacted]

ON THE ROTATION OF A SOLID BODY ROUND A FIXED POINT.

By Arthur Cayley.

(*Continued from* p. 173.)

On the Variation of the Constants, when the body is acted upon by forces.

The dynamical equations of a problem being expressed in the form

$$\frac{d}{dt}\cdot\frac{dT}{d\lambda'}-\frac{dT}{d\lambda}=\frac{dV}{d\lambda},$$

$$\frac{d}{dt}\cdot\frac{dT}{d\mu'}-\frac{dT}{d\mu}=\frac{dV}{d\mu}.$$

$$\frac{d}{dt}\cdot\frac{dT}{d\nu'}-\frac{dT}{d\nu}=\frac{dV}{d\nu}.$$

Suppose the equations obtained from these by neglecting the function V, are integrated; each of the six integrals may be expressed in the form

$$a=f(\lambda,\ \mu,\ \nu,\ \lambda',\ \mu',\ \nu',\ t),$$

where a denotes any one of the arbitrary constants. Assume

$$\frac{dT}{d\lambda'}=u,\quad \frac{dT}{d\mu'}=v,\quad \frac{dT}{d\nu}=w.$$

Then $\lambda',\ \mu',\ \nu'$ may be expressed in terms of $\lambda,\ \mu,\ \nu,\ u,\ v,\ w$, and the integrals may be reduced to the form

$$a=F(\lambda,\ \mu,\ \nu,\ u,\ v,\ w,\ t).$$

These equations may be considered as the integrals of the proposed system, taking into account the terms involving V, provided $a, b, \ldots$ &c. be supposed to become variable. We have, in this case, by Lagrange's theory of the variation of the arbitrary constants, the formulæ

$$\frac{da}{dt}=(a,b)\frac{dV}{db}+(a,c)\frac{dV}{dc}+(a,d)\frac{dV}{dd}+(a,e)\frac{dV}{de}+(a,f)\frac{dV}{df};$$

where

$$(a,b)=\left(\frac{da}{du}\frac{db}{d\lambda}-\frac{da}{d\lambda}\frac{db}{du}\right)+\left(\frac{da}{dv}\frac{db}{d\mu}-\frac{da}{d\mu}\frac{db}{dv}\right)+\left(\frac{da}{dw}\frac{db}{d\nu}-\frac{da}{d\nu}\frac{db}{dw}\right),$$

and in which V is supposed to be expressed as a function of $a,\ b,\ c,\ d,\ e,\ f,\ t$.

Thus the solution of the problem requires the calculation of thirty coefficients (a, b), or rather of fifteen only, since evidently $(a, b) = -(b, a)$. It is known that these coefficients are functions of a, b, c, d, e, f, without t; so that, in calculating them, any assumed arbitrary value, e.g. $t = 0$, may be given to the time.

In practice, it often happens that one of the arbitrary constants, e.g. (a), may be expressed in the form

$$a = F(\lambda, \mu, \nu, u, v, w, t, b, c, d, e, f),$$

where b, c, d, e, f are given functions of $\lambda, \mu, \nu, u, v, w, t$. In this case, it is easily seen that we may write

$$(a, b) = \{(a, b)\} + (c, b)\frac{da}{dc} + (d, b)\frac{da}{dd} + (e, b)\frac{da}{de} + (f, b)\frac{da}{df},$$

where, in the calculation of $\{(a, b)\}$, the differentiations upon a are performed, without taking into account the variability of $b, c \ldots.$

In the particular problem in question, the following are the values of the new variables u, v, w, (*Math. Journal*, memoir already quoted),

$$u = \frac{2}{\kappa}(Ap - \nu Bq + \mu Cr) \ldots\ldots (29),$$

$$v = \frac{2}{\kappa}(\nu Ap + Bq - \lambda Cr),$$

$$w = \frac{2}{\kappa}(-\mu Ap + \lambda Bq + Cr);$$

equations which may also be expressed in the form

$$2Ap = (1 + \lambda^2)\, u + (\lambda\mu + \nu)\, v + (\nu\lambda - \mu)\, w \ldots.(30),$$
$$2Bq = (\lambda\mu - \nu)\, u + (1 + \mu^2)\, v + (\mu\nu + \lambda)\, w,$$
$$2Cr = (\nu\lambda + \mu)\, u + (\mu\nu - \lambda)\, v + (1 + \nu^2)\, w,$$

or putting for shortness

$$\lambda u + \mu v + \nu w = \varpi \ldots\ldots\ldots\ldots(31),$$

these become

$$2Ap = \lambda\varpi + u + \nu v - \mu w \ldots\ldots\ldots (32),$$
$$2Bq = \mu\varpi - \nu u + v + \lambda w,$$
$$2Cr = \nu\varpi + \mu u - \lambda v + w.$$

Whence also

$$2\Omega = \kappa\varpi \ldots\ldots\ldots\ldots\ldots\ldots(33).$$

Substituting the values of Ap, Bq, Cr, given by (30) in the equations (6), we deduce

$$2\mathrm{a} = \lambda\varpi + u - \nu v + \mu w \ldots\ldots\ldots\ldots (34),$$
$$2\mathrm{b} = \mu\varpi + \nu u + v - \lambda w,$$
$$2\mathrm{c} = \nu\varpi - \mu u + \lambda v + w,$$

whence also $$2\,(\mathrm{a}\lambda + \mathrm{b}\mu + \mathrm{c}\nu) = \kappa\varpi \ldots\ldots\ldots\ldots (35),$$

which in fact follows from (33) and (17). And likewise the inverse system,

$$u = \frac{2}{\kappa}\,(\mathrm{a} + \nu\mathrm{b} - \mu\mathrm{c}) \ldots\ldots\ldots\ldots (36),$$

$$v = \frac{2}{\kappa}\,(-\,\nu\mathrm{a} + \mathrm{b} + \lambda\mathrm{c}),$$

$$w = \frac{2}{\kappa}\,(\mu\mathrm{a} - \lambda\mathrm{b} + \mathrm{c}).$$

It is easy to deduce

$$k^2 = \frac{1}{4}\,\kappa\,[u^2 + v^2 + w^2 + \varpi^2] \ldots\ldots\ldots\ldots (37),$$

$$\upsilon = \frac{1}{4}\,[(u^2 + v^2 + w^2) + (1 + \kappa)\,\varpi^2] \ldots\ldots (38).$$

Again, from the equations (10 *bis*),

$$\begin{aligned}\kappa(\mathrm{b}Cr - \mathrm{c}Bq) &= -\,2\lambda\,(\mathrm{a}^2 + \mathrm{b}^2 + \mathrm{c}^2) + 2\mathrm{a}\,(\lambda\mathrm{a} + \mu\mathrm{b} + \nu\mathrm{c}) + 2\,(\mathrm{b}\nu - \mathrm{c}\mu)\,\Omega \\ &= -\,2\lambda k^2 + 2\,(\mathrm{a} + \mathrm{b}\nu - \mathrm{c}\mu)\,\Omega \\ &= -\,2\lambda k^2 + \kappa u\,\Omega\,;\end{aligned}$$

[by equations (36),] *i.e.*

$$\Omega u - \frac{2}{\kappa}\,k^2\lambda = \mathrm{b}Cr - \mathrm{c}Bq \ldots\ldots\ldots\ldots (39),$$

$$\Omega v - \frac{2}{\kappa}\,k^2\mu = \mathrm{c}Ap - \mathrm{b}Cr,$$

$$\Omega w - \frac{2}{\kappa}\,k^2\nu = \mathrm{a}Bq - \mathrm{c}Ap\,;$$

to which many others might probably be joined.

The constants of the problem are a, b, c, h, ϵ, δ. Of these a, b, c are given as functions of λ, μ, ν, u, v, w, by the equations (34); in which ϖ is to be considered as standing for $\lambda u + \mu v + \nu w$. [These determine k^2, which is however given immediately by (37).] As for h, we have

$$h = \frac{1}{A}(Ap)^2 + \frac{1}{B}(Bq)^2 + \frac{1}{C}(Cr)^2 \ldots\ldots (40),$$

where Ap, Bq, Cr are given as functions of $\lambda, \mu, \nu, u, v, w$ by (32), in which also ϖ stands for $\lambda u + \mu v + \nu w$. Again,

$$\epsilon = t - \frac{1}{2}\int \frac{dv}{\nabla},$$

$$\delta = 2\tan^{-1}\frac{\kappa\varpi}{2k} - \frac{1}{4}k\int\frac{(h+\Phi)\,dv}{v\nabla}.$$

In each of which ∇, Φ are functions of v, and of a, b, c, h, partly as entering explicitly into these functions, partly as contained implicitly in p, q, r, which enter into ∇, Φ, and are functions of v, h, k given by (18). After the integration v is to be considered a function of $\lambda, \mu, \nu, u, v, w$ given by (38). Both of the integrals may be supposed taken from a certain value v_0 of v, which may be considered as an absolutely invariable arbitrary constant, since without it we have the right number, six, of arbitrary constants. First to find (a, b), (b, c), and (c, a). From (34) we have

$$\begin{aligned}(\text{a}, \text{b}) = \tfrac{1}{4}\{&(1+\lambda^2)(\mu u - w) - (\lambda u + \varpi)(\lambda\mu + \nu)\\ &+ (\lambda\mu - \nu)(\mu v + \varpi) - (\lambda v + w)(1 + \mu^2)\\ &+ (\nu\lambda + \mu)(u + \mu w) - (\lambda w - v)(\mu\nu - \lambda)\}\\ = \tfrac{1}{2}(\mu u - \lambda v - w - \nu\varpi) &= -\tfrac{1}{2}2\text{c} = -\text{c};\end{aligned}$$

whence the system

$$(\text{b}, \text{c}) = -\text{a}, \quad (\text{c}, \text{a}) = -\text{b}, \quad (\text{a}, \text{b}) = -\text{c} \;\ldots\; (40).$$

Also we may add

$$(k, \text{a}) = \frac{\text{a}}{k}(\text{a}, \text{a}) + \frac{\text{b}}{k}(\text{b}, \text{a}) + \frac{\text{c}}{k}(\text{c}, \text{a}) = 0,$$

or $$(k, \text{a}) = 0, \quad (k, \text{b}) = 0, \quad (k, \text{c}) = 0 \;\ldots\ldots\; (41),$$

which will be useful in calculating some of the following coefficients.

Proceeding to calculate (a, h), (b, h), (c, h). It is seen immediately that

$$(\text{a}, h) = 2\{p(\text{a}, Ap) + q(\text{a}, Bq) + r(\text{a}, Cr)\},$$

where Ap, Bq, Cr, are given by the equations (32), so that

$$\begin{aligned}(\text{a}, Ap) = \tfrac{1}{4}\{&(1+\lambda^2)(\lambda u + \varpi) - (1+\lambda^2)(\lambda u + \varpi)\\ &+ (\lambda\mu - \nu)(\lambda v - w) - (\lambda\mu + \nu)(\lambda v + w)\\ &+ (\nu\lambda + \mu)(v + \lambda w) - (\nu\lambda - \mu)(-v + \lambda w)\}\end{aligned}$$

i.e. $(\text{a}, Ap) = 0$. $\;\ldots\ldots\ldots\ldots\;$ (42).

Similarly

$$\begin{aligned}(\mathrm{a},\, Bq) = \tfrac{1}{4}\{&(1+\lambda^2)(\mu u + w) - (\lambda u + \varpi)(\lambda\mu - \nu)\\ &+ (\lambda\mu - \nu)(\mu v + \varpi) - (\lambda v + w)(1 + \mu^2)\\ &+ (\nu\lambda + \mu)(\mu w - u) - (-v + \lambda w)(\mu\nu + \lambda)\}\end{aligned}$$

i.e. $(\mathrm{a},\, Bq) = 0$, and similarly ...... (43),

$(\mathrm{a},\, Cr) = 0$;

whence $(\mathrm{a},\, h) = 0$, and $\therefore (\mathrm{b},\, h) = 0,\ (\mathrm{c},\, h) = 0$ ...... (44);

also $(k,\, h) = 0,$ ...... (45).

Next we have to determine $(\mathrm{a},\, \epsilon)$, $(\mathrm{b},\, \epsilon)$, $(\mathrm{c},\, \epsilon)$. Here ϵ being a function of $u, v, w, \lambda, \mu, \nu$, a, b, c, h, we must write

$$(\mathrm{a},\, \epsilon) = \{(\mathrm{a},\, \epsilon)\} + (\mathrm{a},\, \mathrm{b})\frac{d\epsilon}{d\mathrm{b}} + (\mathrm{a},\, \mathrm{c})\frac{d\epsilon}{d\mathrm{c}} + (\mathrm{a},\, h)\frac{d\epsilon}{dh},$$

i.e. $(\mathrm{a},\, \epsilon) = \{(\mathrm{a},\, \epsilon)\} + \mathrm{b}\dfrac{d\epsilon}{d\mathrm{c}} - \mathrm{c}\dfrac{d\epsilon}{d\mathrm{b}}.$

But $$\epsilon = t - 2\int\frac{dv}{\nabla};$$

whence $$\{(\mathrm{a},\, \epsilon)\} = -\frac{2}{\nabla}(\mathrm{a},\, v),$$

and v is given immediately as a function of $\lambda, \mu, \nu, u, v, w$, by the equation (38). Hence

$$\begin{aligned}(\mathrm{a},\, v) &= \tfrac{1}{4}[(1+\lambda^2)\{(1+\kappa)u\varpi + \lambda\varpi^2\} - (\lambda u + \varpi)\{u + \lambda(1+\kappa)\varpi\}\\ &\quad + (\lambda\mu - \nu)\{(1+\kappa)v\varpi + \mu\varpi^2\} - (\lambda v + w)\{v + \mu(1+\kappa)\varpi\}\\ &\quad + (\nu\lambda + \mu)\{(1+\kappa)w\varpi + \nu\varpi^2) - (-v + \lambda w)\{w + \nu(1+\kappa)\varpi\}]\\ &= \tfrac{1}{4}\{(1+\kappa)\varpi u - \lambda(1+\kappa)\varpi^2 + \lambda\kappa - \lambda(u^2+v^2+w^2) - u\varpi\}\\ &= \tfrac{1}{4}\{\kappa u\varpi - \lambda\varpi^2 - \lambda(u^2+v^2+w^2)\}\\ &= \tfrac{1}{4}\kappa u\varpi - \frac{k^2\lambda}{\kappa} = \tfrac{1}{2}\left(\Omega u - \frac{2k^2\lambda}{\kappa}\right) \text{ [by (37) and (33)]},\\ &= \tfrac{1}{2}(\mathrm{b}Cr - \mathrm{c}Bq). \qquad \text{...... (46)};\end{aligned}$$

whence $$\{(\mathrm{a},\, \epsilon)\} = -\frac{1}{\nabla}(\mathrm{b}Cr - \mathrm{c}Bq),$$

$$(\mathrm{a},\, \epsilon) = -\frac{1}{\nabla}(\mathrm{b}Cr - \mathrm{c}Bq) + \mathrm{b}\frac{d\epsilon}{d\mathrm{c}} - \mathrm{c}\frac{d\epsilon}{d\mathrm{b}}.$$

The terms $\mathrm{b}\dfrac{d\epsilon}{d\mathrm{c}} - \mathrm{c}\dfrac{d\epsilon}{d\mathrm{b}}$ are evidently of the form $F(v) - F(v_0)$. If therefore we suppose $v = v_0$, we have

$$(\mathrm{a},\, \epsilon) = -\frac{1}{\nabla_0}(\mathrm{b}Cr_0 - \mathrm{c}Bq_0) \text{ .... (47)},$$

if p_0, q_0, r_0, ∇_0 refer to the value v_0 of v, i.e. if

$$Ap_0^2 + Bq_0^2 + Cr_0^2 = h \ldots\ldots\ldots\ldots (48),$$
$$A^2p_0^2 + B^2q_0^2 + C^2r_0^2 = k^2,$$
$$Ap_0\mathrm{a} + Bq_0\mathrm{b} + Cr_0\mathrm{c} = 2v_0 - k^2.$$

[This implies evidently

$$\mathrm{b}\frac{d\epsilon}{d\mathrm{c}} - \mathrm{c}\frac{d\epsilon}{d\mathrm{b}} = \frac{1}{\nabla}(\mathrm{b}Cr - \mathrm{c}Bq) - \frac{1}{\nabla_0}(\mathrm{b}Cr_0 - \mathrm{c}Bq_0),$$

an equation which it is interesting to verify. In fact, from the value of ϵ

$$\mathrm{b}\frac{d\epsilon}{d\mathrm{c}} - \mathrm{c}\frac{d\epsilon}{d\mathrm{b}} = -2\int dv\left(\mathrm{b}\frac{d}{d\mathrm{c}} - \mathrm{c}\frac{d}{d\mathrm{b}}\right)\frac{1}{\nabla} = 2\int dv\,\frac{1}{\nabla^2}\left(\mathrm{b}\frac{d\nabla}{d\mathrm{c}} - \mathrm{c}\frac{d\nabla}{d\mathrm{b}}\right);$$

or we have to shew that

$$\frac{d}{dv}\,\frac{1}{\nabla}(\mathrm{b}Cr - \mathrm{c}Bq) = \frac{2}{\nabla^2}\left(\mathrm{b}\frac{d\nabla}{d\mathrm{c}} - \mathrm{c}\frac{d\nabla}{d\mathrm{b}}\right) = \frac{2}{\nabla^2}\,\delta\nabla;$$

if for shortness,

$$\delta = \mathrm{b}\frac{d}{d\mathrm{c}} - \mathrm{c}\frac{d}{d\mathrm{b}}.$$

Now ∇ containing a, b, c explicitly, and also as involved in p, q, r, we have

$$\delta\nabla = \mathrm{b}pq(A - B) - \mathrm{c}rp(C - A) + \frac{d\nabla}{dp}\delta p + \frac{d\nabla}{dq}\delta q + \frac{d\nabla}{dr}\delta r$$
$$= \mathrm{b}pq(A - B) - \mathrm{c}rp(C - A) + \delta'\nabla$$

suppose. The equation to be verified becomes

$$\nabla\left(\mathrm{b}C\frac{dr}{dv} - \mathrm{c}B\frac{dq}{dv}\right) - (\mathrm{b}Cr - \mathrm{c}Bq)\frac{d\nabla}{dv}$$
$$= 2\{\mathrm{b}pq(A - B) - \mathrm{c}rp(C - A) + \delta'\nabla\}.$$

Now, observing that $\delta k = 0$, we have

$$Ap\delta p + Bq\delta q + Cr\delta r = 0,$$
$$A^2p\delta p + B^2q\delta q + C^2r\delta r = 0,$$
$$A\mathrm{a}\delta p + B\mathrm{b}\delta q + C\mathrm{c}\delta r = -(\mathrm{b}Cr - \mathrm{c}Bq).$$

Also,
$$Ap\frac{dp}{dv} + Bq\frac{dq}{dv} + Cr\frac{dr}{dv} = 0,$$
$$A^2p\frac{dp}{dv} + B^2q\frac{dq}{dv} + C^2r\frac{dr}{dv} = 0,$$
$$A\mathrm{a}\frac{dp}{dv} + B\mathrm{b}\frac{dq}{dv} + C\mathrm{c}\frac{dr}{dv} = 2.$$

Whence evidently

$$\frac{dp}{dv} = \frac{-2}{\mathrm{b}Cr - \mathrm{c}Bq}\,\delta p,\quad \frac{dq}{dv} = \frac{-2}{\mathrm{b}Cr - \mathrm{c}Bq}\,\delta q,\quad \frac{dr}{dv} = \frac{-2}{\mathrm{b}Cr - \mathrm{c}Bq}\,\delta r,$$

or

$$\frac{d\nabla}{dv} = \frac{-2}{\mathrm{b}Cr - \mathrm{c}Bq}\,\delta'\nabla\,;$$

or the equation to be verified is simply

$$\nabla\left(\mathrm{b}C\frac{dr}{dv} - \mathrm{c}B\frac{dq}{dv}\right) = 2\,\{\mathrm{b}pq\,(A - B) - \mathrm{c}rp\,(C - A)\}\,;$$

which follows immediately from the three equations just given for the determination of $\frac{dp}{dv},\ \frac{dq}{dv},\ \frac{dr}{dv}\Big]$.

From which values also

$$(k,\ \epsilon) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (49).$$

Next, to calculate $(h,\ \epsilon)$,

$$(h,\ \epsilon) = \{(h,\ \epsilon)\} + (h,\ \mathrm{a})\frac{d\epsilon}{d\mathrm{a}} + (h,\ \mathrm{b})\frac{d\epsilon}{d\mathrm{b}} + (h,\ \mathrm{c})\frac{d\epsilon}{d\mathrm{c}}\,.$$

But the three last terms being evidently such as to vanish for $v = v_0$, we may neglect them, and consider $(h,\ \epsilon)$ as the value which $\{(h,\ \epsilon)\}$ assumes for this value of v.

Now $\{(h,\ \epsilon)\} = 2p\,\{(Ap,\ \epsilon)\} + 2q\,\{(Bq,\ \epsilon)\} + 2r\,\{(Cr,\ \epsilon)\}$,

where $$\{(Ap,\ \epsilon)\} = -\frac{2}{\nabla}(Ap,\ v),$$

and

$$\begin{aligned}(Ap,\ v) = \tfrac{1}{4}\,[&(1 + \lambda^2)\{(1 + \kappa)\,u\varpi + \lambda\varpi^2\} - (\lambda u + \varpi)\{u + \lambda(1 + \kappa)\varpi\}\\ &+ (\lambda\mu + \nu)\{(1 + \kappa)v\varpi + \mu\varpi^2\} - (\lambda v - w)\{v + \mu(1 + \kappa)\varpi\}\\ &+ (\nu\lambda - \mu)\{(1 + \kappa)w\varpi + \nu\varpi^2\} - (v + \lambda w)\{w + \nu(1 + \kappa)\varpi\}]\\ = \tfrac{1}{4}\,&\{(1 + \kappa)\varpi u + \lambda\kappa\varpi^2 - \lambda(1 + \kappa)\varpi^2 - \lambda(u^2 + v^2 + w^2) - \varpi u\} = (\mathrm{a},\ v)\\ = \tfrac{1}{2}\,&(\mathrm{b}Cr - \mathrm{c}Bq), \qquad \ldots\ldots (50),\end{aligned}$$

whence $$\{(Ap,\ \epsilon)\} = -\frac{1}{\nabla}(\mathrm{b}Cr - \mathrm{c}Bq) \ldots\ldots (51),$$

and therefore $$\{(Bq,\ \epsilon)\} = -\frac{1}{\nabla}(\mathrm{c}Ap - \mathrm{a}Cr),$$

$$\{(Cr,\ \epsilon)\} = -\frac{1}{\nabla}(\mathrm{a}Bq - \mathrm{b}Ap),$$

whence $\{(h,\ \epsilon) = -2$,

and therefore $(h,\ \epsilon) = -2 \ldots\ldots\ldots\ldots\ldots\ldots (52).$

Next, to find (a, δ), (b, δ), (c, δ), (h, δ),

$$\delta = 2 \tan^{-1} \frac{\kappa\varpi}{2k} - k\int \frac{(h+\Phi)\,dv}{v\nabla}$$

$$= \delta' + \delta'' \text{ suppose,}$$

$$(\text{a}, \delta) = (\text{a}, \delta') + (\text{a}, \delta''),$$

$$(\text{a}, \delta') = \frac{k}{\kappa v}(\text{a}, \kappa\varpi) + (\text{a}, k)\frac{d\delta'}{dk}$$

[observing $\kappa^2\varpi^2 + 4k^2 = 4(\Omega^2 + k^2) = 4\kappa v$]

$$= \frac{k}{\kappa v}(\text{a}, \kappa\varpi),$$

where

$$\begin{aligned}(\text{a}, \kappa\varpi) = \tfrac{1}{2}\{&(1+\lambda^2)(\kappa u + 2\lambda\varpi) - (\lambda u + \varpi)\kappa\lambda \\ &+ (\lambda\mu - \nu)(\kappa v + 2\mu\varpi) - (\lambda v + w)\kappa\mu \\ &+ (\nu\lambda + \mu)(\kappa w + 2\nu\varpi) - (-v + \lambda w)\kappa\nu\} \\ = \tfrac{1}{2}&\kappa(u + \lambda\varpi) = Ap - \nu Bq + \mu Cr + \lambda\Omega = \tfrac{1}{2}(\text{a} + Ap)\kappa \ldots (53),\end{aligned}$$

by equations (29), (33), and (10);

or $$(\text{a}, \delta') = \frac{k}{2v}(\text{a} + Ap).$$

Also $$(\text{a}, \delta'') = -k\frac{h+\Phi}{v\nabla}(\text{a}, v) + (\text{a}, \text{b})\frac{d\delta''}{d\text{b}} + \&\text{c}.$$

$$= -\tfrac{1}{2}k\frac{h+\Phi}{v\nabla}(\text{b}Cr - \text{c}Bq) + Fv - Fv_0,$$

whence

$$(\text{a}, \delta) = \frac{k}{2v}\left\{\text{a} + Ap - \frac{h+\Phi}{\nabla}(\text{b}Cr - \text{c}Bq)\right\} + Fv - Fv_0,$$

or putting $v = v_0$,

$$(\text{a}, \delta) = \frac{k}{2v_0}\left\{\text{a} + Ap_0 - \frac{h+\Phi_0}{\nabla_0}(\text{b}Cr_0 - \text{c}Bq_0)\right\} \ldots . (54),$$

and therefore

$$(\text{b}, \delta) = \frac{k}{2v_0}\left\{\text{b} + Bq_0 - \frac{h+\Phi_0}{\nabla_0}(\text{c}Ap_0 - \text{a}Cr_0)\right\},$$

$$(\text{c}, \delta) = \frac{k}{2v_0}\left\{\text{c} + Cr_0 - \frac{h+\Phi_0}{\nabla_0}(\text{a}Bq_0 - \text{b}Ap_0)\right\}.$$

Again, $(h, \delta) = 2p(Ap, \delta) + 2q(Bq, \delta) + 2r(Cr, \delta)$,

$$(Ap, \delta) = (Ap, \delta') + (Ap, \delta''),$$

$$(Ap, \delta') = \frac{k}{\kappa v}(Ap, \kappa\varpi) + (Ap, k)\frac{d\delta'}{dk}$$

$$= \frac{k}{\kappa v}(Ap, \kappa\varpi),$$

$$\begin{aligned}(Ap, \kappa\varpi) = \tfrac{1}{2}\{&(1+\lambda^2)(\kappa u + 2\lambda\varpi) - (\lambda u + \varpi)\kappa\lambda \\ &+ (\lambda\mu + \nu)(\kappa v + 2\mu\varpi) - (\lambda v - w)\kappa\mu \\ &+ (\nu\lambda + \mu)(\kappa w + 2\nu\varpi) - (v + \lambda w)\kappa\nu\} \\ = \tfrac{1}{2}\kappa(u + \lambda\varpi) &= \tfrac{1}{2}\kappa(\mathrm{a} + Ap) \ldots\ldots\ldots\ldots (55);\end{aligned}$$

$$\therefore (Ap, \delta') = \frac{k}{2u}(\mathrm{a} + Ap) \ldots\ldots\ldots\ldots (56),$$

$$(Ap, \delta'') = -k\frac{h+\Phi}{v\nabla}(Ap, v) + \&c.$$

$$= -\tfrac{1}{2}k\frac{h+\Phi}{v\nabla}(\mathrm{b}Cr - \mathrm{c}Bq) + Fv - Fv_0,$$

$$(Ap, \delta) = \frac{k}{2v}\left\{\mathrm{a} - Ap - \frac{h+\Phi}{\nabla}(\mathrm{b}Cr - \mathrm{c}Bq)\right\} + Fv - Fv_0,$$

and similarly for (Bq, δ), (Cr, δ). Substituting, and neglecting the terms which vanish for $v = v_0$,

$$(h, \delta) = \frac{k}{v}\left(\Phi + h - \frac{\Phi + h}{\nabla}\nabla\right),$$

$$\text{i.e.}\quad (h, \delta) = 0 \ldots\ldots\ldots\ldots (57).$$

Lastly, to find (ϵ, δ),

$$(\epsilon, \delta) = \{(\epsilon, \delta)\} + (\mathrm{a}, \delta)\frac{d\epsilon}{d\mathrm{a}} + (\mathrm{b}, \delta)\frac{d\epsilon}{d\mathrm{b}} + (\mathrm{c}, \delta)\frac{d\epsilon}{d\mathrm{c}},$$

where, in $\{(\epsilon, \delta)\}$, the differentiations upon ϵ are supposed not to affect the constants a, b, c. Neglecting the terms which vanish for $v = v_0$,

$$(\epsilon, \delta) = \{(\epsilon, \delta)\},$$

$$\{(\epsilon, \delta)\} = \{(\epsilon, \delta')\} + \{(\epsilon, \delta'')\},$$

$$\{(\epsilon, \delta')\} = [\{(\epsilon, \delta')\}] + (\epsilon, k)\frac{d\delta'}{dk} = [\{(\epsilon, \delta')\}];$$

where, in $[\{(\epsilon, \delta')\}]$, the differentiations upon ϵ and δ do not affect the constants.

$$\{(\epsilon, \delta'')\} = [\{(\epsilon, \delta'')\}] + (\epsilon, \mathrm{a})\frac{d\delta''}{d\mathrm{a}} + \&c.$$

$$\text{i.e.}\quad \{(\epsilon, \delta'')\} = [\{(\epsilon, \delta'')\}]:$$

neglecting the terms which vanish for $v = v_0$,

$$\therefore (\epsilon, \delta) = [\{(\epsilon, \delta')\}] + [\{(\epsilon, \delta'')\}]$$

$$= [\{(\epsilon, \delta')\}];$$

since $$[\{(\epsilon, \delta'')\}] = (v, v)\frac{d\epsilon}{dv}\frac{d\delta''}{dv} = 0.$$

Hence $$(\epsilon, \delta) = -\tfrac{1}{2}\frac{k}{\kappa\nabla v}(v, \kappa\varpi)\ldots\ldots\ldots\ldots(58),$$

$$(v, \kappa\varpi) = \tfrac{1}{2}\begin{Bmatrix} \{u + (1+\kappa)\lambda\varpi\}(2\lambda\varpi + \kappa u) - \{\lambda\varpi^2 + (1+\kappa)\varpi u\}\kappa\lambda \\ + \{v + (1+\kappa)\mu\varpi\}(2\mu\varpi + \kappa v) - \{\mu\varpi^2 + (1+\kappa)\varpi v\}\kappa\mu \\ + \{w + (1+\kappa)\nu\varpi\}(2\nu\varpi + \kappa w) - \{\nu\varpi^2 + (1+\kappa)\varpi w\}\kappa\nu \end{Bmatrix}$$

$$= \tfrac{1}{2}\{2\varpi^2 + \kappa(u^2 + v^2 + w^2) + 2(1+\kappa)(\kappa - 1)\varpi^2$$
$$+ \kappa(1+\kappa)\varpi^2 - \kappa(\kappa-1)\varpi^2 - \kappa(\kappa+1)\varpi^2\}$$
$$= \tfrac{1}{2}\kappa\{(\kappa+1)\varpi^2 + (u^2 + v^2 + w^2)\} = \tfrac{1}{2}4\kappa v = 2\kappa v \ldots(59),$$

therefore

$$(\epsilon, \delta) = -\frac{k}{\nabla_0}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(60).$$

Hence, recapitulating,

$$\left.\begin{aligned}
&(\mathrm{b}, \mathrm{c}) = -\mathrm{a}, \quad (\mathrm{c}, \mathrm{a}) = -\mathrm{b}, \quad (\mathrm{a}, \mathrm{b}) = -\mathrm{c}, \\
&(\mathrm{a}, h) = 0, \quad (\mathrm{b}, h) = 0, \quad (\mathrm{c}, h) = 0, \\
&(\mathrm{a}, \epsilon) = -\frac{1}{\nabla_0}(\mathrm{b}Cr_0 - \mathrm{c}Bq_0), \\
&(\mathrm{b}, \epsilon) = -\frac{1}{\nabla_0}(\mathrm{c}Ap_0 - \mathrm{a}Cr_0), \\
&(\mathrm{c}, \epsilon) = -\frac{1}{\nabla_0}(\mathrm{a}Bq_0 - \mathrm{b}Ap_0), \\
&(h, \epsilon) = -2, \\
&(\mathrm{a}, \delta) = \frac{k}{2v_0}\left\{\mathrm{a} + Ap_0 - \frac{h + \Phi_0}{\nabla_0}(\mathrm{b}Cr_0 - \mathrm{c}Bq_0)\right\}, \\
&(\mathrm{b}, \delta) = \frac{k}{2v_0}\left\{\mathrm{b} + Bq_0 - \frac{h + \Phi_0}{\nabla_0}(\mathrm{c}Ap_0 - \mathrm{a}Cr_0)\right\}, \\
&(\mathrm{c}, \delta) = \frac{k}{2v_0}\left\{\mathrm{c} + Cr_0 - \frac{h + \Phi_0}{\nabla_0}(\mathrm{a}Bq_0 - \mathrm{b}Ap_0)\right\}, \\
&(h, \delta) = 0, \\
&(\epsilon, \delta) = -\frac{k}{\nabla_0},
\end{aligned}\right\}\ldots(61),$$

and therefore

$$\left.\begin{aligned}
\frac{d\mathrm{a}}{dt} &= -\mathrm{c}\frac{dV}{d\mathrm{b}} + \mathrm{b}\frac{dV}{d\mathrm{c}} - \frac{1}{\nabla_0}(\mathrm{b}Cr_0 - \mathrm{c}Bq_0)\frac{dV}{d\epsilon} \\
&\quad + \frac{k}{2v_0}\left\{\mathrm{a} + Ap_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{b}Cr_0 - \mathrm{c}Bq_0)\right\}\frac{dV}{d\delta}, \\
\frac{d\mathrm{b}}{dt} &= -\mathrm{a}\frac{dV}{d\mathrm{c}} + \mathrm{c}\frac{dV}{d\mathrm{a}} - \frac{1}{\nabla_0}(\mathrm{c}Ap_0 - \mathrm{a}Cr_0)\frac{dV}{d\epsilon} \\
&\quad + \frac{k}{2v_0}\left\{\mathrm{b} + Bq_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{c}Ap_0 - \mathrm{a}Cr_0)\right\}\frac{dV}{d\delta}, \\
\frac{d\mathrm{c}}{dt} &= -\mathrm{b}\frac{dV}{d\mathrm{a}} + \mathrm{a}\frac{dV}{d\mathrm{b}} - \frac{1}{\nabla_0}(\mathrm{a}Bq_0 - \mathrm{b}Ap_0)\frac{dV}{d\epsilon} \\
&\quad + \frac{k}{2v_0}\left\{\mathrm{c} + Cr_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{a}Bq_0 - \mathrm{b}Ap_0)\right\}\frac{dV}{d\delta}, \\
\frac{dh}{dt} &= -2\frac{dV}{d\epsilon}, \\
\frac{d\epsilon}{dt} &= \frac{1}{\nabla_0}\left\{(\mathrm{b}Cr_0 - \mathrm{c}Bq_0)\frac{dV}{d\mathrm{a}} + (\mathrm{c}Ap_0 - \mathrm{a}Bq_0)\frac{dV}{d\mathrm{b}}\right. \\
&\quad \left. + (\mathrm{a}Bq_0 - \mathrm{b}Ap_0)\frac{dV}{d\mathrm{c}}\right\} + 2\frac{dV}{dh} - \frac{k}{\nabla_0}\frac{dV}{d\delta}, \\
\frac{d\delta}{dt} &= -\frac{k}{2v_0}\left[\left\{\mathrm{a} + Ap_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{b}Cr_0 - \mathrm{c}Bq_0)\right\}\frac{dV}{d\mathrm{a}}\right. \\
&\quad + \left\{\mathrm{b} + Bq_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{c}Ap_0 - \mathrm{a}Cr_0)\right\}\frac{dV}{d\mathrm{b}} \\
&\quad \left. + \left\{\mathrm{c} + Cr_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{a}Bq_0 - \mathrm{b}Ap_0)\right\}\frac{dV}{d\mathrm{c}}\right] + \frac{k}{\nabla_0}\frac{dV}{d\epsilon},
\end{aligned}\right\} \quad \ldots\ldots (62),$$

to which we may join $\dfrac{dk}{dt} = \dfrac{dV}{d\delta}$ .... (63).

ON THE DIAMETRAL PLANES OF A SURFACE OF THE SECOND ORDER.

By Arthur Cayley.

Let $U = Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gxz + 2Hxy = 0$, be the equation of a surface of the second order referred to its centre, and let $ax + a'y + a''z = 0$ be the equation of one of its diametral planes; then, as usual,

$$(A-u)\,\alpha + H\alpha' + G\alpha'' = 0,$$
$$H\alpha + (B-u)\,\alpha' + F\alpha'' = 0,$$
$$G\alpha + F\alpha' + (C-u)\,\alpha'' = 0,$$

which are equivalent to two independent equations, and consequently capable of determining the ratios $\alpha : \alpha' : \alpha''$, provided that u satisfy the cubic equation that is obtained by eliminating $\alpha, \alpha', \alpha''$ from the three equations.

We have from the second and third, from the third and first, and from the first and second equations respectively,

$$\alpha : \alpha' : \alpha'' = \mathfrak{A}_{,} : \mathfrak{H}_{,} : \mathfrak{G}_{,} = \mathfrak{H}_{,} : \mathfrak{B}_{,} : \mathfrak{F}_{,} = \mathfrak{G}_{,} : \mathfrak{F}_{,} : \mathfrak{C}_{,}\,;$$

where, if
$$\mathfrak{A} = BC - F^2,$$
$$\mathfrak{B} = CA - G^2,$$
$$\mathfrak{C} = AB - H^2,$$
$$\mathfrak{F} = GH - AF,$$
$$\mathfrak{G} = HF - BG,$$
$$\mathfrak{H} = FG - CH.$$

$\mathfrak{A}_{,}, \mathfrak{B}_{,}, \mathfrak{C}_{,}, \mathfrak{F}_{,}, \mathfrak{G}_{,}, \mathfrak{H}_{,}$ are what these become when A, B, C are changed into $A-u$, $B-u$, $C-u$, so that

$$\mathfrak{A}_{,} = \mathfrak{A} - (B+C)\,u + u^2,$$
$$\mathfrak{B}_{,} = \mathfrak{B} - (C+A)\,u + u^2,$$
$$\mathfrak{C}_{,} = \mathfrak{C} - (A+B)\,u + u^2,$$
$$\mathfrak{F}_{,} = \mathfrak{F} + Fu,$$
$$\mathfrak{G}_{,} = \mathfrak{G} + Gu,$$
$$\mathfrak{H}_{,} = \mathfrak{H} + Hu.$$

Hence the equation $\alpha x + \alpha' y + \alpha'' z = 0$ may be written in the three forms

$$\mathfrak{A}_{,}x + \mathfrak{H}_{,}y + \mathfrak{G}_{,}z = 0,$$
$$\mathfrak{H}_{,}x + \mathfrak{B}_{,}y + \mathfrak{F}_{,}z = 0,$$
$$\mathfrak{G}_{,}x + \mathfrak{F}_{,}y + \mathfrak{C}_{,}z = 0;$$

or, what comes to the same thing, as follows,

$$\mathfrak{A}x + \mathfrak{H}y + \mathfrak{G}z + u\,(Ax + Hy + Gz) + vx = 0,$$
$$\mathfrak{H}x + \mathfrak{B}y + \mathfrak{F}z + u\,(Hx + By + Fz) + vy = 0,$$
$$\mathfrak{G}x + \mathfrak{F}y + \mathfrak{C}z + u\,(Gx + Fy + Cz) + vz = 0,$$

in which for shortness v has been written instead of

$$u^2 - (A+B+C)\,u.$$

The elimination of u, v from these equations gives a result $\Theta = 0$, where Θ is a homogeneous function of the third order

in x, y, z; and this equation, it is evident, must belong to the three diametral planes jointly, *i.e.* Θ must be the product of three linear factors, each of which equated to zero would correspond to a diametral plane. Thus the system of diametral planes is given by

$$\Theta = \begin{vmatrix} \mathfrak{A}x + \mathfrak{H}y + \mathfrak{G}z, & Ax + Hy + Gz, & x \\ \mathfrak{H}x + \mathfrak{B}y + \mathfrak{F}z, & Hx + By + Fz, & y \\ \mathfrak{G}x + \mathfrak{F}y + \mathfrak{C}z, & Gx + Fy + Cz, & z \end{vmatrix} = 0,$$

or developing the determinant, as follows,

$$\begin{aligned} \Theta = & (G\mathfrak{H} - H\mathfrak{G})\, x^3 + (H\mathfrak{F} - F\mathfrak{H})\, y^3 + (F\mathfrak{G} - G\mathfrak{F})\, z^3 \\ & + \{G(\mathfrak{C} - \mathfrak{B}) - \mathfrak{G}(C - B) - (H\mathfrak{F} - F\mathfrak{H})\}\, yz^2 \\ & + \{H(\mathfrak{A} - \mathfrak{C}) - \mathfrak{H}(A - C) - (F\mathfrak{G} - G\mathfrak{F})\}\, zx^2 \\ & + \{F(\mathfrak{B} - \mathfrak{A}) - \mathfrak{F}(B - A) - (G\mathfrak{H} - H\mathfrak{G})\}\, xy^2 \\ & + \{- H(\mathfrak{C} - \mathfrak{B}) + \mathfrak{H}(C - B) + (F\mathfrak{G} - G\mathfrak{F})\}\, y^2 z \\ & + \{- F(\mathfrak{A} - \mathfrak{C}) + \mathfrak{F}(A - C) + (G\mathfrak{H} - H\mathfrak{G})\}\, z^2 x \\ & + \{- G(\mathfrak{B} - \mathfrak{A}) + \mathfrak{C}(B - A) + (H\mathfrak{F} - F\mathfrak{H})\}\, x^2 y \\ & + (C\mathfrak{B} - B\mathfrak{A} + \mathfrak{A}\mathfrak{C} - C\mathfrak{A} + B\mathfrak{A} - A\mathfrak{B})\, xyz; \end{aligned}$$

or reducing

$$\begin{aligned} \Theta = & \{F(G^2 - H^2) - GH(C - B)\}\, x^3 \\ & + \{G(H^2 - F^2) - HF(A - C)\}\, y^3 \\ & + \{H(F^2 - G^2) - FG(B - A)\}\, z^3 \\ & + \{G(A - B)(B - C) + FH(A + B - 2C) \\ & \qquad\qquad + G(F^2 + G^2 - 2H^2)\}\, yz^2 \\ & + \{H(B - C)(C - A) + GF(B + C - 2A) \\ & \qquad\qquad + H(G^2 + H^2 - 2F^2)\}\, zx^2 \\ & + \{F(C - A)(A - B) + GH(C + A - 2B) \\ & \qquad\qquad + F(H^2 + F^2 - 2G^2)\}\, xy^2 \\ & + \{H(B - C)(C - A) + FG(C + A - 2B) \\ & \qquad\qquad + H(H^2 + F^2 - 2G^2)\}\, y^2 z \\ & + \{F(C - A)(A - B) + GH(A + B - 2C) \\ & \qquad\qquad + F(F^2 + G^2 - 2H^2)\}\, z^2 x \\ & + \{G(A - B(B - C) + HF(B + C - 2A) \\ & \qquad\qquad + G(G^2 + H^2 - 2F^2)\}\, x^2 y \\ & - \{(A - B)(B - C)(C - A) \\ & \qquad\qquad + (B - C)F^2 + (C - A)G^2 + (A - B)H^2\}\, xyz. \end{aligned}$$

In the case of *curves* of the second order, the result is much simpler; we have

$$\Theta = \begin{vmatrix} Ax + Hy, & x \\ Hx + By, & y \end{vmatrix} = 0,$$

i.e.
$$\Theta = H(y^2 - x^2) + (A - B)\,xy = 0,$$

for the equation of the two diameters.

The above formulæ may be applied to the question of finding the diametral planes of the cone circumscribed about a given surface of the second order, (or of the lines bisecting the angles made by two tangents of a curve of the second order). Considering the latter question first: if

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

be the equation of the curve, and α, β the coordinates of the point of intersection of the two tangents, the equation of the pair of tangents is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} - 1\right) - \left(\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} - 1\right)^2 = 0;$$

or making the point of intersection the origin,

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} - 1\right) - \left(\frac{\alpha x}{a^2} + \frac{\beta y}{b^2}\right)^2 = 0,$$

i.e.
$$(\beta x - \alpha y)^2 - (b^2x^2 + a^2y^2) = 0;$$

whence $A = \beta^2 - b^2$, $B = \alpha^2 - a^2$, $H = -\alpha\beta$, and the equation to the lines bisecting the angles formed by the tangents is

$$\alpha\beta\,(x^2 - y^2) - \{\alpha^2 - \beta^2 - (a^2 - b^2)\}\,xy = 0,$$

which is the same for all confocal ellipses; whence the known theorem,

"If there be two confocal ellipses, and tangents be drawn to the second from any point P of the first, the tangent and normal of the first conic at the point P, bisect the angles formed by the two tangents in question."

In the case of surfaces, the equation of the circumscribing cone referred to its vertex as origin, is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} - 1\right) - \left(\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} + \frac{\gamma z}{c^2}\right)^2 = 0;$$

whence
$$\begin{aligned} A &= \beta^2c^2 + \gamma^2b^2 - b^2c^2, \\ B &= \gamma^2a^2 + \alpha^2c^2 - a^2c^2, \\ C &= \alpha^2b^2 + \beta^2a^2 - b^2a^2, \\ F &= -\,a^2\beta\gamma, \\ G &= -\,b^2\gamma\alpha, \\ H &= -\,c^2\alpha\beta. \end{aligned}$$

And thence omitting the factor $b^2c^2\alpha^2 + c^2a^2\beta^2 + a^2b^2\gamma^2 - a^2b^2c^2$,
$$\begin{aligned} \mathfrak{A} &= \alpha^2 - a^2, \\ \mathfrak{B} &= \beta^2 - b^2, \\ \mathfrak{C} &= \gamma^2 - c^2, \\ \mathfrak{F} &= \beta\gamma, \\ \mathfrak{G} &= \gamma\alpha, \\ \mathfrak{H} &= \alpha\beta; \end{aligned}$$

and the equation of the system of diametral planes becomes
$$\begin{aligned} \Theta = 0 = {} & x^3.\alpha^2\beta\gamma\,(c^2 - b^2) + y^3.\beta^2\gamma\alpha\,(a^2 - c^2) + z^3.\gamma^2\alpha\beta\,(b^2 - a^2) \\ & + \gamma\alpha\,\{\alpha^2(c^2 - b^2) + \beta^2(b^2 + c^2 - 2a^2) - \gamma^2(b^2 - a^2) \\ & \qquad + (b^2 - a^2)(c^2 - b^2)\}\,yz^2 \\ & + \alpha\beta\,\{-\,\alpha^2(c^2 - b^2) + \beta^2(a^2 - c^2) + \gamma^2(c^2 + a^2 - 2b^2) \\ & \qquad + (c^2 - b^2)(a^2 - c^2)\}\,zx^2 \\ & + \gamma\alpha\,\{\alpha^2(a^2 + b^2 - 2c^2) - \beta^2(a^2 - c^2) + \gamma^2(b^2 - a^2) \\ & \qquad + (a^2 - c^2)(b^2 - a^2)\}\,xy^2 \\ & - \alpha\beta\,\{\alpha^2(c^2 - b^2) - \beta^2(a^2 - c^2) - \gamma^2(b^2 + c^2 - 2a^2) \\ & \qquad - (a^2 - c^2)(c^2 - b^2)\}\,y^2z \\ & - \beta\gamma\,\{-\,\alpha^2(c^2 + a^2 - 2b^2) + \beta^2(a^2 - c^2) - \gamma^2(b^2 - a^2) \\ & \qquad - (b^2 - a^2)(a^2 - c^2)\}\,z^2x \\ & - \gamma\alpha\,\{-\,\alpha^2(c^2 - b^2) - \beta^2(a^2 + b^2 - 2c^2) \\ & \qquad + \gamma^2(b^2 - a^2) - (c^2 - b^2)(b^2 - a^2)\}\,x^2y \\ & + \{(a^2 - b^2)(b^2 - c^2)(c^2 - a^2) \\ & \quad + (\alpha^4 + \beta^2\gamma^2)(c^2 - b^2) + (\beta^4 + \gamma^2\alpha^2)(a^2 - c^2) + (\gamma^4 + \alpha^2\beta^2)(b^2 - a^2) \\ & \quad + \alpha^2(b^2 - c^2)(2a^2 - b^2 - c^2) + \beta^2(c^2 - a^2)(2b^2 - c^2 - a^2) \\ & \qquad + \gamma^2(a^2 - b^2)(2c^2 - a^2 - b^2)\}\,xyz. \end{aligned}$$

And since this is a function of $a^2 - b^2$, $b^2 - c^2$, and $c^2 - a^2$, the equation is the same for all confocal ellipsoids; whence the known theorem, "The axes of the circumscribing cone having its vertex in a given point P, are tangents to the curves of intersection of the three surfaces, confocal with the given surface, which pass through the point P."

SUR UNE PROPRIÉTÉ DE LA COUCHE ÉLECTRIQUE EN ÉQUILIBRE À LA SURFACE D'UN CORPS CONDUCTEUR.

Par M. J. Liouville.

La méthode la plus générale que l'on connaisse pour former des couches électriques, en équilibre à la surface de corps conducteurs, consiste à considérer une masse M; et le potentiel,

$$V=\iiint\frac{f(x',y',z')\,dx'\,dy'\,dz'}{\Delta},$$

de cette masse, par rapport à un point quelconque (x, y, z), dont la distance au point (x', y', z'), ou à l'élément

$$f(x',y',z')\,dx'\,dy'\,dz',$$

est désignée par Δ. Prenons ensuite une surface de niveau ou d'équilibre relativement à l'attraction de la masse M, et qui entoure cette masse, c'est à dire prenons une surface fermée (A), contenant la masse M dans son intérieur, et pour tous les points de laquelle V conserve une valeur constante. En fin soit $\frac{dV}{ds}ds$ la variation infiniment petite que V éprouve lorsqu' on passe d'un point de cette surface à un point extérieur infiniment voisin situé sur la normale à une distance ds. C'est la dérivée $\frac{dV}{ds}$, multipliée si l'on veut par une constante, qui réglera la loi des densités de l'électricité en équilibre sur un corps conducteur terminé par la surface (A). Plusieurs géomètres sont parvenus, chacun de leur côté, à ce beau théorème; mais c'est George Green qui l'a, je crois, donné le premier dans un excellent mémoire publié en 1828, sous ce titre: *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism.* Je me propose de montrer que la couche électrique en équilibre ainsi obtenue a précisément le même centre de gravité que la masse M.

Plaçons l'origine des coordonnées x, y, z, au centre de gravité de la masse M; et désignons par x_1 une quelconque des coordonnées du centre de gravité de la couche électrique, laquelle sera fournie par la formule

$$x_1\iint\frac{dV}{ds}\,d\omega=\iint x\,\frac{dV}{ds}\,d\omega,$$

où les intégrations s'appliquent à la surface (A) dont l'élément est représenté par $d\omega$. Il s'agit de prouver que $x_1 = 0$.

D' après l'expression de V, on a

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = -4\pi f(x, y, z), \quad \text{ou} = 0,$$

suivant que le point (x, y, z) appartient ou non à la masse M.

Pour plus de simplicité, écrivons toujours

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = -4\pi f(x, y, z),$$

en regardant la fonction $f(x, y, z)$ comme nulle hors de la masse M; et combinons cette équation avec cette autre de forme analogue

$$\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} + \frac{d^2U}{dz^2} = 0,$$

où nous supposons que U est une fonction de x, y, z, qui reste finie et continue ainsi que ses dérivées dans tout l'espace intérieur à (A). Nous aurons

$$V\frac{d^2U}{dx^2} - U\frac{d^2V}{dx^2} + V\frac{d^2U}{dy^2} - U\frac{d^2V}{dy^2} + V\frac{d^2U}{dz^2} - U\frac{d^2V}{dz^2} = 4\pi Uf(x, y, z).$$

Multiplions par $dx\,dy\,dz$, et intégrons dans tout l'espace intérieur à (A). En conservant à ds et à $d\omega$ la même signification que ci-dessus, on trouve, après des transformations bien connues:

$$\iint V\frac{dU}{ds}\,d\omega - \iint U\frac{dV}{ds}\,d\omega = 4\pi \iiint Uf(x, y, z)\,dx\,dy\,dz.$$

Mais l'équation en U est satisfaite par $U = x$; nous avons donc:

$$\iint V\frac{dx}{ds}\,d\omega - \iint x\frac{dV}{ds}\,d\omega = 4\pi \iiint x f(x, y, z)\,dx\,dy\,dz.$$

L' intégrale triple du second membre, divisée par M, donne l'abscisse du centre de gravité de la masse M. Ce centre étant à l' origine des coordonnées, l' intégrale dont nous parlons est nulle. Je vais prouver que l' intégrale $\iint V\frac{dx}{ds}\,d\omega$ l'est aussi. D' abord on peut faire sortir V du signe $\int$, puisque, sur la surface (A), V est constant. Observons ensuite que $\frac{dx}{ds}$ a pour valeur le cosinus de l'angle α que la normale ds fait avec l'axe des x. Notre intégrale

deviendra donc: $V\iint \cos a\, d\omega$. Or l'intégrale $\iint \cos a\, d\omega$ est nulle, d'après un théorême connu, comme composée d'éléments deux à deux égaux et de signes contraires. Ainsi $\iint V\frac{dx}{ds}\, d\omega = 0$. Il reste donc finalement

$$\iint x\,\frac{dV}{ds}\, d\omega = 0,$$

et l'on en conclut $x_1 = 0$, ce qu'il fallait démontrer.

TOUL, 4 *Juillet*, 1846.

NOTE ON THE PRECEDING PAPER.

By WILLIAM THOMSON.

[*Extracted from a Letter to* M. *Liouville.*]

"... THE demonstration which you have given has led me to this other theorem, that the mass M, and the shell surrounding it, have the same principal axes, through any point.

To demonstrate this, let $U = yz$ in the formula which you have given. Then, since, if we denote by K the constant value of V at the shell, we have

$$\iint V\frac{dU}{ds}\, d\omega = K\iint \frac{dU}{ds} = 0,^*$$

we find

$$\iint yz\,\frac{-dV}{ds}\, d\omega = 4\pi \iiint yz.f(x, y, z)\, dx\, dy\, dz \;\ldots\ldots\; (1),$$

which proves the proposition enunciated.

If we take $U = x^2$, we find

$$V\frac{d^2U}{dx^2} - U\frac{d^2V}{dx^2} + V\frac{d^2U}{dy^2} - U\frac{d^2V}{dy^2} + V\frac{d^2U}{dz^2} - U\frac{d^2V}{dz^2}$$
$$= 2V + 4\pi x^2 f(x, y, z):$$

from which, observing that

$$\iint V\left(\frac{dU}{dx}\, dy\, dz + \frac{dU}{dy}\, dz\, dx + \frac{dU}{dz}\, dx\, dy\right)$$
$$= K\iiint \left(\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} + \frac{d^2U}{dz^2}\right) dx\, dy\, dz$$
$$= 2\, K \iiint dx\, dy\, dz;$$

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* See First Series, Vol. III. p. 203, (8).

we deduce

$$\frac{1}{4\pi}\iint x^2 \frac{-dV}{ds}\,d\omega = \frac{1}{2\pi}\iiint (V-K)\,dx\,dy\,dz$$
$$+\iiint x^2 f(x,y,z)\,dx\,dy\,dz.$$

Let A, B, C be the moments of inertia of the mass M round the axes of coordinates, and A_1, B_1, C_1 those of the shell, round the same axes, it being supposed that the quantity of matter of the shell is the same as that of M;* the preceding equation, and the two others which correspond relatively to the axes of y and z, are with this notation,

$$A_1 = Q + A,\ B_1 = Q + B,\ C_1 = Q + C \ldots\ldots (2),\dagger$$

where $$Q, = \frac{1}{2\pi}\iiint (V-K)\,dx\,dy\,dz,$$

is a quantity which is independent of the position of the origin.

From equations (2), we have

$$B - C = B_1 - C_1,\ C - A = C_1 - A_1,\ A - B = A_1 - B_1 \ldots (3).$$

A demonstration of your theorem and of the theorems expressed by the equations (1) and (3) may be arrived at by comparing the expressions for the equal potentials ‡ produced by the mass M, and the shell at very distant points."

St. Peter's College, July 15, 1846.

ACTION OF A FORCE WHOSE DIRECTION ROTATES IN A PLANE.

By Andrew Bell.

This paper treats of the motion of a physical point acted on by a constant force whose direction passes through the point, and has a uniform angular motion in one plane.

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* In this case the "density" of the distribution at any point of the shell will be equal to $\frac{1}{4\pi}\cdot\frac{-dV}{ds}$. See Vol. III., p. 75.

† If the origin be taken at the centre of gravity, and the axes of coordinates principal axes of M, (and therefore of the shell, according to the proposition enunciated above,) these equations shew that the "central ellipsoid" (see note to p. 202) for the shell is confocal with that for the body M.

‡ A shell constructed round the mass M, in the manner described by M. Liouville, with a quantity of matter equal to M, exerts the same force upon points without the shell, as was proved first by Green, (see also Vol. III., p. 75); and, since the potential of each vanishes at an infinite distance, it follows that the two bodies produce equal potentials at every point without the shell.

[If a rocket be made to revolve round a horizontal axis, and then, being ignited, be allowed to fall freely, its horizontal motion will be of the kind considered in this paper.]

Let Ox, Oy be rectangular axes in the same plane with the material point and with the direction of the force; x, y the co-ordinates of the point at any instant of time t, and ϕ the constant accelerating force making an angle θ with the axis x, the line of its direction revolving from right to left, or so as to increase θ, and let the force when $\theta < \frac{1}{2}\pi$, be directed from the origin, so as to increase the co-ordinates.

The resolved parts of ϕ parallel to the axes are respectively

$$P = \phi \cos \theta, \quad Y = \phi \sin \theta.$$

If T be the time of revolution of the force, and t that of describing θ, then

$$2\pi : \theta = T : t; \quad \text{hence} \quad \theta = \frac{2\pi t}{T}:$$

or, if $\dfrac{2\pi}{T} = r$, then $\theta = rt$.

Hence the equations of motion now become

$$\frac{d^2x}{dt^2} = X, \qquad \frac{d^2y}{dt^2} = Y,$$

or
$$\frac{d^2x}{dt^2} = \phi \cos rt, \quad \frac{d^2y}{dt^2} = \phi \sin rt.$$

The first integrals of these equations are

$$\frac{dx}{dt} = \frac{\phi}{r} \sin rt + a, \quad \frac{dy}{dt} = -\frac{\phi}{r} \cos rt + b.$$

Since these equations give the values of the velocities of the point in the directions of the axes, the constants will be determined by assigning values to these velocities at any instant. Let the components of its velocity in the directions of x and y be v' and v'' when $t = 0$; then will $a = v'$ and $b = v'' + \dfrac{\phi}{r}$, and consequently

$$\frac{dx}{dt} = \frac{\phi}{r} \sin rt + v', \quad \frac{dy}{dt} = \frac{\phi}{r}(1 - \cos rt) + v''.$$

Integrating again,

$$x = -\frac{\phi}{r^2} \cos rt + v't + a, \quad y = \frac{\phi}{r^2}(rt - \sin rt) + v''t + b.$$

If the point is at the origin when $t = 0$, then

$$x = \frac{\phi}{r^2}(1 - \cos rt) + v't, \quad y = \frac{\phi}{r^2}(rt - \sin rt) + v''t.$$

Let m, n be such numbers that $v' = \frac{m}{r}\phi$, $v'' = \frac{n}{r}\phi$, then

$$x = \frac{\phi}{r^2}(mrt + 1 - \cos rt), \quad y = \frac{\phi}{r^2}\{(1 + n)\,rt - \sin rt\}.$$

This system of equations is that of a species of oblique cycloid, or rather of a series of such cycloids, the line of whose bases passes through the origin, and is expressed by the eqnation

$$x = \frac{m\phi}{r}\,y = cy, \quad \text{if } c = \frac{m\phi}{r}.$$

Since $1 - \cos rt$ is never negative, the value of x is never less than that obtained from the equation $x = cy$, so that no part of the series of equal curves lies between the line of the bases and the axes of y. The points in which the curves meet the line of the bases, will be found by assuming $rt = 2e\pi$, where e is any integer; for then

$$x = \frac{m\phi}{r}\,t = ct,$$

which indicates a point of this line. The successive points of meeting are found by giving e the consecutive integral values $1, 2, 3, \ldots$. The corresponding values of y are

$$y = 2e\,(1 + n)\,\pi\,\frac{\phi}{r^2}.$$

The axis is the greatest value of the absciss of the curve reckoned from the base line, and is

$$x' = \frac{2\phi}{r^2},$$

and the length of the oblique base projected on the ordinate is

$$= 2\,(1 + n)\,\pi\,\frac{\phi}{r^2}.$$

The type of the curve is the common, the curtate, or prolate cycloid, according as n is zero, negative, or positive.

If when $t = 0$, $v' = 0$, or the point have merely an initial motion in the direction of the axis y, this motion is represented by the system

$$x = \frac{\phi}{r^2}(1 - \cos rt), \quad y = \frac{\phi}{r^2}\{(1 + n)\,rt - \sin rt\}$$

These equations are the general equation of the cycloid, and belong to a series of equal cycloidal curves whose bases lie in the axis of y, and whose axes are parallel to the axis of x.

The length of the axis is $\frac{2\phi}{r^2}$, and of the base $2\pi(1+n)\frac{\phi}{r^2}$.

It can easily be proved, by determining when $\frac{dy}{dx}$ is $= 0$, or $= \infty$, that according as n is zero, positive, or negative, the curves are common, prolate, or curtate cycloids.

Since the last equation becomes that of the common cycloid when $n = 0$, it appears that the general equation to the cycloid is deducible from that of the common cycloid by adding to the value of y the term $\frac{n}{r}\phi t$, which is proportional to the time or to θ; consequently this term indicates the impression, on a point moving in a common cycloid, of a uniform velocity, in the direction of the base, and $= \frac{n}{r}\phi t$, in addition to the component due to its oscillation. Hence a common cycloidal pendulum may be made to oscillate in a prolate or curtate cycloid by impressing on it and its cycloidal cheeks an initial uniform velocity in the direction of its base, according as the direction of this velocity is towards the side to which the pendulum is to move, or towards the opposite side; the cheeks being constrained to retain their uniform velocity.

The result of this investigation establishes the fact, that a constant and uniformly rotating force is capable of producing a progressive motion, and it also affords another remarkable physical property of the cycloid.

MATHEMATICAL NOTES.

I. LET $$A^3 - 3AB = D^3:$$

and, consequently, $B = -\frac{1}{3A}(D^3 - A^3)$;

then the equation at line 3, p. 249 of the second volume of the former series of this Journal becomes

$$y = -\frac{1}{3A} \cdot \frac{D^3 - A^3}{D - A}.$$

[redacted]

The further reductions are obvious. By writing aD for D (where a is one of the values of $1^{\frac{1}{3}}$) we obtain the three values of y.

J. C.

Devereux Court, March 17, 1846.

II. If
$$A = aa' - bb' - cc', \qquad D = bc' + cb',$$
$$B = bb' - cc' - aa', \qquad E = ca' + ac',$$
$$C = cc' - aa' - bb', \qquad F = ab' + ba';$$
then
$$ABC - AD^2 - BE^2 - CF^2 + 2DEF =$$
$$(a^2 + b^2 + c^2)(aa' + bb' + cc')(a'^2 + b'^2 + c'^2).$$

Under the same conditions
$$(A + B)(B + C)(C + A) - 2DEF =$$
$$(A + B)F^2 + (B + C)D^2 + (C + A)E^2.$$

H. (1).

III. *On the Equation of Payments.*—In the tenth No. of the *Cambridge Mathematical Journal* the Theory of the Equation of Payments is briefly considered. The method there followed, in the case of simple interest, leads to the result ordinarily employed, as an approximate solution; whereas it is in fact as much entitled to be considered exact as any solution obtained on the supposition of simple interest can be. The reason of this misapprehension has been overlooked in the above-mentioned paper, as well as in the common treatises on Algebra.

Suppose that two sums, s_1 and s_2, are due at two periods, t_1 and t_2 respectively; that it is required to find the period T at which the sum $s_1 + s_2$ may be paid without injury to either party, simple interest being allowed.

There are three methods of making the arrangement, which at first sight appear equally fair.

1. The time T ought to be such, that the present worth of s_1 due at t_1, together with the present worth of s_2 due at t_2, shall be equal to the present worth of $s_1 + s_2$ due at T.

2. It ought to be such, that the interest of s_1 from the time t_1 to the time T, shall be equal to the discount of s_2 for the interval between T and t_2.

3. Or lastly, it ought to be such that s_1, with its interest from t_1 to t_2, together with s_2, shall be equal to $s_1 + s_2$ with its interest during the interval between T and t_2.

If these methods are equally just, they ought to give the same value of T. Let us see whether they do so.

Take r for the rate of simple interest—supposed the same for each of the sums. Then, according to the usual formulæ, the first method will give rise to the equation

$$\frac{s_1}{1+rt_1}+\frac{s_2}{1+rt_2}=\frac{s_1+s_2}{1+rT};$$

whence $$T=\frac{s_1t_1+s_2t_2+rt_1t_2(s_1+s_2)}{s_1+s_2+r(s_1t_2+s_2t_1)}\text{ ....(1).}$$

The second arrangement is, in symbolical language,

$$s_1(T-t_1)r=\frac{s_2r(t_2-T)}{1+r(t_2-T)},$$

or $T^2s_1r-T\{s_1+s_2+(t_1+t_2)rs_1\}+s_1t_1+s_2t_2+t_1t_2rs_1=0$ ...(2).

By the third method,

$$s_1\{1+r(t_2-t_1)\}+s_2=(s_1+s_2)\{1+r(t_2-T)\};$$

from which we have $$T=\frac{s_1t_1+s_2t_2}{s_1+s_2}\text{ .... (3).}$$

It appears then that these three modes of determining T will not give the same result: and the reason of this we will explain.

But first it may be observed, that the results themselves would indicate that the third is the only correct one: for it is clear that the termination of the time T ought to be fixed with reference to the terminations of t_1 and t_2, independently of the epoch, which we choose to call *present*, and from which we calculate the *present worth*; in other words, t_2-T or $T-t_1$ ought to be independent of the absolute values of t_1 and t_2, and to depend only upon the interval t_2-t_1. This, it may be easily seen, is the case in (3), but not in (1) or (2).

The reason of the discrepancy will be found in our having made our calculations upon the supposition of *simple* interest, a supposition which implies that the interest is never added to the principal and made a part of it. But in our first method this condition has been violated; for instead of the actual case we have substituted the following imaginary one. If A receive $\frac{s_1}{1+rt_1}$ *now*, it is the same thing to him (*i.e.* he will be in the same position at any future time) as if he receive s_1 at the end of t_1. There is, however, this difference:—if he receive $\frac{s_1}{1+rt_1}$ now, he will have $\frac{s_1}{1+rt_1}+\frac{s_1rt_1}{1+rt_1}$

at the time t_1, but upon the former only of these quantities is he to receive interest from the time t_1, *because the interest is simple:* whereas, if he receive s_1 at the time t_1, he will have the same sum at that time as before, but will receive interest *on the whole* of it from the time t_1.

The imaginary case then which we have substituted for the actual one, and which we have expressed in equation (1), is not equivalent to it, and we cannot therefore rely upon the result.

For the same reason the result in the second case is not correct. It requires no further explanation than to say, that discount implies the calculation of the present worth, and so introduces an error similar to that in the first method. But in the third case, the creditor will be in the same position at the time t_2, and therefore at any subsequent time, as if the two separate payments were made.

If r be supposed so small that the terms of which it is a factor may be neglected, equations (1) and (2) will agree with (3).

Since we have shewn the disagreement of our results in the three cases to arise from the fact of simple interest being considered, it ought to disappear when the interest is compound. We should thus have, for the first case,

$$\frac{s_1}{(1+r)^{t_1}} + \frac{s_2}{(1+r)^{t_2}} = \frac{s_1+s_2}{(1+r)^T},$$

therefore $$s_1(1+r)^{t_2-t_1} + s_2 = (s_1+s_2)(1+r)^{t_2-T} \; \ldots\ldots \; (4);$$

for the second,

$$s_1(1+r)^{T-t_1} - s_1 = s_2 - \frac{s_2}{(1+r)^{t_2-T}},$$

therefore $$s_1(1+r)^{t_2-t_1} + s_2 = (s_1+s_2)(1+r)^{t_2-T} \; \ldots\ldots \; (5);$$

and for the third,

$$s_1(1+r)^{t_2-t_1} + s_2 = (s_1+s_2)(1+r)^{t_2-T} \; \ldots\ldots \; (6);$$

and the three resulting equations now coincide.

H. Y.

END OF VOL. I.

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GENERAL INDEX TO THE FIRST SERIES.

GEOMETRY OF TWO DIMENSIONS.

* In references to vol. i. the pages of the first edition are indicated.

GEOMETRY OF THREE DIMENSIONS.

ALGEBRA AND TRIGONOMETRY.

DIFFERENTIAL CALCULUS AND CALCULUS OF FINITE DIFFERENCES.

MECHANICS.

ASTRONOMY.

LIGHT AND SOUND.

HEAT, ELECTRICITY, AND MAGNETISM.

MISCELLANEOUS.

SCIENTIFIC JOURNALS. (*Feb. 23, 1846.*)

Journal fur die reine und angewandte Mathematik (*In zwanglosen Heften*). Herausgegeben von A. L. CRELLE, Berlin. Mit thätiger Beförderung hoher Königlich-Preussischer Behörden.

BAND XXX. 1845, Heft I.—1. Mémoire sur les Hyperdéterminants. Par *A. Cayley* à Cambridge, (Traduction d' un Mémoire Anglais inséré dans le "Cambridge Mathematical Journal," avec quelques additions de l' auteur).—2. Neue Eigenschaft der Gleichung, mit deren Hülfe man die seculären Störungen der Planeten bestimemt. Von Herrn Dr. *Carl Wilhelm Borchardt*.—3. Sulla condizione di uguaglianzia di due radici dell' equazione cubica, dalla quale dipendono gli assi principali di una superficie del second' ordine. Dal. Sig. Profess. *C. G. I. Jacobi*. (Estratto dal giornale arcadico tome XCVIX.)—4. Uber ein leichtes Verfahren die in der Théorie der Säcularstörungen vor kommenden Gleichungen numerisch aufzulzösen. Von Herrn Professor Dr. *C. G. I. Jacobi*.—5. Ein Lehrsatz von Kegelschnitten. Von Herrn Prof. *Umpfenbach* zu Giessen. Fac-simile einer von dem Herrn *Benoni Friedländer* zu Berlin dem Herausgeber dieses Journals gefälligst mitgetheilten Handschrift von Leibnitz. Heft II.—6. Teoremi relativi alle coniche inscritte e circoscritte. Del Sig. cav. *J. Steiner*, Professore nell' Università e membro dell' Academia di Berlino. (Estratto dal giornale arcadico di Roma tomo XCVIX.—7. Uber die Divisoren gewisser Formen der Zahlen, welche aus der Theorie der Kreistheilung entstehen. (Von Herrn Dr. *E. E. Kummer*, Professor an der Universität zu Breslau.)—8. Neues Theorem der analytischen Mechanik. Von Herrn *C. G. J. Jacobi*, Prof. und Akademiker zu Berlin. (Aus den Monatsberichten der Königl. Akademie der Wissenschaften zu Berlin vom Jahre 1838.)—9. Uber die Additionstheoreme der Abelschen Integrale zweiter und dritter Gattung. Von Herrn Professor Dr. *C. G. J. Jacobi*.—10. Uber die Darstellung einer Reihe gegebner Werthe durch eine gebrochne rationale Function. Von Herrn Professor Dr. *C. G. J. Jacobi* zu Berlin.—11. Neue Darstellung der Resultante der Elimination von z aus zwei algebraischen Gleichungen $f(z)=0$ und $\phi(z)=0$ vermittelst der Werthe welche die Functionen $f(z)$ und $\phi(z)$ für gegebne Werthe von z annehmen. Von Dr. *G. Rosenhain*, Privatdocenten in Breslau —12. Uber die Kreistheilung und ihre Anwendung auf die Zahlentheorie. Auszug eines Schreibens an die Berliner Akad. d. W. von Herrn Professor Dr. *C. G. J. Jacobi* zu Berlin. (Abgedruckt aus den Monatsberichten der Königl. Akad. der Wiss. zu Berlin vom Jahre 1837.)—13. Note sur les fonctions Abéliennes par M. *C. G. J. Jacobi*, lue le 29 Mai, 1843. (Tiré du Bulletin de la classe physico-mathématique de l'Acad. Imp. des sciences de St. Pétersbourg, tome II. no. 7.) Fac-simile einer von dem Herrn *Benoni Friedländer* zu Berlin dem Herausgeber gefälligst mitgetheilten Hanfischrift von *Monge*.

BAND XXXI. 1846. Heft I.—1. Entwicklung eines symmetrischen Ausdrucks für ded Grad einer durch Elimination hervorgehenden Gleichung. (Von Herrn *Ferd. Minding*, Professor an der Universität zu Dorpat.) (Gelesen in der Sitzung der Petersburger Akademie der Wissenschaften am $\frac{24\ \text{Nov.}}{6\ \text{Decbr.}}$ 1843, und aus dem Bulletin übersetzt.)—2. Nuove applicazioni del Calcolo Integrale relative alla quadratura delle Superficie curve, e cubatura de solidi. Memoria (Dal Sign. D. Barnaba *Tortolini*, Professore di Mathematiche trascendenti a l' Università di Roma).—3. Auflösungen und Beweise einer Reihe von Aufgaben und Lehrsätzen der ebenen Geometrie. (Von Herrn *A. Jacobi* zu Breslau, Premier-Lieutenant a. D.)—4. Einige geometrische Aufgaben. (Von Herrn Prof. *Lehmus* in Berlin.)—5. Geometrische Lehrsätze und Aufgaben. (Von Herrn Prof. *J. Steiner* in Berlin.) Fac-simile einer Handschrift von *Gal Galilāi*. Aus "Memorie e lettere ined. di G. G. Parte Prima 1587–1616. Modena, 1818. Heft II.—6. Auflösungen und Beweise einer Reihe von Aufgaben und Lehrsätzen der ebenen Geometrie. (Von Herrn *A. Jacobi* zu Breslau, Premier-Lieutenant a. D.—Schluss der Abhandlung No. 3 im vorigen Hefte.)—7. Grundzüge zu einer rein geometrischen Theorie der Curven, mit Anwendung einer rein geome-

trischen Analyse. (Von Herrn Dr. *Herrman Grassmann*. Lehrer der Mathematik zu Stettin.)—8. Summation der Reihe

$$\frac{1}{(b+a)^{1+\rho}}+\frac{1}{(b+2a)^{1+\rho}}+\frac{1}{(b+3a)^{1+\rho}}+\ldots \text{ für } \rho=0.$$

(Von Herrn Dr. *Heine* zu Bonn.)—9. Recherches sur les surfaces isothermes et sur l'attraction des ellipsoides (par M. *Ch. Despeyrous* à Paris, Docteur es-sciences).—10. Note sur la division abregée en arithmétique. (*Par l'éditeur.*)—11. Elementare Herleitung des Newtonschen Gesetzes aus den Keplerschen Gesetzen der Planetenbewegung. (Von Herrn *A. F. Möbius*, Professor in Leipzig.)—12. Beweis eines geometrischen Satzes. (Von dem Premier-Lieutenant a. D. Herrn *A. Jacobi* zu Breslau.) Fac-simile einer auf der Königlichen Bibliothek zu Berlin befindlichen Handschrift von *Condorcet.*

[The third and fourth *Hefte* of the thirtieth volume will shortly be published.]

The contents of the tenth volume (1845) of Liouville's *Journal de Mathématiques*, will be published in our next Number.

Archiv der Mathematik und Physik mit besonderer Rücksicht auf die Bedürfnisse der Lehrer an höhern Unterrichtsanstalten. Herausgegeben von Johann August Grunert, Professor zu Greifswald. Griefswald, 1845.

Theil VI. Heft. 3. XXXV. Geometrischer Beweis des Satzes, dass jeder algebraischen Gleichung mit Einer Unbekannten durch einen complexen Werth dieser Unbekannten Genüge geleistet werden kann. Von. Herrn Dr. *T. Wittstein* zu Hannover Nachschrift des Herausgebers.—XXXVI. Ueber das Princip des kleinsten Zwangs und die damit zusammenhängenden mechanischen Principe. Von Herrn Professor Dr. *Reuschle* am Gymnasium zu Stuttgart.—XXXVII. Untersuchungen über den sogenannten berganlaufenden Doppelkegel. Von dem Herrn Prof. Dr. *F. Stegmann* an der Universität zu Marburg.—XXXVIII. Ueber die Projection einer geraden Linie auf einer Ebene, auf einer Fläche überhaupt, und auf der Oberfläche eines elliptischen Sphäroids insbesondere. Von dem *Herausgeber.*—XXXIX. Ist $\int\frac{dx}{x}=lx+\text{const.}$, oder $=\frac{1}{2}\,l(x^2)+\text{const}$? Von dem Herrn Dr. *O. Schlömilch*, Privatdocenten an der Universität zu Jena.—XL. Uebungsaufgaben für Schüler.—XLI. Miscellen.—XXIII. Literarischer Bericht.

The Mathematician. Edited by William Rutherford, F.R.A.S., and Stephen Fenwick. Vol. II. No. 1, price 3*s*. 6*d*. London, Nov. 1845.

1. Application of Algebra to Modern Geometry.—2. On the Summation of Infinite Series.—3. A Property relative to Surfaces which pass through the same eight points in space.—4. New Demonstrations of the Theorems of Pascal and Brianchon.—5. Mathematical Note.—6. Computation of Logarithms and Anti-logarithms.—7. On Confocal Conic Sections.—8. On the General Reduction of certain Integrals.—9. Direct Solution of a Spherical Problem.—10. Mathematical Note.—11. Horner on Algebraic Transformation.—12. Demonstration of a Geometrical Theorem.—13. Mathematical Notes.—Mathematical Exercises (*continued*).—14. Solutions of Mathematical Exercises.

SCIENTIFIC JOURNALS. (*May* 11, 1846.)

Journal für die reine und angewandte Mathematik (*In zwanglosen Heften*). Herausgegeben von A. L. CRELLE, Berlin. Mit thätiger Beförderung hoher Königlich-Preussischer Behörden.

BAND XXX. 1845, Heft III.—14. Beiträge zur Theorie der elliptischen Functionen. Von Herrn Dr. phil. *G. Eisenstein* zu Berlin. I. Ableitung des biquadratischen Fundamentaltheorems aus der Theorie der Lemniscatenfunctionen, nebst Bemerkungen zu den Multiplicationsund Transformationsformeln. II. Neuer Beweis der Summationsformeln.—15. Démonstration d'un théorème de Mr. *Slonimsky* sur les nombres, avec une application de ce théorème au calcul de chiffres. Par l'éditeur.—16. Uber die gleichseitigen Dreiecke, welche um ein gegebenes Dreieck gelegt werden können. Von dem Herrn Conrector *Fasbender* zu Iserlohn.—17. Untersuchungen über die Wahrscheinlichkeitsrechnung. Von Herrn Dr. *Öttinger*, Prof. ord. an der Universität zu Freiburg im Br. (Fortsetzung des Aufsatzes No. 16. im dritten und No. 21. im vierten Heft 26ten Bandes.)—18. Uber einige, die elliptischen Functionen betreffenden Formeln. Von Herrn Professor Dr. *C. G. J. Jacobi* zu Berlin.—19. Uber eine Eigenschaft der Krümmungshalbmesser der Kegelschnitte. Von Herrn Prof. *J. Steiner* zu Berlin.—20. Lehrsätze und Aufgaben. Von Herrn Prof. *J. Steiner* zu Berlin. Fac-simile einer auf der Königl. Bibliothek zu Berlin befindlichen Handschrift von *Kästner*. Heft IV. 21. De integrali $-\int_0^x \frac{\log(1-a)}{a}\,da$. Auctore Dr. *Schaeffer* Berol. 22. Untersuchungen über die Wahrscheinlichkeitsrechnung. Von Herrn Dr. *Ottinger*, Prof. ord. an der Universität zu Freiburg im Br. (Fortsetzung des Aufsatzes No. 16. im dritten, No. 21. im vierten Heft 26ten und No. 17. im 3ten Heft 30ten Bandes.) 23. Uber eine Eigenschaft der Leitstrahlen der Kegelschnitte. Von Herrn Prof. *J. Steiner* zu Berlin.—24. Intorno agl' intimi movimenti osservati nei muri dell' Osservatorio di Modena. Memoria del Sgr. Dottor *Antonio Bernardi*.—25. Demonstrationes theorematum ad superficies curvas spectantium. Auctore *F. Joachimsthal*, doct. phil.—Druckfehler-Verzeichnifs.—26. Inhalts-Verzeichnifs I. der *dritten zehn* Bände dieses Journals, 21 bis 30, herausgegeben in den Jahren 1841 bis 1845; nach alphabetischer Ordnung der Namen der Verfasser.—27. Inhalts-Verzeichnifs II. der *erstein dreifsig* Bände dieses Journals; nach den Gegenständen.—Fac-simile einer Handschrift von *Fermat*. Aus dem " Journal des savants, an 1839."

BAND. XXX. Heft III.—13. Ueber die Anzahl und die Form der Bedingungsgleichungen, unter welchen eine gewöhnliche Differentialgleichung zwischen zwei Variabeln n ter Ordnung und von der Form

$$V = y_n \phi\,(x, y, y_1, y_2, \ldots\ldots y_{n-1}) + \psi\,(x, y, y_1, y_2, \ldots\ldots y_{n-1}) = 0$$

das unmittelbare Differentiations-Ergebniss einer nach der allgemeinen Constante aufgelöseten analogen Differentialgleichung $(n-1)$ ter Ordnung ist. (Von Herrn Prof *Raabe* zu Zürich.)—14. Sur quelques théorème de la géométrie de position. (Par Mr. *A. Cayley* de Cambridge.)—15. Problème de géométrie analytique. (Par Mr. *Cayley* de Cambridge.)—16. Zwei Beweise für die Existenz der Wurzeln der höhern algebraischen Gleichungen. (Von Herrn *J. C. Ullherr*, Professor an der polytechnischen Schule zu Nürnberg.)—17. Ueber die Summirung der beiden Reihen

$$\text{(a)}\quad \gamma_0 - n_1\gamma_1 + n_2\gamma_2 - \text{etc.} + (-1)^n\gamma_n,$$
$$\text{(b)}\quad \gamma_0 + n_1\gamma_1 + n_2\gamma_2 + \text{etc.} + \gamma_n,$$

in welchen die Grössen γ willkührlich und die Coëfficienten Binomialcoëfficienten des ganzen Exponenten n sind, mittels höherer Differenzen und Summen. (Von dem Herrn Gymnasiallehrer *F. Arndt* zu Stralsund.)—18. Nova solutio problematis determinandi multitudinem numerorum, qui ad numerum aliquem sint primi eoque minores. (Auctore *Friderico Arndt*, Sundiae.)—19. Entwickelung der Summe der n ten Potenzen des natürlichen Zahlen nach den Potenzen des Index mittelst des *Taylor*'schen Lehrsatzes. (Von dem Herrn Gymnasiallehrer *Arndt* zu Stralsund.)—20. Ueber die *Bernoulli*'sche Methode summirbare Reihen zu finden. (Von dem Herrn Gymnasiallehrer *Arndt* zu

Stralsund.)—21. Nova methodus determinandi multitudinem radicum congruentiae $x^e \equiv 1$ (mod. M) aliaque ad hanc materiam spectantia. Auctore *Friderico Arndt*, Sundiae.—Fac-simile einer von dem Herrn *Benoni Friedlaender* dem Herausgeber gefälligst mitgetheilten Handschrift von *W. Herschel*. Heft. IV. 22. Grundzüge einer allgemeinen Theorie der höhern Congruenzen, deren Modul eine reelle Primzahl ist. Von Herrn Oberlehrer *Schönemann* am Gymnasio zu Brandenburg a. d. H.—23. Demonstratio duorum theorematum Gaussianis his generaliorum: I. *Productum ex omnibus radicibus primitivis moduli imparis* p *unitate sec.* p *congruum est, excepto casu, in quo* $p=3$. II. *Summa omnium radicum primitivarum moduli primi imparis* p *est* $\equiv 0$, *quando* $p-1$ *per quadratum aliquod divisibilis est; quando vero per nullum quadratum divisibilis, summa est* $\equiv \pm 1$, *prout multitudo factorum ipsius* $p-1$ *primorum est* $\left.\begin{matrix}\text{par}\\ \textit{aut}\ \text{impar}\end{matrix}\right\}$. Auctore *Friderico Arndt*, Sundiae.—24. Demonstratio nova theorematis Wilsoniani a summo *Gauss* hoc modo generalius enunciati: "*Productum omnium numerorum ad numerum quemcunque M primorum eoque inferiorum unitati negativae aut positivas sec. M congruum est: et quidem negative sumenda est unitas, quando M potestas numeri primi imparis vel ejus duplum, vel denique 4, positive autem in omnibus casibus reliquis.*" Auctore *Friderico Arndt*, Sundiae.—25. Disquisitiones de residuis cujusvis ordinis. Auctore *Friderico Arndt*, Sundiae.—26. Bemerkungen über die Verwandlung der irrationalen Quadratwurzel in einen Kettenbruch. Von Herrn Dr. *Arndt* zu Stralsund.—Fac-simile einer auf der Königlichen Bibliothek zu Berlin befindlichen Handschrift von *Lalande*.

Journal de Mathématiques pures et appliquées, ou Recueil Mensuel de Mémoires sur les diverses parties des Mathématiques. Publié par Joseph Liouville, Membre de l'Académie des Sciences et du Bureau des Longitudes.

TOME X. 1845.—Réflexions sur les principes fondamentaux de la théorie des nombres; par M. *Poinsot*.—Nouvelles remarques sur les courbes du troisième ordre; par M. *A. Cayley*, de Cambridge.—Mémoire sur les diverses propriétés des surfaces du deuxième ordre déduites de la théorie des focales; par M. *B. Amiot*.—Démonstration d'un théorème d'analyse; par M. *William Thomson*.—Méthode géométrique pour les rayons de courbure d'une certaine classe de courbes; par M. *Abel Transon*.—Note à l'occasion du Mémoire précédent; par M. *Chasles*.—Sur quelques intégrales multiples; par M. *A. Cayley*.—Sur les deux formes

$$x^2+y^2+z^2+t^2,\ x^2+2y^2+3z^2+6t^2;$$

par M. *J. Liouville*.—Equations numériques. Recherche des facteurs commensurables du second degré; par M. *Finck*.—Application de la théorie des transcendantes elliptiques à la rectification d'une classe étendue de courbes planes; par M. *William Roberts*.—Note sur la transformation et l'intégration d'une classe d'équations différentielles simultanées à plusieurs variables; par M. *E. Brassinne*.—Construction des rayons de courbure des courbes décrites dans le mouvement d'une figure plane qui glisse sur son plan; par M. *Chasles*. Note sur les lois élémentaires de l'électricité statique; par M. *William Thomson*.—Sur diverses questions d'analyse et de physique mathématique; par M. *J. Liouville*.—Sur les fonctions de Laplace, qui résultent du développement de l'expression $\{a^2-2aa'\,[\cos\omega\cos\phi+\sin\omega\sin\phi\cos(\theta-\theta')]+a'^2\}^{-\frac{1}{2}}$;

par M. *Jacobi*.—Sur les exponentielles successives d'Euler et les logarithmes des différentes ordres des nombres; par. M. *J.-H. Grillet*.—Addition à la Note sur quelques intégrales multiples, insérée dans la cahier d'avril; par M. *A. Cayley*. Mémoire sur les courbes à double courbure et les surfaces développables; par M. *A. Cayley*.—Note sur deux systèmes généraux de trajectoires orthogonales; par M. *Michael Roberts*.—Mémoire sur la représentation géométrique des fonctions elliptiques et ultra-elliptiques; par M. *J.-A. Serret*.—Addition au Mémoire précédent; par. M. *J.-A. Serret*.—Rapport sur ce Mémoire; par M. *J. Liouville*.—Note de M. *J. Liouville*.—Mémoire sur quelques propriétés géométriques relatives aux fonctions elliptiques; par M. *William Roberts*.—Note sur l'intégration de l'équation différentielle

$$(A+A'x+A''y)(xdy-ydx)-(B+B'x+B''y)\,dy+(C+C'x+C''y)\,dx=0;$$

par M. *Lebesgue.*—Note sur les principes de la Mécanique; par M. *Abel Transon.*—Sur une propriété générale d'une classe de fonctions; par M. *J. Liouville.*—Note relative à l'instabilité de l'équilibre d'un système de points matériels; par M. *Jules Vieille.*—Sur le principe du dernier multiplicateur et sur son usage comme nouveau principe général de mécanique; par M. *Jacobi.*—Mémoire de Géométrie (deuxième partie); par M. *Auguste Miquel.*—Développements sur une classe d'équations relatives à la représentation géométrique des fonctions elliptiques; par M. *J.-A. Serret.*—Extrait d'une Lettre de M. *William Thomson* à M. Liouville.—Théorie des points singuliers dans les courbes planes algébriques; par M. *C. Briot.*—Note sur la formule de Taylor; par M. *Caqué*—Démonstration d'un théorème de M. Chasles; par M. *A. Cayley.*—Mémoire sur les fonctions doublement périodiques; par M. *A. Cayley.*—Note sur les courbes elliptiques de la première espèce; par M. *J.-A. Serret.*—Théorie géométrique des centres multiples; par M. *Philippe Breton.*—Sur l'application des transcendantes elliptiques à ce problème connu de la géométrie élémentaire: "Trouver la relation entre la distance des centres et les rayons de deux cercles dont l'un est circonscrit à un polygone irrégulier et dont l'autre est inscrit à ce même polygone;" par M. *C.-G.-J. Jacobi.*—Remarques sur les transcendantes elliptiques et abéliennes; par M. *Eisenstein.*—Extrait d'une Lettre de M. *William Roberts* à M. Liouville.—Note sur une intégrale définie; par M. *William Roberts.*—Sur un Mémoire de M. Serret, relatif à la représentation des fonctions elliptiques; par *J. Liouville.*—Théorèmes de Géométrie; par M. *Michael Roberts.*

Tome XI. 1846, No. i. Sur quelques propriétés des lignes géodésiques et des lignes de courbure de l'ellipsoïde; par M. *Michael Roberts.*—Sur les lignes géodésiques et les lignes de courbure des surfaces du second degré; par M. *Chasles.*—Démonstration géométrique relative à l'équation des lignes géodésiques sur les surfaces du second degré; par M. *J. Liouville.*—Application des transcendantes elliptiques aux polygones sphériques, qui sont inscrits à un petit cercle de la sphère, et circonscrits à un autre petit cercle, simultanément; par M. *Richelot.* No. ii. Sur le nombre des divisions à effectuer pour obtenir le plus grand commun diviseur entre deux nombres entiers; par M. *Athanase Dupré.*—Mémoire de géométrie (troisième partie); par M. *Auguste Miquel.*—Démonstration d'une formule de M. Dirichlet.—Remarques sur quelques expressions du nombre π; par M. *Lebesgue.*

The Mathematician. Edited by William Rutherford, F.R.A.S., and Stephen Fenwick. Vol. II. No. 2. March, 1845.

1. Propositions on the Conic Sections.—2. On Involution.—3. On the Theorems in Space analogous to a certain Property of Curves of the Second Order.—4. Simplifications in Elementary Trigonometry.—5. Relations between the Radii of certain Circles.—6. Note on a new Geometrical Theorem.—7. Connection between the Formulæ in Plane and Spherical Trigonometry.—8. On the Reduction of certain Definite Integrals.—9. On Surfaces of the Second Order.—10. On the Tangents to Algebraic Curves.—11. Application of Algebra to the Modern Geometry.—12. On the General Reduction of certain Integrals.—13. The Modern Geometry.—14. Locus of the Equation

$$(ay + bx + c)(a'y + b'x + c') = (dy + ex + f)(d'y + e'x + f').$$

15. Note on the Tangent to the Parabola.—16. Solutions of Mathematical Exercises.—17. Mathematical Exercises (*continued*).

(GLASGOW, *Oct.* 17, 1846.)

SCIENTIFIC JOURNALS.

Journal de Mathématiques pures et appliquées, ou Recueil Mensuel de Mémoires sur les diverses parties des Mathématiques. Publié par Joseph Liouville, Membre de l'Académie des Sciences et du Bureau des Longitudes.

TOME XI. (1846) No. III. Note sur l'évaluation de l'aire de la surface nommée, dans l'optique, surface d'élasticité; par M. *William Roberts.*—Sur un théorème de M. Joachimsthal, relatif aux lignes de courbure planes; par *J. Liouville.*—Théorie géométrique de la lemniscate et des courbes elliptiques de la première classe; par M. *J. A. Serret.*—Sur l'équation

$$\frac{d^2y}{dt^2} = \frac{y}{(e^t + e^{-t})^2};$$

par M. *Besge.*—Extrait d'une Lettre adressée à M. Hermite; par M. *C. G. J. Jacobi.*—Construction des caustiques par réflexion sur les courbes planes, le point lumineux étant dans le plan de la courbe; par M. *J. H. Grillet.*—Nouvelles démonstrations des deux équations relatives aux tangentes communes à deux surfaces du second degré homofocales; Et propriétés des lignes géodésiques et des lignes de courbure de ces surfaces; par M. *Chasles.*—Notes sur quelques questions de priorité, au sujet d'un Mémoire de M. Mac Cullagh; par M. *Chasles.*—No. IV. Notes sur quelques questions de priorité, au sujet d'un Mémoire de M. Mac Cullagh; par M. *Chasles.* (Fin.)—Extrait d'une Lettre adressée à M. Liouville; par M. *William Roberts.*—Remarques sur les systèmes de droites dans l'espace; par M. *J. Bouquet.*—Note sur le théorème de M. Cauchy relatif au développement des fonctions en series; par M. *Ernest Lamarle.*—Expression numérique des intégrales définies qui se présentent, quand on cherche les termes généraux du développement des coordonnées d'une planète, dans son mouvement elliptique; par M. *F. Lefort.*—No. V. Note sur les centres des lignes et des surfaces algébriques; par M. *Breton* (*de Champ*).—Sur l'évaluation de quelques intégrales définies par des fonctions elliptiques; par M. *William Roberts.*—Note sur l'attraction; par M. *C. Briot.*—Sur l'interpolation; par M. *E. Brassinne.*—Sur les trajectoires qui coupent, sous un angle constant, les courbes méridiennes des surfaces de révolution; par M. *l'Abbé Aoust.*—No. VI. Note sur les équations d'équilibre d'un système de forces dirigées d'une manière quelconque dans l'espace; par M. *R. Lobatto.*—Sur les intégrales définies

$$\int_0^\infty \frac{e^{-\beta x} \,.\, x^{m-1}\,dx}{1+x^2}, \quad \int_0^\infty \frac{\cos \beta x \,.\, x^{m-1}\,dx}{1+x^2}, \quad \int_0^\infty \frac{\sin \beta x \,.\, x^{m-1}\,dx}{1+x^2};$$

par M. *A. F. Svanberg.*—Note sur quelques intégrales multiples; par M. *William Roberts.*—Démonstration d'un théorème de Poisson; par M. *William Roberts.*—Note sur un problème de mécanique; par M. *E. Catalan.*—Note sur la propriété de la cycloïde, d'être la seule tautochrone dans le vide; par M. *E. L. Guillon.*—Lettres sur diverses questions d'analyse et de physique mathématique concernant l'ellipsoïde, adresseés à *M.* P. H. Blanchet; par *J. Liouville.* (Première Lettre.)—Extrait d'une Lettre adressée à *M.* J. Steiner, par M. *C. G. J. Jacobi.*

Journal für die reine und angewandte Mathematik (*In zwanglosen Heften*). Herausgegeben von A. L. Crelle, Berlin. Mit thätiger Beförderung hoher Königlich-Preussischer Behörden.

Band XXXII. (1846) Heft I.—1. Théorémes généraux sur les dérivées d'un ordre quelconque, de certaines fonctions très générales. Par M. *O. Schlömilch*, docteur es-sciences er lecteur des mathématiques à l'université de Jena.—2. Uber den Werth, welchen das bestimmte Integral $\int_0^{2\pi} \frac{d\phi}{1 - A\cos\phi - B\sin\phi}$ für beliebige imaginäre Werthe von A und B annimmt. Von Herrn Prof. Dr. *C. G. J. Jacobi* zu Berlin.—3. Mémoire sur les différentes manières de se servir de l'élasticité de l'air atmosphérique comme force motrice sur les chemins de fer. Une de ces manières constitue les chemins de fer *atmosphériques* proprement dits. Par l'éditeur.—4. Beiträge zur theorie der elliptischen functionen. Von Herrn Dr. phil. *G. Eisenstein* zu Berlin. (Fortsetzung der Abhandlung Nr. 14, Band 30, Heft 3.) III. Fernere Bemerkungen zu den transformations-formeln.—5. Notiz über partialbrüche. Von Herrn *G. Eisenstein*, Dr. phil. zu Berlin.—6. Uber Lehrsätze, von welchen die bekannten sätze über parallele curven besondere fälle sind. Von Herrn Prof. *J. Steiner* zu Berlin. (Auszug aus einer am 26 März in der hiesigen Akad. der Wiss. gehaltenen Vorlesung.)—7. Sur un moyen général de vérifier l'expression du potentiel relatif à une masse quelconque, homogène ou hétérogène. Par Mr. *G. Lejeune Dirichlet* à Berlin.—8. Über die stabilität des gleichgewichts. Von Herrn Prof. *G. Lejeune Dirichlet* zu Berlin. (Auszug aus einer am 22 Januar 1846 in der Königl. Akademie der Wissenschaften gelesenen Abhandlung.)—9. Eine Bemerkung zur Zahlentheorie. Von Herrn Prof. Dr. *Stern* zu Göttingen.—10. Zur näherungsweisen Kreis-Quadratur.—Fac-simile einer auf der Konigl. Bibliothek zu Berlin befindlichen Handschrift von *Maupertuis*. Heft II. 11. Von denjenigen Moduln, welche Potenzen von Primzahlen sind. Von Herrn Oberlehrer Dr. *Schönemann* am Gymnasio zu Brandenburg an der Havel. (Fortsetzung und Schlufs der Abhandlung No. 22, im vorigen Bande.)—12. Variationum quas elementa motus perturbati planetarum subeunt nova et facilis evolutio. Auct. *Aug. Ferd. Möbius*, Prof. ord. Lipsiæ.—13. Sur quelques propriétés des déterminants gauches. Par Mr. *A. Cayley* de Cambridge.—14. Mémoire sur les différentes manières de se servir de l'élasticité de l'air atmosphérique comme force motrice sur les chemins de fer. Une de ces manières constitue les chemins de fer *atmosphériques* proprement dits. Par l'éditeur. (Suite du No. 3 du cahier précédant.)—15. Beweis des Satzes, dafs jede nicht fünfeckige Zahl eben so oft in eine gerade als ungerade Anzahl verschiedener Zahlen zerlegt werden kann. Von Herrn Prof. Dr. *C. G. J. Jacobi* zu Berlin.—16. Extrait d'une lettre adressée à M. *Hermite*. Par M. *C. G. J. Jacobi*.—17. Geometrische Lehrsätze. Von Herrn Prof. *J. Steiner* zu Berlin. (Auszug aus einer am 27 Nov. 1845, in der Akad. der Wissenschaften zu Berlin gehaltenen Vorlesung.)—Fac-simile einer von dem Herrn Geheimen Oberbergrath *Karsten* zu Berlin dem Herausgeber gefälligst mitgetheilten Handschrift von *W. J. G. Karsten*. Heft III. 18. Uber die Vertauschung von Parameter und Argument bei der dritten Gattung der Abelschen und höhern Transcendenten. Von Herrn Prof. Dr. *C. G. J. Jacobi* zu Berlin.—19. Uber einige der Binominalreihe analoge Reihen. Von Herrn Prof. Dr. *C. G. J. Jacobi* zu Berlin.—20. Verwandlung von Reihen in Kettenbrüche. Auszug eines Schreibens des Herrn Dr. *E. Heine*, Privatdocenten zu Bonn, an den Herrn Prof. *C. G. J. Jacobi* in Berlin.—21. Uber die Reihe

$$1 + \frac{(q^\alpha - 1)(q^\beta - 1)}{(q-1)(q^\gamma - 1)} x + \frac{(q^\alpha - 1)(q^{\alpha+1} - 1)(q^\beta - 1)(q^{\beta+1} - 1)}{(q-1)(q^2-1)(q^\gamma - 1)(q^{\gamma+1} - 1)} x^2 + \ldots$$

Aus einem Schreiben des Herrn Dr. *Heine* zu Bonn an den Herrn Prof. *Lejeune Dirichlet*.—22. Uber die Reduction des Integrals $\int \frac{fx\,dx}{\sqrt{\pm(1-x^6)}}$ auf elliptische

[redacted]

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FELLOW OF ST. PETER'S COLLEGE, CAMBRIDGE,

AND PROFESSOR OF NATURAL PHILOSOPHY IN THE UNIVERSITY OF GLASGOW.

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INDEX TO VOL. II.

GEOMETRY OF THREE DIMENSIONS.

ALGEBRA.

DIFFERENTIAL AND INTEGRAL CALCULUS.

MECHANICS.

MECHANICS—*continued*,

OPTICS.

HEAT, ELECTRICITY, AND MAGNETISM.

MISCELLANEOUS.

Notes and other passages enclosed in brackets [ ], are insertions made by the Editor.

The date of actual publication of any article, or portion of an article, may be found by referring to the first page of the sheet in which it is contained. The Author's private date is frequently given, at the end of an article.—ED.

ERRATA.

IN the memoir "On a Multiple Integral connected with the theory of Attractions," in the denominator of the value of U given by the formula (14), p. 221, *for* $\Gamma(\frac{1}{2}n-q)$ *read* $\Gamma(\frac{1}{2}n+q)$, and in the next line *for* $\Gamma(\frac{1}{2}n+q)$ *read* $\Gamma(\frac{1}{2}n-q)$.

THE

CAMBRIDGE AND DUBLIN

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ON THE ATTRACTION OF A SOLID OF REVOLUTION ON AN EXTERNAL POINT.

By George Boole.

I propose in this paper to determine the most general integral of which the differential equation

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = 0 \quad \ldots\ldots\ldots\ldots (1)$$

is susceptible, when u is the potential of a solid of revolution on an external point, in such manner that the component attractions on that point are represented by $-\frac{du}{dx}, -\frac{du}{dy}, -\frac{du'}{dz}$, respectively, and to consider the physical application of the result. This idea has been already applied in the case of the sphere, the integral being then a function of the distance of the attracted point from the centre; and Professor Challis has, I think, made a similar use of the equation, to determine, under certain circumstances, the motion of an incompressible fluid. But it has not, apparently, occurred to any one to apply a more general form of the integral except in Laplace's series. Allusion is sometimes made to a complete solution of the equation obtained by Poisson, but I have not been so fortunate as to meet with it. Certainly it does not appear to be involved in the solution of the well-known equation of elastic fluids. A form of the general integral which I have obtained, is too complex for physical applications, and is for the present reserved. But the case referred to, as the subject of this paper, admits of separate and comparatively easy discussion.

Let z be the axis of revolution of the solid, and r or $\sqrt{(x^2 + y^2)}$ the distance of the attracted point from that axis;

then u will be a function of z and r. The transformed equation is easily found to be

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \frac{d^2u}{dz^2} = 0 \ldots\ldots\ldots\ldots (2),$$

which we proceed to discuss.

Writing the equation in the form

$$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} + r^2\frac{d^2u}{dz^2} = 0 \ldots\ldots\ldots (3),$$

let $r = \epsilon^\theta$, and let the symbol $\frac{d}{d\theta}$ be represented by D, then

$$r\frac{d}{dr} = D. \quad r^2\frac{d^2}{dr^2} = D(D-1);$$

$$\therefore \quad r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} = D^2u,$$

and the *symbolical* form of the differential equation is

$$D^2u + \frac{d^2}{dz^2}\epsilon^{2\theta}u = 0 \ \ldots\ldots\ldots\ldots (4).$$

The method which we shall employ in the solution of this equation, is that developed in the *Philosophical Transactions* for 1844, Part II. (On a General Method in Analysis), and partially explained in the first Number of this *Journal* (On Laplace's Equation). We shall first obtain the complete integral by series. We shall then deduce a particular solution, in the form of a definite integral, and shall examine the relation which it bears to the different parts of the general solution.

The equation $D^2u = 0$ would give

$$u = A + B\theta \ \ldots\ldots\ldots\ldots\ldots (5).$$

Substitute this value in (4); regarding A and B as variable parameters, we have

$$D^2A + \frac{d^2}{dz^2}\epsilon^{2\theta}A + 2DB + \left(D^2B + \frac{d^2}{dz^2}\epsilon^{2\theta}B\right)\theta = 0,$$

which affords the system of equations

$$D^2A + \frac{d^2}{dz^2}\epsilon^{2\theta}A + 2DB = 0,$$

$$D^2B + \frac{d^2}{dz^2}\epsilon^{2\theta}B = 0.$$

Of which the complete solution is

$$\left.\begin{aligned} A &= a_0 + a_2\varepsilon^{2\theta} + a_4\varepsilon^{4\theta} + \dots \\ B &= b_0 + b_2\varepsilon^{2\theta} + b_4\varepsilon^{4\theta} + \dots \end{aligned}\right\} \dots\dots\dots\dots (6),$$

in which a_0 and b_0 are arbitrary functions of z, and the remaining coefficients are connected by the relations

$$m^2 a_m + \frac{d^2}{dz^2} a_{m-2} + 2mb_m = 0 \dots\dots\dots\dots (7),$$

$$m^2 b_m + \frac{d^2}{dz^2} b_{m-2} = 0 \dots\dots\dots\dots (8).$$

Hence, writing r for ε^θ, and $\log r$ for θ, we have

$$u = A + B \log r \dots\dots\dots\dots (9),$$

wherein

$$A = a_0 + a_2 r^2 + a_4 r^4 + \dots$$
$$B = b_0 + b_2 r^2 + b_4 r^4 + \dots$$

the relations (7) and (8) giving, as the law of derivation of the coefficients,

$$b_m = -\frac{1}{m^2} \frac{d^2}{dz^2} b_{m-2} \dots\dots\dots\dots (10),$$

$$a_m = -\frac{1}{m^2} \frac{d^2}{dz^2} a_{m-2} - \frac{2}{m} \frac{d^2}{dz^2} b_{m-2} \dots\dots\dots\dots (11).$$

This is the complete integral of the equation in series. We may remark that when $b_0 = 0$, all the succeeding values of b_m vanish, and the relation (11) gives

$$a_m = -\frac{1}{m^2} \frac{d^2}{dz^2} a_{m-2},$$

which is of the same form as (10). *Hence, if B vanishes, A assumes the general form of B.*

We will now deduce a particular solution in the form of a definite integral, and for this purpose, resuming (4), we have, on operating with the factor D^{-2},

$$u + \frac{1}{D^2} \frac{d^2}{dz^2} \varepsilon^{2\theta} u = 0 \dots\dots\dots\dots (12),$$

we retain no constants in the second member, because the equation which, after reduction, we shall actually integrate, will be of the second order, and will give us the proper number of arbitrary functions (*Phil. Trans.* p. 249).

Assume as the transformed equation

$$v + \frac{1}{D(D-1)} \frac{d^2}{dz^2} \varepsilon^{2\theta} v = 0 \dots\dots\dots\dots (13).$$

A being a series already determined, but reducible to a definite integral, viz.

$$A = \int_0^\pi d\theta\, \psi(z + r\cos\theta\sqrt{-1})\ldots\ldots.(20),$$

whenever the definite integral in (19) vanishes.

Now if we suppose the second member of (19) to represent the potential of a solid of revolution on an external point, it is necessary that the definite integral in the second term should be assumed to vanish; for otherwise the value of u would be infinite, were the attracted point in the axis of revolution, since $r = 0$ renders $\log(r)$ infinite. We have therefore

$$u = A = \int_0^\pi d\theta\, \psi(z + r\cos\theta\sqrt{-1}) \text{ by (20).}$$

Let $f(z)$ represent the potential on any exterior point z, in the axis of revolution, then

$$f(z) = \int_0^\pi d\theta\, \psi(z) = \pi\psi(z).$$

$$\therefore\quad \psi(z) = \frac{1}{\pi} f(z),$$

whence
$$u = \frac{1}{\pi}\int_0^\pi d\theta\, f(z + r\cos\theta\sqrt{-1})\ldots\ldots(21),$$

which is the expression required.

Some interesting consequences flow from this theorem. If the potential of a solid of revolution on every external point in the axis be constant, we have

$$f(z) = c,$$

$$u = \frac{1}{\pi}\int_0^\pi c\,d\theta = c.$$

Hence the potential on points out of the axis will be constant also, and the attraction will vanish. We have examples of this case in some closed shells and hollow cylinders of infinite length, as respects points situate on their hollow interiors. Points exterior to the outer surface are not continuous with the above, and require a separate determination of the arbitrary function. But if the surface is not closed, however small may be the aperture, or if the cylinder is of finite length, all points within the concave or without the convex surface are to be considered as included in one application of the general formula.

It would be interesting to verify the general theorem of this paper, by applying it to the case of a circular ring, and comparing the result with the one obtained by ordinary integration. I shall simply indicate the equation, the truth of which would for this purpose require independent proof.

Let a be the diameter of the ring, the centre being the origin of coordinates. Then, ϕ representing an arc of the ring, the rest as before, the potential on the attracted point is easily found to be

$$u = \int_0^{2\pi} \frac{d\phi}{\sqrt{(a^2 - 2ar\cos\phi + r^2 + z^2)}} = 2\int_0^{\pi} \frac{d\phi}{\sqrt{(a^2 - 2ar\cos\phi + r^2 + z^2)}}.$$

Now the potential of the ring on the point z in the axis is $\frac{2\pi}{\sqrt{(a^2+z^2)}}$, therefore, by the general theorem,

$$u = \frac{1}{\pi}\int_0^{\pi} \frac{2\pi\, d\theta}{\sqrt{\{a^2 + (z + r\cos\theta\sqrt{-1})^2\}}}.$$

$$= 2\int_0^{\pi} \frac{d\theta}{\sqrt{\{a^2 + (z + r\cos\theta\sqrt{-1})^2\}}}.$$

Equating these expressions, we have

$$\int_0^{\pi} \frac{d\theta}{\sqrt{\{a^2 + (z + r\cos\theta\sqrt{-1})^2\}}} = \int_0^{\pi} \frac{d\phi}{\sqrt{\{a^2 - 2ar\cos\phi + r^2 + z^2)\}}} \;..(22).$$

The discovery of relations like the above among definite integrals expressing in common the amount of some physical consequence, is not the least curious of the applications of the theorem.

Lincoln, Aug. 18, 1846.

ON A CERTAIN SYMBOLICAL EQUATION.

By George Boole.

In those preliminary researches on the Equation of Laplace's Functions, by which I was led to the method of solution exemplified in the first number of this *Journal*, a remarkable equation presented itself, which has appeared to me to be deserving of special and separate notice. This equation is a symbolical one, and it admits of two conjugate solutions, if the expression may be allowed, which are also purely symbolical; *i.e.* their validity does not depend on the significance of the symbols which they involve, but only on the truth of the laws of their combination. One interpretation of those symbols gives us Laplace's equation, but a more general interpretation than this is possible in the Integral Calculus, and there is perhaps, for I have not examined the question, an interpretation in the Calculus of Finite Differences. Practically, the solutions of Laplace's equation, which we are thus made acquainted with, are of little utility,

as compared with the one which I have already given; but they throw an interesting light on the subject of Symbolical Algebra, and serve to illustrate some general doctrines in Analysis.

The equation which we shall consider is the following, viz.

$$\pi_m \pi_n u + q\rho u = 0 \ldots\ldots\ldots\ldots\ldots (1),$$

in which u is the quantity to be determined, and the symbols π_m, π_n, ρ, applied to any subject u, combine according to the two laws

$$\pi_m \rho = \rho \pi_{m+1}, \quad \pi_n \rho = \rho \pi_{n+1} \ldots\ldots\ldots (2),$$

$$\pi_m \pi_n = \pi_n \pi_m + a(n-m)\rho \ldots\ldots\ldots\ldots (3).$$

The equations (2) are seen to be expressions of one law, π_m, π_n, differing only in the constants m and n. We suppose a to be an arbitrary constant.

Assume $u = \pi_{m+1} v$; we have by (1)

$$\pi_m \pi_n \pi_{m+1} v + q\rho \pi_{m+1} v = 0;$$

$$\therefore \quad \pi_m \pi_n \pi_{m+1} v + q\pi_m \rho v = 0 \quad \text{by (2)},$$

$$\pi_n \pi_{m+1} v + q\rho v = 0.^{*}$$

But $\pi_n \pi_{m+1} = \pi_{m+1} \pi_n + a(m+1-n)\rho$, by (3); therefore, on substitution,

$$\pi_{m+1}\pi_n v + \{q + a(m-n+1)\}\rho v = 0.$$

Let $v = \pi_{m+2} w$; then, by inspection,

$$\pi_{m+2}\pi_n w + \{q + a(m-n+1) + a(m-n+2)\}\rho w = 0.$$

Continuing these transformations, it is evident that, if we suppose in the original equation

$$u = \pi_{m+1}\pi_{m+2} \ldots \pi_{m+r} v,$$

we shall have

$$\pi_{m+r}\pi_n v + \{q + a(m-n+1) + a(m-n+2) \ldots + a(m-n+r)\}\rho v = 0.$$

Or $$\pi_{m+r}\pi_n v + \left\{q + ar(m-n) + a\frac{r(r+1)}{2}\right\}\rho v = 0.$$

If it then be possible by an integer value of r to satisfy the equation

$$q + ar(m-n) + a\frac{r(r+1)}{2} = 0 \ldots\ldots\ldots (4),$$

we shall have $$\pi_{m+r}\pi_n v = 0,$$

$$v = \pi^{-1}_n \pi^{-1}_{m+r} 0;$$

* Strictly $\pi_n \pi_{m+1} v + q\rho v = \pi^{-1}_m 0$. The assumption in the text is lawful, if the result (5) gives the requisite number of arbitrary constants; a condition which is satisfied in the example adduced.

[redacted]

$$\therefore \quad u = \pi_{m+1}\pi_{m+2} \ldots \pi_{m+r}\pi_n^{-1}\pi^{-1}_{m+r}0 \ldots\ldots\ldots (5),$$

which is a complete solution of the equation proposed.

As the equation determining r has two roots, it may be inferred that there are two solutions, which may be denominated as conjugate to each other. The existence and character of the second solution will be most distinctly presented by the following analysis.

Resuming the equation

$$\pi_m\pi_n u + q\rho u = 0.$$

Let us suppose $u = \pi_m^{-1}v$, then

$$\pi_m\pi_n\pi_m^{-1}v + q\rho\pi_m^{-1}v = 0$$

$$\{\pi_n\pi_m + a(n-m)\rho\}\,\pi_m^{-1}v + q\rho\pi_m^{-1}v = 0, \text{ by (3)},$$

$$\pi_n v + \{q + a(n-m)\}\,\rho\pi_m^{-1}v = 0,$$

$$\pi_{m-1}\pi_n v + \{q + a(n-m)\}\,\pi_{m-1}\rho\pi_m^{-1}v = 0.$$

But, by (2), $\pi_{m-1}\rho = \rho\pi_m$. Substituting,

$$\pi_{m-1}\pi_n v + \{q + a(n-m)\}\,\rho v = 0.$$

Hence it is evident that the compound substitution

$$u = \pi_m^{-1}\pi^{-1}_{m-1} \ldots \pi^{-1}_{m-r+1}v \ldots\ldots\ldots (6),$$

would give

$$\pi_{m-r}\pi_n v + \{q + a(n-m) \ldots + a(n-m+r-1)\}\,\rho v = 0 \ldots (7).$$

Or $$\pi_{m-r}\pi_n v + \left\{q + a(n-m)\,r + a\,\frac{r(r-1)}{2}\right\}\rho v = 0.$$

Hence if we determine r by the equation

$$q + a(n-m)r + a\,\frac{r(r-1)}{2} = 0 \ldots\ldots\ldots (8),$$

we shall have $$\pi_{m-r}\pi_n v = 0,$$

$$v = \pi_n^{-1}\pi^{-1}_{m-r}0,$$

$$u = \pi_m^{-1}\pi^{-1}_{m-1} \ldots \pi^{-1}_{m-r+1}\pi_n^{-1}\pi^{-1}_{m-r}0 \ldots\ldots (9);$$

so that the two conjugate solutions, exhibited at one view, are

$$\left.\begin{array}{l} u = \pi_{m+1}\pi_{m+2} \ldots \pi_{m+r}\pi_n^{-1}\pi^{-1}_{m+r}0 \\ u = \pi_m^{-1}\pi^{-1}_{m-1} \ldots \pi^{-1}_{m-r+1}\pi_n^{-1}\pi^{-1}_{m-r}0 \end{array}\right\} \ldots\ldots (10),$$

the values of r in the two cases being respectively determined by the equations

$$\left.\begin{array}{l} q + a(m-n)\,r + a\,\dfrac{r(r+1)}{2} = 0 \\ q + a(n-m)\,r + a\,\dfrac{r(r-1)}{2} = 0 \end{array}\right\} \ldots (11).$$

The roots of the one equation are evidently those of the other with changed signs. If both solutions are available, each equation will have a positive and a negative root, the former belonging to the solution with which it is connected, the latter with its sign changed to the conjugate solution.

It is an obvious corollary from the above, that if a and β be constants, then the solution of the equation

$$(\pi_m + a)(\pi_n + \beta)\, u + q\rho u = 0 \;\ldots\ldots\ldots\; (12),$$

will be be exhibited in either of the conjugate forms,

$$\left.\begin{aligned} u &= (\pi_{m+1} + a)(\pi_{m+2} + a)\ldots(\pi_{m+r} + a)\pi_n + \beta)^{-1}(\pi_{m+r} + a)^{-1}\, 0 \\ u &= (\pi_m + a)^{-1}(\pi_{m-1} + a)^{-1}\ldots(\pi_{m-r+1} + a)^{-1}(\pi_n + \beta)^{-1}(\pi_{m-r} + \beta)^{-1}\, 0 \end{aligned}\right\}\ldots(13),$$

the values of r being determined as before.

It remains to seek an interpretation of our symbols, and for this purpose let us assume

$$\pi_m = \phi(\mu)\frac{d}{d\mu} + m\phi'(\mu),\ \rho = \phi(\mu) \;\ldots\ldots\; (14).$$

Then $\pi_m \rho u = \left\{\phi(\mu)\dfrac{d}{d\mu} + m\phi'(\mu)\right\}\phi(\mu)\, u,$

$$= \phi(\mu)^2\frac{du}{d\mu} + \phi(\mu)\,\phi'(\mu)\, u + m\phi(\mu)\,\phi'(\mu)\, u,$$

$$= \phi(\mu)\left\{\phi(\mu)\frac{d}{d\mu} + (m+1)\,\phi'(\mu)\right\} u,$$

$$= \rho\pi_{m+1} u \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (15).$$

Secondly

$$\pi_m \pi_n u = \left\{\phi(\mu)\frac{d}{d\mu} + m\phi'(\mu)\right\}\left\{\phi(\mu)\frac{d}{d\mu} + n\phi'(\mu)\right\} u,$$

$$= \left\{\phi(\mu)\frac{d}{d\mu}\right\}^2 u + (m+n)\,\phi(\mu)\,\phi'(\mu)\frac{d}{d\mu}$$

$$+ n\phi(\mu)\,\phi''(\mu)\, u + mn\,\{\phi'(\mu)\}^2 u,$$

$$\pi_n \pi_m u = \left\{\phi(\mu)\frac{d}{d\mu}\right\}^2 u + (m+n)\,\phi(\mu)\,\phi'(\mu)\frac{du}{d\mu}$$

$$+ m\phi(m)\,\phi''(\mu)\, u + mn\,\{\phi'(\mu)\}^2 u:$$

therefore $(\pi_m\pi_n - \pi_n\pi_m)\, u = (n - m)\,\phi(\mu)\,\phi''(\mu)\, u,$

$$= (n - m)\,\rho\phi''(\mu)\, u.$$

Or, dropping the subject, u

$$\pi_m\pi_n = \pi_n\pi_m + (n - m)\,\rho\phi''(\mu)\, u \;\ldots\ldots\; (16);$$

and that this may be identical with (3), we must have

$$\phi''(\mu) = a)$$

therefore $$\phi(\mu) = \frac{a}{2}\mu^2 + c_1\mu + c_2 \; \ldots\ldots\ldots\ldots \; (17),$$

c_1 and c_2 being arbitrary constants. Hence the laws of combination (2) and (3) are satisfied, if we assume

$$\pi_m = \left(\frac{a}{2}\mu^2 + c_1\mu + c_2\right)\frac{d}{d\mu} + m(a\mu + c_1), \rho = \frac{a}{2}\mu^2 + c_1\mu + c_2 \ldots (18);$$

and we are at liberty to substitute these values in the general equation (12), and in its conjugate solutions (13).

The equation of Laplace's Functions will be a particular case of the equation thus transformed. For, assume in (12) and (18),

$$m = 0, n = 0, a = -2, c_1 = 0, c_2 = 1, \alpha = \frac{d}{d\phi}\sqrt{-1}, \beta = -\frac{d}{d\phi}\sqrt{-1};$$

we have $$\pi_0 = (1-\mu^2)\frac{d}{d\mu}, \quad \rho = 1 - \mu^2,$$

$$\left\{(1-\mu^2)\frac{d}{d\mu} + \frac{d}{d\phi}\sqrt{-1}\right\}\left\{(1-\mu^2)\frac{d}{d\mu} - \frac{d}{d\phi}\sqrt{-1}\right\}u + q(1-\mu^2)u = 0.$$

Or $$(1-\mu^2)\frac{d}{d\mu}(1-\mu^2)\frac{du}{d\mu} + \frac{d^2u}{d\phi^2} + q(1-\mu^2)u = 0,$$

which, on assigning a proper value to q, is Laplace's equation, the solutions being

$$u = \left(\pi_1 + \frac{d}{d\phi}\sqrt{-1}\right)\left(\pi_2 + \frac{d}{d\phi}\sqrt{-1}\right)\ldots\left(\pi_r + \frac{d}{d\phi}\sqrt{-1}\right)$$
$$\left(\pi_0 - \frac{d}{d\phi}\sqrt{-1}\right)^{-1}\left(\pi_r + \frac{d}{d\phi}\sqrt{-1}\right)^{-1} 0 \ldots (19),$$

$$u = \left(\pi_0 + \frac{d}{d\phi}\sqrt{-1}\right)^{-1}\left(\pi_{-1} + \frac{d}{d\phi}\sqrt{-1}\right)^{-1}\ldots\left(\pi_{-r+1} + \frac{d}{d\phi}\sqrt{-1}\right)^{-1}$$
$$\left(\pi_0 - \frac{d}{d\phi}\sqrt{-1}\right)^{-1}\left(\pi_{-r} - \frac{d}{d\phi}\sqrt{-1}\right)^{-1} 0 \ldots (20);$$

where, in general, $\pi^\lambda = (1-\mu^2)\dfrac{d}{d\mu} + 2\lambda\mu$;

and the values of r are given respectively by the equations

$$q - r(r+1) = 0,$$
$$q - r(r-1) = 0.$$

If $q = n(n+1)$, as in our previous paper, then $r = n$ in the first solution, and $n + 1$ in the second.

[redacted]

We may remark that the process of reduction might have been so ordered as to have eliminated the last operating factor in each solution. This would have detracted from the generality of the first solution, in which there is but one other inverse factor, but not of the second. We have therefore for the sake of symmetry retained the factor in both. To shew how it might have been evaded, let us resume the equations (6) and (7), and writing the second in the form

$$\pi_n\pi_{m-r}v + \{q + a(n-m) \ldots + a(n-m+r-1) + a(n-m+r)\}\rho v = 0,$$

or $$\pi_n\pi_{m-r}v + \left\{q + a(n-m)(r+1) + a\frac{r(r+1)}{2}\right\} = 0,$$

we have $$\pi_n\pi_{m-r}v = 0, \quad \text{or } v = \pi_{m-r}^{-1}\pi_n^{-1}0,$$

if $$q + a(n-m)(r+1) + a\frac{r(r+1)}{2} = 0.$$

Hence $u = \pi_m^{-1}\pi_{m-1}^{-1}\ldots\pi_{m-r+1}^{-1}\pi_{m-r}^{-1}\pi_n^{-1}0$, or writing $r-1$, for r

$$u = \pi_m^{-1}\pi_{m-1}^{-1}\ldots\pi_{m-r+1}\,\pi_n^{-1}0 \ldots\ldots\ldots (21),$$

if $$q + a(n-m)r + a\frac{r(r-1)}{2} = 0,$$

which differs only from the solution before obtained by the last factor.

The direct operations implied in the above solutions will involve differentiation, and the inverse ones the solution of a partial differential equation of the first order. We shall not exhibit the results, as it is clear that neither of the solutions can be freed from integral signs, but shall only remark that the second solution, freed as above from its last factor, is equivalent to the result obtained by Mr. Hargreave in the *Philosophical Transactions*.

The investigation we have entered upon is chiefly valuable, as presenting to us what will be thought a very curious chapter in symbolical algebra, and introducing us to the family of which Laplace's equation is a member. But it must be confessed that they are an interesting rather than an amiable group.

To give completeness to my former paper, I ought to have illustrated the general value of P_n deduced from the integral by actually calculating a few coefficients. This is an extremely simple matter, as all the operations are direct. It must be remembered that the product $1.2.\ldots.p$ like $\Gamma(p+1)$ becomes 1 when $p = 0$.

Lincoln, Aug. 18, 1846.

INVESTIGATION OF CERTAIN PROPERTIES OF THE ELLIPSOID.

By Thomas Weddle, Newcastle-upon-Tyne.

*Conjugate points and diameters—Conjugate diametral and tangent planes—Conjugate parallelepipeds.**

Let
$$\left.\begin{array}{ll} l_1^2+m_1^2+n_1^2=1, & l_1l_2+m_1m_2+n_1n_2=0 \\ l_2^2+m_2^2+n_2^2=1, & l_1l_3+m_1m_3+n_1n_3=0 \\ l_3^2+m_3^2+n_3^2=1, & l_2l_3+m_2m_3+n_2n_3=0 \end{array}\right\}\ldots\ldots(\mathrm{A}).$$

Now these are the same relations as those that obtain among the directing cosines of three straight lines mutually at right angles, hence we must likewise have

$$\left.\begin{array}{ll} l_1^2+l_2^2+l_3^2=1, & l_1m_1+l_2m_2+l_3m_3=0 \\ m_1^2+m_2^2+m_3^2=1, & l_1n_1+l_2n_2+l_3n_3=0 \\ n_1^2+n_2^2+n_3^2=1, & m_1n_1+m_2n_2+m_3n_3=0 \end{array}\right\}\ldots\ldots(\mathrm{B}).$$

Also, from Lagrange's formulas, we get (Gregory's *Solid Geom.* p. 51),

$$\left.\begin{array}{lll} \pm\, l_1=m_2n_3-m_3n_2, & \pm\, l_2=m_3n_1-m_1n_3, & \pm\, l_3=m_1n_2-m_2n_1 \\ \pm\, m_1=l_3n_2-l_2n_3, & \pm\, m_2=l_1n_3-l_3n_1, & \pm\, m_3=l_2n_1-l_1n_2 \\ \pm\, n_1=l_2m_3-l_3m_2, & \pm\, n_2=l_3m_1-l_1m_3, & \pm\, n_3=l_1m_2-l_2m_1 \end{array}\right\}$$
$$\ldots\ldots(\mathrm{C}).\dagger$$

Let
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\ldots\ldots\ldots\ldots\ldots(1)$$

be the equation to an ellipsoid, and $(x_1y_1z_1)$, $(x_2y_2z_2)$, $(x_3y_3z_3)$ three conjugate points on it. The equations to the three conjugate tangent planes touching at these points will be

$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}+\frac{z_1z}{c^2}=1\ldots\ldots\ldots\ldots(2),$$

$$\frac{x_2x}{a^2}+\frac{y_2y}{b^2}+\frac{z_2z}{c^2}=1\ldots\ldots\ldots\ldots(3),$$

$$\frac{x_3x}{a^2}+\frac{y_3y}{b^2}+\frac{z_3z}{c^2}=1\ldots\ldots\ldots\ldots(4).$$

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* Def. *Conjugate points* are the (three) extremities of conjugate diameters, and *conjugate tangent planes* touch the ellipsoid at conjugate points. Also a *conjugate parallelepiped* circumscribing an ellipsoid has its faces parallel to conjugate diametral planes. These terms are convenient, and they seem to be appropriate.

† [In all these formulæ the same sign must be used; the upper in the case when the two systems of axes (l_1, l_2, l_3), (l_1, m_1, n_1) are similarly arranged, and the lower sign when the arrangements are inverse, as, for instance, would be the case if one system were the image of the other in a mirror whose plane passes through the origin.]

[redacted]

The diameter conjugate to (2) is $\frac{x}{x_1} = \frac{y}{y_1} = \frac{z}{z_1}$, and as this line is the intersection of the diametral planes parallel to (3) and (4), we must have

$$\frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} + \frac{z_1z_2}{c^2} = 0 \dots\dots\dots\dots (5),$$

and

$$\frac{x_1x_3}{a^2} + \frac{y_1y_3}{b^2} + \frac{z_1z_3}{c^2} = 0 \dots\dots\dots\dots (6).$$

By similarly considering the diameters conjugate to (3) or (4), we shall get one of the equations just deduced, together with the following,

$$\frac{x_2x_3}{a^2} + \frac{y_2y_3}{b^2} + \frac{z_2z_3}{c^2} = 0 \dots\dots\dots\dots (7).$$

Moreover the conjugate points being on the surface of the ellipsoid, we have

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1 \dots\dots\dots\dots (8),$$

$$\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} + \frac{z_2^2}{c^2} = 1 \dots\dots\dots\dots (9),$$

$$\frac{x_3^2}{a^2} + \frac{y_3^2}{b^2} + \frac{z_3^2}{c^2} = 1 \dots\dots\dots\dots (10).$$

Now, if in (A) we substitute $\frac{x_1}{a}, \frac{y_1}{b}, \frac{z_1}{c}, \frac{x_2}{a}$, &c. for l_1, m_1, n_1, l_2, &c., we shall get (8, 9, 10, 5, 6, 7), hence, as the results of this substitution are true, the equations (B) and (C) will still be true after undergoing the same transformation. We shall thus, after an obvious reduction, get the following equations,

$$x_1^2 + x_2^2 + x_3^2 = a^2 \dots\dots\dots\dots (11),$$

$$y_1^2 + y_2^2 + y_3^2 = b^2 \dots\dots\dots\dots (12),$$

$$z_1^2 + z_2^2 + z_3^2 = c^2 \dots\dots\dots\dots (13),$$

$$x_1y_1 + x_2y_2 + x_3y_3 = 0 \dots\dots\dots\dots (14),$$

$$x_1z_1 + x_2z_2 + x_3z_3 = 0 \dots\dots\dots\dots (15),$$

$$y_1z_1 + y_2z_2 + y_3z_3 = 0 \dots\dots\dots\dots (16).$$

$$\pm\frac{x_1}{a} = \frac{y_2z_3 - y_3z_2}{bc} \quad \pm\frac{x_2}{a} = \frac{y_3z_1 - y_1z_3}{bc} \quad \pm\frac{x_3}{a} = \frac{y_1z_2 - y_2z_1}{bc} \dots (17).$$

$$\pm\frac{y_1}{b} = \frac{x_3z_2 - x_2z_3}{ac} \quad \pm\frac{y_2}{b} = \frac{x_1z_3 - x_3z_1}{ac} \quad \pm\frac{y_3}{b} = \frac{x_2z_1 - x_1z_2}{ac} \dots (18).$$

$$\pm\frac{z_1}{c} = \frac{x_2y_3 - x_3y_2}{ab} \quad \pm\frac{z_2}{c} = \frac{x_3y_1 - x_1y_3}{ab} \quad \dots y_1 \dots (19).$$

Many of the preceding relations, (5)....(19), among the coordinates of three conjugate points are very neat; and some of them, so far as is known to me, have not been noticed before. They facilitate the investigation of several interesting properties of the ellipsoid, as will be shown below. It will be observed that these equations do not require the axes to be rectangular; they hold if the ellipsoid be referred to any system of conjugate diameters. In the following investigations, however, I shall suppose the axes to coincide with the principal diameters.

It may not be amiss to give the verbal statement of the geometrical properties implied in the three groups of equations (11, 12, 13), (14, 15, 16), and (17, 18, 19).

The first group (11, 12, 13) signifies that,

(A) If three conjugate points be projected on any diametral plane by lines drawn parallel to the diameter conjugate to this plane, the sum of the squares of the three lines of projection is equal to the square of the semidiameter.

The second group (14, 15, 16) shews that

(B) If from the points of projection mentioned in (A) lines be drawn parallel to, and be terminated by, any two conjugate diameters in the diametral plane, thus forming three parallelograms, the sum of two of these parallelograms is equal to the third.

Also from the third group (17, 18, 19), we have the following theorem:

(C) Let the parallelogram constructed on any two of three conjugate diameters, as well as the extremity of the third diameter, be projected, as in (A), on the plane of any two diameters of a second conjugate system. As the projection of the parallelogram is to the parallelogram constructed on the two diameters of the second system, so is the line which projects the extremity of the diameter to half the third diameter of the second system.

Let r_1, r_2, r_3 be the radii drawn to the three conjugate points, then will $r_1^2 = x_1^2 + y_1^2 + z_1^2$, &c.; wherefore, adding (11, 12, 13), we have

$$r_1^2 + r_2^2 + r_3^2 = a^2 + b^2 + c^2 \quad \ldots\ldots\ldots (20).$$

Hence

(D) The sum of the squares of any system of conjugate diameters is equal to the sum of the squares of the principal diameters.

Moreover (Gregory's *Solid Geom.* p. 17), the volume (V)

of a parallelepiped of which three contiguous edges meet at the origin and terminate in the three points $(x_1y_1z_1)$, $(x_2y_2z_2)$, $(x_3y_3z_3)$, is

$$V = (x_2y_3 - x_3y_2)\,z_1 + (x_3y_1 - x_1y_3)\,z_2 + (x_1y_2 - x_2y_1)\,z_3$$

$$= (19)\,\frac{ab}{c}\left\{z_1^2 + z_2^2 + z_3^2\right\} = (13)\ abc \ldots\ldots\ldots (21).$$

Now the volume of the conjugate parallelepiped of which the conjugate tangent planes (2, 3, 4) are adjacent faces, is evidently eight times that just found; hence, we infer that

(E) Each conjugate parallelepiped circumscribing an ellipsoid, is equal to that constructed on the principal diameters.

Let $\widehat{r_1r_2}$ denote the angle included between the radii r_1 and r_2. The area of the parallelogram of which r_1 and r_2 are contiguous sides is $r_1r_2 \sin \widehat{r_1r_2}$, and the projections of this area on the planes of yz, zx, and xy, are $y_1z_2 - y_2z_1$, $x_2z_1 - x_1z_2$, and $x_1y_2 - x_2y_1$; hence, by the theory of projections (Gregory's *Solid Geom.* p. 14),

$$r_1^2r_2^2 \sin^2 \widehat{r_1r_2} = (y_1z_2 - y_2z_1)^2 + (x_2z_1 - x_1z_2)^2 + (x_1y_2 - x_2y_1)^2;$$

or reducing, by means of (17, 18, 19),

$$r_1^2r_2^2 \sin^2 \widehat{r_1r_2} = a^2b^2c^2\left\{\frac{x_3^2}{a^4} + \frac{y_3^2}{b^4} + \frac{z_3^2}{c^4}\right\}$$

Similarly $$r_1^2r_3^2 \sin^2 \widehat{r_1r_3} = a^2b^2c^2\left\{\frac{x_2^2}{a^4} + \frac{y_2^2}{b^4} + \frac{z_2^2}{c^4}\right\} \ldots (22).$$

and $$r_2^2r_3^2 \sin^2 \widehat{r_2r_3} = a^2b^2c^2\left\{\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} + \frac{z_1^2}{c^4}\right\}.$$

Add these and reduce by (11, 12, 13),

$$r_1^2r_2^2 \sin^2 \widehat{r_1r_2} + r_1^2r_3^2 \sin^2 \widehat{r_1r_3} + r_2^2r_3^2 \sin^2 \widehat{r_2r_3} = b^2c^2 + a^2c^2 + a^2b^2 \ldots (23),$$

which amounts to another well-known theorem; namely

(F) The sum of the squares of the parallelograms formed by each pair of conjugate diameters is equal to the sum of the squares of the rectangles under each pair of the principal diameters. Or, the sum of the squares of the faces of any conjugate parallelepiped is equal to the sum of the squares of the faces of the parallelepiped described on the principal diameters.

If p_1, p_2, and p_3 be the perpendiculars from the centre on the three conjugate tangent planes (2, 3, 4), we shall have

$$\left.\begin{aligned}\frac{1}{p_1^2} &= \frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} + \frac{z_1^2}{c^4}\\ \frac{1}{p_2^2} &= \frac{x_2^2}{a^4} + \frac{y_2^2}{b^4} + \frac{z_2^2}{c^4}\\ \frac{1}{p_3^2} &= \frac{x_3^2}{a^4} + \frac{y_3^2}{b^4} + \frac{z_3^2}{c^4}\end{aligned}\right\} \dots\dots\dots\dots (24).$$

By addition, and (11, 12, 13),

$$\frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \dots\dots\dots (25).$$

Hence

(G) The sum of the squares of the reciprocals of the perpendiculars from the centre of an ellipsoid on three conjugate tangent planes is equal to the sum of the squares of the reciprocals of the principal semidiameters.

From (22) and (24), we have

$$r_1 r_2 \sin \widehat{r_1 r_2} = \frac{abc}{p_3}, \quad r_1 r_3 \sin \widehat{r_1 r_3} = \frac{abc}{p_2}, \quad r_2 r_3 \sin \widehat{r_2 r_3} = \frac{abc}{p_1} \dots (26).$$

We might, however, have deduced (23) otherwise. After having established (25) and (26), the latter of which is evidently only another form of (21), eliminate p_1, p_2, and p_3 between them, and we have (23) at once.

The locus of the intersections of three conjugate tangent planes will obviously be obtained by eliminating x_1, y_1, z_1, x_2, &c. from (2, 3, 4) by means of some of the succeeding equations. Now this elimination is immediately effected by taking the sum of the squares of (2, 3, 4) and reducing by (11)...(16),

$$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 3 \dots\dots\dots\dots\dots (27).$$

Hence

(H) The locus of the intersections of conjugate tangent planes to an ellipsoid is a concentric similar ellipsoid whose principal diameters are to those of the given ellipsoid as $\sqrt{3}:1$.

A different investigation of this may be seen in Gregory's *Solid Geom.* p. 269.

The theorem (H) may evidently be varied by saying

(I) Every conjugate parallelepiped circumscribing an ellipsoid is inscribed in a concentric similar ellipsoid.

We shall next establish the following theorem:

(K) If from the centre O of an ellipsoid any line be drawn intersecting the spheres described on the principal diameters $A'A$, $B'B$, $C'C$ in the points P, Q, R; and from P, Q, R

equation of that cone, from which, to get that of its reciprocal cone, we have but to change the three coefficients into their reciprocals, which will give

$$\frac{x^2}{A-I_1}+\frac{y^2}{B-I_1}+\frac{z^2}{C-I_1}=0\;\ldots\ldots\ldots\;(a),$$

where I_1 has the value above; and this, with $r^2=x^2+y^2+z^2$, will be the equations required.

This conic being a function of I and r, is therefore variable in magnitude, position, and figure, as these quantities vary, and will consequently follow in its successive changes of state no continuous law, if their variations be independent of each other; but if they be connected by any relation, the conic, depending then on but a single variable, will in passing through its several states describe a surface, which will obviously be of that extensive and very important class of central surfaces, of such frequent occurrence in physical investigations, those viz. whose intersections with every concentric sphere are all sphero-conics similarly placed, and which are contained all in the equation

$$\phi(a).x^2+\chi(a).y^2+\psi(a).z^2=0\;\ldots\ldots\ldots\;(\beta);$$

where $a=x^2+y^2+z^2$, and when ϕ, χ, and ψ, are functions of any forms whatever, given, known, or to be determined, as the case may be: to this class belongs the wave surface of light in biaxal crystals, and it was from the present consideration that Professor Mac Cullagh obtained the symmetrical equation, and deduced many properties of that interesting surface. In the present case, to find the equation of the surface generated by the conics when the relation between I and r is given, we have but to substitute for I_1 in (a) its value in terms of r, (I being known in terms of the same from the given relation), and then in the result to change r into its value $\sqrt{(x^2+y^2+z^2)}$.

For instance, let the axes which generate the systems of enveloping cylinders for different spheres round the centre of gravity be all equimomental; then shall we have $I=$ a constant, and the equation will be

$$\frac{x^2}{x^2+y^2+z^2-a^2}+\frac{y^2}{x^2+y^2+z^2-b^2}+\frac{z^2}{x^2+y^2+z^2-c^2}=0\ldots(\gamma),$$

where $\qquad a^2,\, b^2,\, c^2=\dfrac{I-A}{M},\ \dfrac{I-B}{M},\ \dfrac{I-C}{M}$, respectively.

As a second instance, let the axes be all *isochronal*, that is, such that a body would vibrate as a pendulum round them

all in the same time, then shall we have $\frac{I}{Mr}$ (= the radius of oscillation) = a constant = l; therefore $I_1 = M.r(l-r)$, and the equation will be

$$\frac{x^2}{a^2 - r(l-r)} + \frac{y^2}{b^2 - r(l-r)} + \frac{z^2}{c^2 - r(l-r)} = 0 \ldots\ldots (\delta),$$

where $a^2, b^2, c^2 = \frac{A}{M}, \frac{B}{M}, \frac{C}{M}$ = the squares of the three principal radii of gyration, and $r = \sqrt{(x^2 + y^2 + z^2)}$.

For a third instance, see next article.

14 Every system of equimomental axes, which are all equidistant from the centre of gravity, will obviously be also an isochronal system; whatever therefore has been said respecting such an equimomental system is true of such an isochronal system (13).

The direction of an equimomental system being given, the axes must lie all on a circular cylinder round the centre of gravity; but since every axis drawn at random in a body is isochronal with a parallel axis, in a plane passing through the centre of gravity, and distant from that point by an interval equal to its radius of oscillation, the rectangle under the distances of the two axes from the centre being equal to the square of their common radius of gyration; if the direction of an isochronal system be given, the axes will separate into two distinct systems, lying on two circular cylinders round the centre of gravity, whose radii for a given direction will be reciprocally proportional to each other, and which will therefore coincide only in the particular case when their common time of vibration is a minimum for that direction, in which case the radius of oscillation, or the length of the equivalent simple pendulum, is double the radius of gyration.

Taking, therefore, the ellipsoid reciprocal to the momental ellipsoid at the centre of gravity, the axes of absolute minimum time will describe a circular cylinder round the minimum axis of that ellipsoid with a radius equal to that semi-axis; and a cylinder similarly described round the maximum axis will be the locus of the axes of maximum minimorum times.

But for every intermediate value of the minimum time the axes will no longer lie on a single cylinder, but will group themselves on an infinite number of cylinders of equal radii, which will envelope all the same sphere round the centre of gravity, and whose axes passing all through that centre will nerate a central equimomental cone.

The sides of this cone coinciding with a system of equal radii of the central ellipsoid, and every radius of that surface being the reciprocal of the coincident radius of gyration, the radius of the sphere enveloped by all the cylinders will be therefore equal to the reciprocal of the central radii which generate the equimomental cone, and the sphero-conic envelope of all their great circles of contact will be the intersection with that sphere of the reciprocal cone.

The equation of the surface generated by all these conics for the different systems of cylinders corresponding to different values of the minimum time of vibration may be immediately obtained.

For, in the equation (α) of the preceding article, we have I_1 connected with r by the relation $I_1 = Mr^2$; eliminating therefore that quantity, we get for the surface required the equation

$$\frac{x^2}{x^2+y^2+z^2-a^2}+\frac{y^2}{x^2+y^2+z^2-b^2}+\frac{z^2}{x^2+y^2+z^2-c^2}=0 \dots (\epsilon);$$

where $a^2, b^2, c^2 = \frac{A}{M}, \frac{B}{M}, \frac{C}{M}$, a, b, c being the principal radii of gyration.

15. We may observe that the equation just determined is the same as Mr. Haughton, Fellow of Trinity College, Dublin, has found for the surface locus of the feet of central perpendiculars on all the tangent planes to the biaxal wave surface whose semi-axes are a, b, c. And that such should be the case may be readily shewn, for the surface at any point of a body, the squares of whose radii represent their movements of inertia, is the same as the surface of elasticity in the wave theory of light, (both surfaces will be in all respects similar at those points of a body where the three principal moments of inertia are proportional to the three principal elasticities of the medium); so that the surface round the centre of gravity, locus of the extremities of all the radii of gyration, is identical with the surface of elasticity in a medium whose three principal elasticities are proportional to the three principal central moments of inertia: the ***apsidals,**** therefore, of these two surfaces are also identical;

* Through a fixed point, taken arbitrarily in space, planes being drawn in all directions intersecting a given surface, and on a perpendicular to each erected at the same point, portions being taken equal to all the apsidal radii vectors of the curve of section, the surface locus of the extremities of all these portions is called the apsidal of the given surface. Professor Mac-

but of the former surface the apsidal is obviously (vide note *infra*) the surface whose equation has just been found; and of the latter, the apsidal is the locus of feet of perpendiculars on the wave surface, whose generating ellipsoid has for semi-axes those of the surface of elasticity.

Isochronal, like equimomental axes, if no otherwise restricted, are obviously infinite in number; like the latter, they are confined within certain limits in the neighbourbood of the centre of gravity; and as in the case of moments of inertia, a central surface may at every point of a body be readily found, whose radii shall represent the times of vibration round them*, and whose intersections with concentric spheres of different radii will give us the system of cones of isochronal axes for each point of the body; but this surface, not being of the second order, is little likely to prove interesting.

Before closing this article and proceeding to our more immediate subject, that of principal axes, it may be satisfactory to state to the reader that the properties we have been considering in this and in the two preceding articles are intimately connected with that subject, though at first sight they would seem to be altogether foreign to it. To shew that this is the case, we will here, in anticipation, state

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Cullagh, to whom the name is due and who was the first to consider this class of surfaces, has given the following method of constructing them and finding their equations. With its centre at the given point let a sphere of arbitrary radius be described intersecting the surface, then will every tangent plane to the cone which from the centre subtends the curve of intersection determine a section of the surface of which the side of contact will be an apsidal radius vector; the reciprocal cone will therefore intersect the same sphere in a curve belonging to the apsidal surface, so that to find the equation of that surface we have but to eliminate the parameter r, from the equation of the reciprocal cone, by means of that of the sphere $r^2=x^2+y^2+z^2$. He has also shewn that every radius vector of the given surface, the corresponding radius of its apsidal, and the two perpendiculars on the tangent planes at the extremities of these radii, are respectively two and two, equal and at right angles to each other, and that all four lie always in the same plane, thus affording an obvious method of determining the tangent plane at any point of an apsidal when we know it for the corresponding point of the given surface; and shewing also that the apsidals (from the common pole) of two surfaces which are sphero-polar reciprocals to each other, are themselves also reciprocal surfaces. These results he has applied, in general, to the biaxal wave surface, which is the apsidal of an ellipsoid; and the last, in particular, to the two biaxal wave surfaces, apsidals to two reciprocal ellipsoids.

* In the same way as the radii of the momental ellipsoid at every point of a body represent the angular velocities with which the body would revolve round the coincident fixed axes, in consequence of impressed impulses having for all of them the same moment; which is obvious, as the angular velocities round the different axes will be then inversely as their moments of inertia, that is, directly as the squares of the radii.

a property of principal axes, which will be hereafter established and its consequences developed, but which will serve here to manifest the truth of the present assertion.

On every circular cylinder round an axis passing through the centre of gravity of any body there always exist two generatrices diametrically opposite to each other which enjoy the property of being principal axes, while in general no other generatrix of the same cylinder possesses that property; and these two particular generatrices are always those in which it is ultimately intersected by the consecutive *equimomental* cylinder which circumscribes the *same* sphere round the centre of gravity.

Assuming for the present the truth of this property, we see immediately that—

If to any sphere of arbitrary radius, described round the centre of gravity of a body, a system of tangent planes be drawn parallel to the system of planes tangent to any central *equimomental* cone, then will the developable surface envelope of that system of planes possess always the property, that its edges will be all principal axes. This theorem is due to Professor Thomson.

The curve of contact of every such developable surface with the sphere it envelopes being obviously a sphero-conic of the class we have been considering in the preceding articles, we hence see immediately a general property of that whole system of conics.—If each be made the curve of contact of a developable surface circumscribing the sphere on which it lies, then will that developable surface always possess the property that all its edges will be principal axes.

Taking now the whole system of developable surfaces thus related to the system of sphero-conics which generate the surface (ϵ), and observing that if a developable surface be at the same time circumscribed to a sphere and to any other surface whatever, then will its curve of contact with the former be a curve on the surface locus of feet of perpendiculars let fall from the centre of the sphere upon all the tangent planes to the latter, we get, remembering the opening remark of the present article, from the equation (ϵ) the following interesting property of principal axes.

In every body that particular system of axes which possess the twofold property of being axes of minimum time of vibration for their respective directions, and of being also principal axes, will always admit of an enveloping surface, and that envelope will be always the biaxal wave surface apsidal to the ellipsoid of gyration round the centre of gravity.

Again, for the same reason, from the equation (γ), which is exactly of the same form as (ϵ), and to which therefore the remark at the beginning of the present article equally applies, we get, by considering the system of developable surfaces similarly related to the system of conics which generate that surface, the property of principal axes discovered by Professor Thomson (see *Cambridge and Dublin Mathematical Journal*, vol. I. p. 203, Art. 21), a property for which he also (in a letter to the author) gave independently the present solution.

Every system of axes in a body which possess the twofold property of being both equimomental and principal will also envelope a biaxal surface round the centre of gravity, the apsidal of a surface of the second order concentric and coaxal with the ellipsoid of gyration round that point, and for different values of the common moment of inertia the whole system of apsidals envelopes of the different systems of equimomental principal axes will be always such that the generating surfaces of the second order, obviously all forming a concentric and coaxal system, will be all confocal, being (as is evident from the values of a, b, c there given) the system of surfaces of the second order conjugate to the surfaces of the system confocal with the ellipsoid of gyration, and therefore forming themselves also another confocal system, the well-known system conjugate to the former.

Professor Mac Cullagh, to whom indeed the whole theory of apsidals is due, was the first who considered geometrically the properties of a system of biaxal surfaces such as the present, the apsidals of a system of confocal surfaces of the second order. The author of the present paper hopes, in a subsequent article, in which the properties of the above system of envelopes will be exclusively considered, to be able to introduce to the reader's notice some remarkable and interesting properties of the important class of surfaces forming that system: but for the present we must discontinue a subject which was only introduced for the purpose of creating an interest in the properties discussed in the preceding article, and which, from its fundamental property not having been established but only assumed, could not but be considered as premature.

16. But to return to our more immediate subject, that of Principal Axes. From the important property of the centre of gravity with respect to parallel axes, it appears that we can find the moments of inertia round all axes assumed at pleasure in a body, if we know them for all axes through the

centre of gravity, or, which is the same thing, if we know the momental ellipsoid for that point. Hence, by means of that ellipsoid we may construct the momental ellipsoid for any other point we please of the body, and may therefore, by means of the same, find the principal axes at every point.

Now the equation of that ellipsoid contains six constants, which if known, we may consider the surface as determined: if, therefore, the body be terminated by any regular surface, and that it either be homogeneous, or that its density vary according to any regular law, we may assume arbitrarily three rectangular coordinate axes through the centre of gravity, and actually calculate for these the three moments of inertia and the three sums $\Sigma xydm$, $\Sigma zxdm$, $\Sigma yzdm$; or we may calculate the moments of inertia for any six axes making known angles with the same, and then equate the results to their known values in terms of the six constants, which will give us six linear equations to find these constants.

But as most bodies are altogether irregular, both in form and density, this method is seldom practicable, and we must have recourse to the experimental method, which holds in all cases and for all bodies.

By making the body (whatever be its nature) vibrate as a pendulum round any number of axes, assumed at pleasure, we can, from the observed time of a small oscillation round each axis of suspension, from the known distances of these axes from the centre of gravity, and from the known weight of the body, determine* the moments of inertia round the parallel axes through the centre of gravity; and these found for a sufficient number of axes will enable us to construct the momental ellipsoid for that point.

When therefore a body is given, we may consider as also given (since it may be always found) its momental ellipsoid at its centre of gravity, and consequently every curve or surface which is geometrically connected with that ellipsoid, or which may be derived therefrom.

We may therefore consider as given with every body the concentric ellipsoid (of which we have often made mention

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* By means of the well-known formula $T = \pi\sqrt{\frac{l}{g}}$, T being the time of vibration, and l the length of the equivalent simple pendulum; for, in this equation substituting for l its value $\frac{I + M.d^2}{M.d}$, I being the moment round the parallel central axis, we at once get $I = \frac{W.d.T^2}{\pi^2} - M.d^2$, $W = Mg$ being the weight of the body.

and which is of primary importance in every thing relating to the present subject) sphero-polar reciprocal to the momental ellipsoid round its centre of gravity, and every curve and surface geometrically connected therewith.

The momental ellipsoid round the centre of gravity being such that the squared reciprocals of its radii, multiplied each by the mass of the body, are equal to their moments of inertia, it follows that the reciprocal ellipsoid is such that the squares of the central perpendiculars on its tangent planes, multiplied each by the mass of the body, are equal to their moments of inertia; that ellipsoid round the centre of gravity of every body is therefore called the *ellipsoid of gyration*, and its semi-axes a, b, c, coinciding with the three central principal axes, are obviously the three principal radii of gyration.

At every point of a body, indeed, it is easy to see directly that the envelope of a plane, whose perpendicular distance therefrom is such that its square multiplied by the mass of the body is equal to the moment of inertia round that perpendicular, will be an ellipsoid, the reciprocal of the momental ellipsoid at the same point; for, let $a_1 b_1 c_1$ be the three particular distances which coincide with the principal axes, $A_1 B_1 C_1$ the three principal moments, and $\alpha_1 \beta_1 \gamma_1$ the direction angles of any other perpendicular p_1; then we shall have, moment of inertia round $p_1 = A_1 \cos^2 \alpha_1 + \beta_1 \cos^2 \beta_1 + C_1 \cos^2 \gamma_1$, that is, $M \,.\, p_1^2 = M \,.\, (a_1^2 \cos^2 \alpha_1 + b_1^2 \cos^2 \beta_1 + c_1^2 \cos^2 \gamma_1)$; the envelope therefore of the plane is the ellipsoid $a_1 b_1 c_1$.

We might have set out with establishing the existence of this latter ellipsoid at every point of a body, and by its aid have arrived at the results of the preceding articles; but by means of the momental ellipsoids they have been obtained perhaps more simply. Both ellipsoids, however, are important, and possess their advantages each over the other; for instance, in the problem of the rotation of a rigid body round a fixed point, which has been completely solved both by Poinsot and by Professor Mac Cullagh, the momental ellipsoid which has been employed by the former exhibits the whole motion with a clearness which could not be surpassed; and the reciprocal ellipsoid which has been used by the latter gives immediately, and without the trouble of any transformation, the elliptic integrals which express the circumstances of the motion, such as the times of oscillation, revolution, &c. Also in the more general case, when any external forces are acting, it exhibits more clearly the action of the different forces, centrifugal and external, in every position of the body.

In the particular case when the point is the centre of

gravity, the reciprocal ellipsoid is such that the central perpendiculars on its tangent planes are all equal to the coincident radii of gyration—that particular surface therefore (as above stated) is called the ellipsoid of gyration, and with every curve and surface derived from it, is to be considered as given with every given body. It is important in all that relates to the present subject, for by means of its varieties bodies may be conveniently classified, and we may always (as we shall presently see) suppose the remainder of a body, whatever be its nature, cut away, and confine our attention to that surface alone.

17. Having proved the existence at every point of a body of at least three principal axes mutually at right angles to each other, and having stated that there exists in every body an infinite number of points forming a curve for all of which the number of principal axes is infinite; we proceed now to the determination of their directions and of their moments of inertia at any particular point given or arbitrarily assumed in a body; to the consequent determination of that particular system of points which admit of an infinite number of principal axes; and subsequently, to the development of the general laws which govern the distribution in space of that class of lines in every body which enjoy the property of being principal axes.

Towards these objects we have the following theorem—

The dynamical principal axes at any point of a body are the geometrical principal axes of the cone which from that point as vertex envelopes the ellipsoid of gyration.

For the axes of that cone, which is the envelope of all the tangent planes drawn to the ellipsoid from the given point O, coincide with the axes of its reciprocal cone, and the axes of the latter cone, which is the locus of all the perpendiculars erected at the same point to those tangent planes, are the principal axes at that point; for all the sides of that cone are equimomental axes (6).

To shew this, let $\alpha\beta\gamma$ be the direction angles of the indefinite perpendicular P, erected at O to any tangent plane thence drawn to the ellipsoid, and let p be the length of the perpendicular upon that plane from the centre of gravity G, and q and r the distances of O from p and G respectively; then we shall have (P and p being parallel axes, distant by an interval q, and the latter passing through the centre of gravity,)

$$\text{Moment of inertia round } P = \text{moment round } p + Mq^2$$
$$= Mp^2 + Mq^2 = M(p^2 + q^2) = Mr^2 \dots\dots\dots \text{(VI.)};$$

which being independent of the position of the tangent plane $(\alpha\beta\gamma)$ shews that the perpendiculars, that is the sides of the reciprocal cone, are all equimomental axes.

The construction thus indicated for determining the principal axes at every point of a body, enables us to conceive their directions at each point, and their relative positions at different points, as easily as could perhaps be desired. It also confirms the anticipations of Art. 11, respecting the symmetrical distribution of principal axes in the eight regions of space determined by the three principal planes through the centre of gravity.

18. By using the known property, that the three axes of the cone real or imaginary which from any vertex envelopes an ellipsoid, are the normals to the three surfaces of the second order confocal with the ellipsoid which pass through that vertex, we may substitute for the above construction another, virtually the same, but affording the advantage of enabling us to apply to principal axes in general the known properties of those lines which are normals to a system of confocal surfaces of the second order. Hence,

The principal axes at any point of a body are the normals to the three surfaces of the second order confocal with the ellipsoid of gyration which pass through that point.

This latter construction has been directly established by Professor Mac Cullagh from the dynamical property of principal axes, as follows:

Let O be any point of a body, G its centre of gravity, OX, OY, OZ the normals to the three surfaces of the second order confocal with the ellipsoid of gyration (a, b, c) which pass through O, and GP, GQ, GR the three central perpendiculars on the corresponding tangent planes at O to these three surfaces (all these lines being supposed produced indefinitely). He has shewn, that if the body revolve round either of the three normals (OX), then will the centrifugal forces of all its elements compound a single resultant passing through O.

ω being the angular velocity of rotation, let O' be the point on the ellipsoid of gyration where the normal $O'X'$ is parallel to OX, and P' the point where the tangent plane at O' meets (at right angles) the perpendicular GP; then, from a known property of confocal surfaces, will the area of the right-angled triangle $GO'P'$ be equal to that of the right-angled triangle GOP, and both triangles will lie in the same plane containing the three parallel lines GP, OX, $O'X'$.

dm being an element of the body at any point *o*, let *r* and *q* be the points where a plane through *o* perpendicular to the parallel axes *OX*, *GP* meets these axes respectively, and let *op* be a line drawn in that plane from *o* parallel and equal to the distance *OP*; the centrifugal force of *dm*, $\omega^2.or.dm$, which acts from *r* to *o* along the line *ro*, may be resolved into two; $\omega^2.oq.dm$ passing through the axis *GP*, and directed perpendicular to that axis from *q* to *o*; and $\omega^2.op.dm$ acting parallel to the line *OP*, in the direction from *O* to *P*.

A similar resolution being effected for every point of the body, we shall have the whole system of centrifugal forces arising from the rotation round *OX* replaced by two different and distinct systems of new forces; a system of parallel forces, $\omega^2.OP.dm$, acting all in the same direction from *O* to *P*, which are the same as if all the elements of the body were acted upon by equal and parallel accelerating forces, equal each to the quantity $\omega^2.OP$, parallel all to the line *OP*, and acting all from *O* to *P*; and a system of forces, $\omega^2.oq.dm$, passing all through the axis *GP*, and acting perpendicularly out from that axis, which are precisely the same in magnitude and direction as would arise from a rotation with the same angular velocity, ω, round *GP* as axis; the former of these systems will compound a single resultant, passing through the centre of gravity, parallel to the line *OP*, acting from *O* to *P*, and equal to the quantity $\omega^2.M.OP$; and the latter system, transferred all to the centre of gravity, will be again replaced by a system of forces passing all through that point, and by a system of moments in planes passing all through *GP*, the system of transferred forces will all equilibrate round the centre (2), and the system of moments will compound a resultant moment passing through the axis *GP*, the same in magnitude, plane, and direction as would result from a rotation round that axis with the angular velocity ω.

Now the ellipsoid of gyration being the reciprocal of the momental ellipsoid at the centre of gravity, the central perpendicular *p*, on the tangent plane at any point of either, and the radius *r*, drawn to the point of contact, coincide respectively with, and are the reciprocals of, the radius *r*′, drawn to the corresponding point of the other, and the perpendicular *p*′, on the tangent plane at the same. Hence, from (4) it appears that, If a body revolve round the central perpendicular (*p*) on any tangent plane to its ellipsoid of gyration, the plane of the resultant centrifugal moment, passing through that perpendicular, will be that of the radius (*r*) and perpendicular, the direction of that moment will be from the radius

towards the perpendicular, and its magnitude will be $\omega^2.M.$ (2 area of right-angled triangle rp).

The whole system of centrifugal forces arising from the rotation round OX, may therefore be finally replaced by a single force $\omega^2.M.OP$, passing through the centre of gravity, and acting in the direction from O' towards P', and by a moment $\omega^2.M.GP'.O'P'$, in the plane $GO'P'$, and acting in the direction from O' to P'; but, the plane $GO'P'$ coinciding with the plane GOP, and the rectangle $GP'.O'P'$ being equal to the rectangle $GP.OP$, the moment is equivalent to a moment in the plane GOP, equal to $\omega^2.M.GP.OP$, and acting from O to P, that is, to two equal and parallel forces $\omega^2.M.OP$, one passing through O and acting from O to P, and the other passing through the centre of gravity and acting in the opposite direction; the latter of these forces will destroy the single force to which it is equal and directly opposed, and we shall have remaining, as the final equivalent to the whole system of centrifugal forces arising from the rotation round OX, but a single force passing through O.*

Hence the normal at any point of a surface of the second order described in a body confocal with its ellipsoid of gyration is a principal axis at that point. And a similar proof holding for the other normals shews that the principal axes at any point of a body are the normals to the three surfaces of the second order confocal with its ellipsoid of gyration which pass through that point.

19. If the cone, real or imaginary, which from O as vertex envelopes the ellipsoid, be of revolution, the first of these constructions (17) shews that such a point admits of an infinite number of principal axes in the plane perpendicular to the internal axis of the cone; but if it be not of revolution, the point will admit of only three.

In general therefore (A, B, C being all unequal) the great majority of points in a body admit of but three principal axes; for when the three principal central moments are all unequal, the cone which envelopes the ellipsoid of gyration is generally not of revolution.

But for every surface of the second order there exist two different and distinct systems, real or imaginary, of enveloping cones of revolution, and the loci of their vertices are the two real focal conics of the surface.

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* The principles involved in the above demonstration were investigated by Professor Mac Cullagh for a different purpose: he merely applied them incidentally to the present question.

Hence, in every body there exist two continuous and distinct series of points which admit of an infinite number of principal axes (5), and the loci of these two series are both plane curves, of the second order, lying each in a principal plane through the centre of gravity, one an ellipse in the plane of AB, the other an hyperbola in the plane of AC, the real focal conics viz. of the ellipsoid of gyration a, b, c.

At every point of these two curves the normal plane is that which contains the infinite number of principal axes; for the tangent at any point on either of its focal conics is the internal axis of the cone of revolution which from that point as vertex envelopes any surface of the second order.

Lest imaginary cones might be here considered as a difficulty, it may be well to shew that the other construction (18) leads readily to the same results.

For, the focal conics common to a confocal system of surfaces of the second order, bound portions of the principal planes in which they lie, which are the infinitely flat surfaces of the system, and the transition state from one species to another; all planes, therefore, which pass through a tangent to a focal conic are tangent planes to these particular surfaces, and the perpendiculars to these planes at the point of contact which there generate the normal plane to that curve, are all normals to the same.

Two, therefore, of the three surfaces confocal with the ellipsoid abc, which pass through any point on either of its focal conics, admit of an infinite number of normals, lying all in the normal plane to the conic at that point; the body, therefore, at such a point admits of an infinite number of principal axes in that plane. (This is the general property of which particular cases were established on other principles in 12.)

20. It was proved above (17) that the cone reciprocal to that which from any point O envelopes the ellipsoid of gyration abc, is a cone of equimomental axes for that point; more generally, the whole system of cones reciprocal to those which from O as their common vertex envelope the whole system of surfaces confocal with that ellipsoid, will be the system of equimomental cones for that point (6).

For, every equimomental system of cones will be all coaxal, and will have all the same cyclic planes (6), its reciprocal system of cones will be therefore all coaxal, and will have all the same focal lines; but this is the known property of the system of cones which from any vertex envelope a system of confocal surfaces of the second order.

The same may be proved directly, which will thus, conversely, establish the more difficult property of confocal surfaces.

For, let a', b', c' be the semiaxes of any surface confocal with abc, then shall we have $a'^2 = a^2 + \delta$, $b'^2 = b^2 + \delta$, $c'^2 = c^2 + \delta$ where δ may have any value from $+\infty$ to $-a^2$); then, denoting by α', β', γ' the direction angles of P' the perpendicular erected at O to any tangent plane thence drawn to this surface, by p' the perpendicular from the centre on that tangent plane, by q' the distance between P' and p', and by r (as before) the distance of O from the centre, we shall have (p' and P' being parallel, the former passing through G),

$$\begin{aligned} \text{moment of inertia round } P' &= \text{moment round } p' + Mq'^2 \\ &= M.(a^2\cos^2\alpha' + b^2\cos^2\beta' + c^2\cos^2\gamma') + M.q'^2 \\ &= M(p'^2 - \delta) + M.q'^2 = M(p'^2 + q'^2) - M.\delta \\ &= M.r^2 - M.(a'^2 - a^2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(VII.)}; \end{aligned}$$

which, being a quantity independent of the position of the tangent plane, shews that P' generates a cone of equimomental axes. The system of cones generated by the perpendiculars are therefore all equimomentatl, consequently (6) they are all coaxal and concyclic; the enveloping system of cones, their reciprocals, are therefore all coaxal and confocal.

(From the above theorem, of which that in (17) is a particular case, we may, if in constructing for principal axes by means of the latter at a point *within* the ellipsoid of gyration the imaginary enveloping cone prove a source of difficulty, substitute for that ellipsoid any other confocal with it to which the point shall be external, and thus get a real coaxal cone.)

21. In the cases of the three particular confocal surfaces of the system which pass through O, the enveloping cones degenerate into three planes, the tangent planes to the surfaces, and their reciprocals therefore into the three normals; these normals are therefore the three common axes of all the enveloping cones, that is, the principal axes at O (18).

By substituting, therefore, in the known expressions which give the directions of the normal at any point of a surface of the second order in terms of the coordinates of the point and the semi-axes of the surface, the three values of δ which correspond to these three surfaces, we shall have the direction cosines of the principal axes at any point O expressed analytically in terms of known quantities.

Again, by putting into the quantity $M(r^2 - \delta)$, (which VII. expresses the value of the moment of inertia common to the

cone of perpendiculars whose parameter is δ,) the same three values of δ, we shall have the moments of inertia round the normals to the three surfaces, that is, the three principal moments at O.

Having got, indeed, the directions of the principal axes at any given point, we get at once the principal moments also from the known distances and moments of the parallel axes through the centre of gravity; but the above has the advantage of giving the values of the moments without requiring that the directions of the axes be previously known.

The three values of δ, which thus give the principal axes and the principal moments at any given point O, are the roots of a cubic equation; for, let xyz be the coordinates of O parallel to abc, and $a_{,}a_{,,}a_{,,,}$ the primary semi-axes of the three confocals, where $a_{,}^2 = a^2 + \delta_{,}$, $a_{,,}^2 = a^2 + \delta_{,,}$, $a_{,,,}^2 = a^2 + \delta_{,,,}$, then are these three values of δ obviously got from the equation

$$\frac{x^2}{a^2+\delta} + \frac{y^2}{b^2+\delta} + \frac{z^2}{c^2+\delta} - 1 = 0 \ldots\ldots (\text{VIII.})$$

A cubic in which the successive substitutions for δ in the left-hand member of the quantities $+\infty$, $-c^2$, $-b^2$, and $-a^2$, give results with the respective signs $- + - +$, and which has, therefore, three real roots lying between these limits, as indeed from the nature of moments of inertia it could not but have.

Denoting now by $I_{,}$, $I_{,,}$, $I_{,,,}$ the three principal moments at O, we have

$$\left.\begin{aligned} I_{,} &= M\,.\,(r^2-\delta_{,}) = M\,.\,r^2 - M\,.\,(a_{,}^2-a^2) \\ I_{,,} &= M\,.\,(r^2-\delta_{,,}) = M\,.\,r^2 - M\,.\,(a_{,,}^2-a^2) \\ I_{,,,} &= M\,.\,(r^2-\delta_{,,,}) = M\,.\,r^2 - M\,.\,(a_{,,,}^2-a^2) \end{aligned}\right\} \ldots\ (\text{IX.}),$$

which we therefore know when we can find $a_{,}$, $a_{,,}$, $a_{,,,}$.

22. If the point were such that two of its principal moments of inertia were equal, then should two of these quantities $a_{,}$, $a_{,,}$, $a_{,,,}$ be equal, and this is the case for every point on either of the focal conics of abc. Hence the result obtained before (19), that in every body there exists two curves, loci of points, that admit of an infinite number of principal axes, viz. the focal conics of the ellipsoid of gyration.

If the point were such that the three principal moments were equal, then should $a_{,}$, $a_{,,}$, $a_{,,,}$, be all three equal, or the point should be upon both focal conics; but this in general would be impossible, for the focal conics of an ellipsoid have never a point in common, except in the particular case of a *prolate*

spheroid, where they both meet at the two foci. In the majority of bodies, therefore, there exists no point for which all axes are principal, and even in the limited class to which such points are confined (those, viz. for which two of the three central principal moments are equal and both less than the third) there never exist more than two; these are always on the central principal axis of unequal moment, and at opposite sides of and equidistant from the center of gravity, and are the foci of the ellipsoid of gyration which is then a prolate spheroid, which results were obtained before on other principles in (12).

That in such bodies all axes are principal at the two foci of its ellipsoid of gyration appears also from the general construction for principal axes at any point. For the two particular surfaces confocal with a prolate spheroid which pass through its foci, are both infinitely slender surfaces of revolution, one a prolate spheroid and the other an hyperboloid of two sheets, and the summits of these two surfaces meet at the foci; all planes, therefore, passing through these points are tangent planes to these two particular surfaces, and consequently all lines which pass through either point of cantact are there normals to the same: for both points, therefore, all axes are principal.

Their common distance on the axis of unequal moment A is easily obtained in terms of the two moments A and B; for calling it c, we have $c^2 = a^2 - b^2 = \dfrac{A - B}{M}$, and therefore $c = \pm\sqrt{\left(\dfrac{A - B}{M}\right)}$, its well known value (5).

In the limiting case, when the ellipsoid of gyration is a sphere, that is, in bodies for which AB and C are all three equal, there will obviously be but one such point, the centre of gravity itself.

23. The equation which contains the whole system of equimomental cones at any given point O referred to their principal axes may be readily obtained in terms of $a_{,}$, $a_{,,}$, $a_{,,,}$, the primary semi-axes of the three confocals which pass therethrough.

For, let a_0 be the primary semi-axes of any fourth surface confocal with abc, and I_0 the moment of inertia common to all the sides of the cone reciprocal to that which from O envelopes that surface, then (6) will the equation of that cone referred to its principal axes be

$$(I_{,} - I_0)\,\xi^2 + (I_{,,} - I_0)\,\eta^2 + (I_{,,,} - I_0)\,\zeta^2 = 0,$$

The system of cones contained i
reciprocal to those which from the sa
system of surfaces confocal to abc, w
from the equation of the former system
direct investigation of which is by no
have merely to change the coefficients
their reciprocals, which in the present

$$\frac{\xi^2}{a_{,}^2 - a_0^2} + \frac{\eta^2}{a_{,,}^2 - a_0^2} + \frac{\zeta^2}{a_{,,,}^2 - a_0}$$

for the equation referred to its axes of
any given vertex envelopes the ellipsoid
different values to a_0, of the system of
same vertex envelope the system of
a_0, b_0, c_0, or with abc.

This equation (investigated directly)
Fellow of Trinity College, Dublin,
Mac Cullagh, who obtained it indepe
most successfully employed by the latte
lem of ellipsoidal attraction on an ext
extensively important in the analytic
surfaces, as shewing immediately that f
cones which envelope such a system of
and confocal.*

If the point O be such that the system
are all of revolution, then must some
$a_{,}$, $a_{,,}$, $a_{,,,}$ be equal; this gives us the r
that the loci of the vertices of such a sy
local conics of the ellipsoid of gyration.

If O be the centre of gravity, then will the cones enveloping the system of ellipsoids be all imaginary, while those which envelope the system of hyperboloids of both kinds will become the asymptotic cones of that system of surfaces. Hence in every body the system of cones reciprocal to the system of equimomental cones at its centre of gravity, will be the asymptotic cones to the system of hyperboloids confocal with its ellipsoid of gyration.

Substituting in (XI.) the values of $a_{,}^2$, $a_{,,}^2$, $a_{,,,}^2$, which correspond to this particular point (viz. $a_{,}^2 = a^2 - c^2$, $a_{,,}^2 = a^2 - b^2$, and $a_{,,,}^2 = a^2 - a^2 = 0$), we get the equation of this particular system of cones, viz.

$$\frac{x^2}{a_0^{\,2}} + \frac{y^2}{b_0^{\,2}} + \frac{z^2}{c_0^{\,2}} = 0,$$

the axes of ξ, η, and ζ coinciding here with those of x, y, and z, and $a_0^{\,2}$, $b_0^{\,2}$, $c_0^{\,2}$ being equal respectively to $a^2 + \delta_0$, $b^2 + \delta_0$, $c^2 + \delta_0$, δ_0 being the general value of δ.

This equation is *à priori* evident, for the annihilation of the absolute term in the common equation of a system of confocal surfaces gives that of a system of confocal cones, which are the asymptotic cones to the system of surfaces, and which have obviously for their common focal lines the asymptotes of the focal hyperbola.

As in every other system of confocal cones, these cones intersect two and two at right angles, which is manifest from their equation, or from the known property, that the tangent plane through any side of a cone of the second order makes equal angles with the two vector planes which contain each that side and one of the focal lines of the cone, from which it immediately appears that the two tangent planes through any one of the four intersecting sides of any two confocal cones (which lines evidently make equal angles with the common axes of these cones) make equal angles both with the same two planes, and are therefore at right angles to each other.

24. If the point O be at an infinite distance then will one of the three confocal surfaces which pass through it (the ellipsoid) be an infinite sphere, whose normal at that point will therefore pass through the centre, and the other two (the hyperboloids) will there coincide with their asymptotic cones, and will therefore have at that point the same normals as these cones. Hence, for a point at infinity, the principal axes may be found by a geometric construction quite elementary.

For, draw through the centre of gravity a line in the direction of the infinitely distant point, this will be one of the principal axes for that point; the others will be, of course, in a plane perpendicular to it: to find their directions in that plane, draw through the line two planes passing through the asymptotes to the focal hyperbola of the ellipsoid of gyration, and then two other planes bisecting the two supplemental angles, acute and obtuse, between the former; these latter will intersect the infinitely distant plane in the principal axes sought.

If the direction line of the infinitely distant point be one of the asymptotes themselves, the above construction failing leaves the remaining principal axes indeterminate: but that is precisely what ought to take place, for, at an infinitely distant point, the focal hyperbola coincides with its asymptote, and all points of that curve admit of an infinite number of principal axes.

In the case of an infinitely distant point, the analytic determination of the principal axes is also complete, which, as depending on the solution of a cubic equation, can, in the general case, be hardly considered as such.

For one of the roots of the cubic (VIII.) which gives the three values of δ, (on which everything depends), being in this case infinite, that equation is immediately depressible into a quadratic, and therefore completely solvible.

To find that quadratic, let $\alpha\,\beta\,\gamma$ be the direction angles of the infinitely distant point, and for $x\,y\,z$, in equation VIII., substituting $r\cos\alpha$, $r\cos\beta$, $r\cos\gamma$, let that equation be divided by r^2, and that quantity then made infinite, this will give

$$\frac{\cos^2\alpha}{a^2+\delta}+\frac{\cos^2\beta}{b^2+\delta}+\frac{\cos^2\gamma}{c^2+\delta}=0\ldots\ldots\ldots\ldots(\text{XII.})$$

the quadratic for δ whose roots are the values required.

For an infinitely distant point, the system of enveloping cones (XI.), the reciprocals of the equimomental system of cones for that point will degenerate obviously into a system of cylinders; these being coaxal and confocal, as in the general case, will intersect any plane perpendicular to their common axis in a system of concentric coaxal and confocal conics. Hence, in a body, if we orthographically project upon any plane taken at random, the whole system of surfaces of the second order confocal with its ellipsoid of gyration, the outlines of the projections will form a system of confocal conics, of which the common centre will be the projection of the centre of gravity.

If from any point on the circle locus of the intersections of its pairs of rectangular tangents, we draw two tangents to one of these conics selected at will in the plane of projection, a plane drawn through the point perpendicular to one of these tangents will obviously touch the projected surface. Hence, by (VII.), A_0 being the primary semiaxis of that surface, and r_0 the distance of the point from the centre of gravity, which distance is obviously the same for all points of the circle in question, we have moment of inertia round that tangent $= Mr_0^2 - M(a_0^2 - a^2) =$ a constant quantity. Hence we know that all the tangents to any one of the projected conics are equimomental axes.

And conversely, the envelope in any plane taken at random in a body of a system of equimomental axes will be a conic, an ellipse, or hyperbola, as the case may be. And by giving different values to the common moment of inertia, the whole system of conics in the same plane will be confocal with the orthographic projection on that plane of the ellipsoid of gyration. This, it will be remembered, was proved before from other principles in (9), and as its consequences were there discussed at some length, we need here consider the property no further.

If the infinitely distant point be on the focal hyperbola, the enveloping cylinders being the limit to a system of cones of revolution, will be all of revolution round the focal asymptote, and the projection of the surfaces on any plane, perpendicular to that asymptote, will be all concentric circles, which last was also proved in the article just referred to.

Since for an infinitely distant point $a_{\prime}$ is infinite, the equation of the equimomental system of cones for such a point will be reduced to $\xi^2 = 0$ (as it ought from the analogy of the general case) except in the cases where a_0 is also nearly infinite; these cases are infinite in number and for them all, since the difference between two infinities does not necessarily vanish, we shall have for the curves of intersection of the cones by planes $\xi =$ a very large constant, the equation $\eta^2 + \zeta^2 =$ a constant, which curves, for different values of the constant, are therefore a system of concentric circles.

The directions of the principal axes depending all the while on $a_{\prime\prime}$ and $a_{\prime\prime\prime}$ which are finite and accurately determinable, we have, in the omission of these quantities in comparison with others infinitely great, the explanation of the apparent paradox noticed in (7).

25. In the general case, the point O being anywhere at all. If in the equations of the two systems of cones (X.) and

(XI.) we give to a_0 the particular value $a_0 = a_{,,}$, the equimomental cone (X.) will degenerate into two planes, the common cyclic planes, viz. of that system of cones and the corresponding enveloping cone, consequently into two right lines, these lines, there is no difficulty in seeing, are the two rectilinear generatrices of the hyperboloid of one sheet $a_{,,}$, confocal with the ellipsoid of gyration which passes through O.

In fact, once we know that the enveloping cones are confocal, we readily see what must be their focal lines; but the latter admits of a simple geometric solution, which at the same time proves directly the former.

For, if through any line, tangent to a surface of the second order, we draw two tangent planes to any confocal surface, these planes will make equal angles with the tangent plane to the original surface through the same touching line. This is manifest, for that tangent plane is a principal plane of the cone, which, from its point of contact as vertex, envelopes the confocal surface, and two tangent planes to a cone which intersect in a principal plane make equal angles with that plane.

Making now any point upon an hyperboloid of one sheet the vertex of a cone enveloping any confocal surface, let two planes through the generatrices of the hyperboloid at the point intersect in any side of the cone, these planes being tangent planes to the hyperboloid will make equal angles with the tangent plane to the confocal surface which is drawn through the side of contact, that is, with the tangent plane to the cone through that side.

The generatrices, therefore, possess with respect to the enveloping cone the property, that the vector planes passing through them and intersecting in any side of the cone, make equal angles with the tangent plane through that side. And this being the known property of the focal lines in a cone of the second order tells us that the cones, which from any vertex envelope a system of confocal surfaces, are confocal, having all the same focal lines, viz. the generatrices of the confocal hyperboloid of one sheet which passes through their vertex.

If in the general equation (XI.) for any point O, we put in successively for a_0^2 the quantities $a^2 - c^2$, $a^2 - b^2$, and $a^2 - a^2$, we shall get the three particular cones which from that vertex pass through the focal conics of the ellipsoid of gyration; these cones have been called, by Professor Mac Cullagh, *focal cones*, with respect to the surface of the point from which they diverge; the last of the three is

of course imaginary as passing through an imaginary curve, but the other two, as passing through the focal ellipse and hyperbola, are real: like every other pair of confocal cones they intersect at right angles; being coaxal, their four lines of intersection (which as passing through the two focal conics are called *bifocal lines*) make equal angles with each of the three common axes, that is, with each of the three principal axes at their vertex; and if through the centre three planes be drawn perpendicular to these same three axes, they also will make equal angles with the four bifocal lines, and will intercept on each of those lines three portions, which, measured from the vertex, are equal to the primary semiaxes of the three confocals which pass through that point, the four equal portions intercepted on the four lines by each plane being equal to the primary semiaxes of the surface to whose normal that plane is perpendicular.

These theorems (which readily appear from the equations of the two cones) are due to Professor Mac Cullagh, who has given them in his Examination papers, Dublin University Calendar: indeed the necessary limits of a paper like the present will oblige us, in what follows, to assume for the most part the known properties of confocal surfaces, referring the reader, to whom they may not be familiar, to the above papers, to a paper on Surfaces of the Second Order by the same author, in the Proceedings of the Royal Irish Academy, Part VIII., and to the memoir on the subject in Chasles' History of Geometry.

26. The following geometric construction, evident from the above principles, for determining the three principal axes at any given point O of a body, and also the three quantities $a_{,}$, $a_{,,}$, $a_{,,,}$, which give immediately (IX.) the three principal moments for the same, is perhaps the simplest that the nature of the case admits of, when we consider that the analytic solution of the same problem depends upon finding the roots of a cubic equation.

From O as vertex describe two cones passing through the focal ellipse and hyperbola of the ellipsoid of gyration, and intersecting therefore in four bifocal lines: the three pairs of planes which pass through these four lines will then intersect in the principal axes of O, and the three planes through the centre of gravity perpendicular to the three axes thus determined will intercept on each of the same four lines three portions which measured from O are equal to the primary semi-axes of the three confocals which pass through that point, that is, to $a_{,}$, $a_{,,}$, $a_{,,,}$, the three quantities required.

This construction possesses over those of (18) and (19) the advantages not only of simplicity and of giving the magnitudes of the principal moments at the same time with the directions of the principal axes, but also of being itself necessary to the completion of both; for to find the axes of a given cone of the second order, the simplest geometric construction is perhaps the following, which is immediately and evidently derivable from the above property of every system of two cones, which from a common vertex pass through two conics, of which one is the focal of the other: taking arbitrarily any plane section of the cone, describe the conic focal of that section; this will pierce the cone in four points; join these with the vertex, and through the four joining lines send the three pairs of planes which contain them two and two; each pair will intersect in an axis of the cone.

In the particular case when the point is at an infinite distance, we have seen (24) that the construction for principal axes becomes exceedingly simple, and other particular cases will be presently noticed in which the construction is also elementary: of this the reason is evident; for in the general case, to find the principal axes at a given point depends on finding the axes of a surface of the second order; which problem, though always reducible to finding the axes of a cone, cannot be solved by elementary geometry: on the contrary, in all the particular cases for which the construction is simple, we shall see that one principal axis is always given, and thus to find the others depends only on finding the axes of a given conic, and for this the construction is of course elementary.

[*To be continued*].

ON THE INTEGRATION OF CERTAIN EQUATIONS IN FINITE DIFFERENCES.

By the Rev. Brice Bronwin.

The method of integrating certain equations in finite differences, which is illustrated in this paper by a few examples, is, I believe, quite new. The process is moreover very simple and easy. A few subsidiary formulæ are required, which must first be given.

$$\left.\begin{aligned} x\Delta^n y &= \Delta^n \{(x-n)y\} - n\Delta^{n-1}y \\ x^2\Delta^n y &= \Delta^n\{(x-n)^2 y\} - n\Delta^{n-1}\{(2x-2n+1)y\} + n(n-1)\Delta^{n-2}y \end{aligned}\right\}\ldots(a).$$

Substitute for

$$\Delta^n\{(x-n)y\},\ \Delta^n\{(x-n)^2 y\},\ \Delta^{n-1}\{(2x-2n+1)y\}$$

their values given by the known theorem

$$\Delta^n (PQ) = Q\Delta^n P + \frac{n}{1}\Delta Q \Delta^{n-1} P_1 + \frac{n(n-1)}{1.2}\Delta^2 Q \Delta^{n-2} P_2 + \&c.,$$

and these formulæ will be verified,

$$\left.\begin{aligned} x\Delta^{-n}y &= \Delta^{-n}\{(x+n)y\} + n\Delta^{-n-1}y \\ x^2\Delta^{-n}y &= \Delta^{-n}\{(x+n)^2 y\} + n\Delta^{-n-1}\{(2x+2n+1)y\} + n(n+1)\Delta^{-n-2}y \end{aligned}\right\}\ldots(b).$$

These may be verified in a similar manner to the last by the known theorem

$$\Sigma^n (PQ) = Q\Sigma^n P - \frac{n}{1}\Delta Q \Sigma^{n+1} P_1 + \frac{n(n+1)}{1.2}\Delta^2 Q \Sigma^{n+2} P_2 - \&c.$$

We are now prepared to proceed to some examples.

Let $(1-x^2)\Delta^2 y + p(p+1)y = 0$, ($p$ a positive integer)...(1).

Make $y = \Delta^{p-1}u$, and the above becomes

$$\Delta^{p+1}u - x^2\Delta^{p+1}u + p(p+1)\Delta^{p-1}u = 0.$$

Putting for $x^2\Delta^{p+1}u$, its value given by the second of (a), there results

$$\Delta^{p+1}\{1-(x-p-1)^2\}u + (p+1)\Delta^p(2x-2p-1)u = 1;$$

or $\Delta\{1-(x-p-1)^2\}u + (p+1)(2x-2p-1)u = \Delta^{-p}0.$

But we will neglect $\Delta^{-p}0$, and find each particular integral separately. The last result may be put under the form

$$\{1-(x-p)^2\}u_{x+1} - \{1+p+p^2-x^2\}u_x = 0.$$

Make $\dfrac{1+p+p^2-x^2}{1-(x-p)^2} = f(x+1)$. Then changing x into $x-1$, we have $u_x - f(x)u_{x-1} = 0$; and integrating we obtain

$$u_x = Cf(x)f(x-1)f(x-2)\ldots\ldots = CPf(x) \text{ suppose,}$$

which is the integral always given.

Again, make $y = \Delta^{-p-2}u$. With this value (1) becomes

$$\Delta^{-p}u - x^2\Delta^{-p}u + p(p+1)\Delta^{-p-2}u = 0.$$

Put for $x^2\Delta^{-p}u$ its value from the second of (b), and we have

$$\Delta^{-p}\{1-(x+p)^2\}u - p\Delta^{-p-1}(2x+2p+1)u = 0,$$

or $$\Delta\{1-(x+p)^2\}u - p(2x+2p+1)u = 0;$$

which, as before, may be put under the form

$$\{1-(x+p+1)^2\}u_{x+1} - (1+p+p^2-x^2)u_x = 0.$$

Make $\dfrac{1+p+p^2-x^2}{1-(x+p+1)^2} = f_1(x+1)$, and change x into $x-1$; the last equation becomes $u_x - f_1(x)u_{x-1} = 0$, which give $u_x = C_1 P f_1(x)$.

Therefore the complete integral of (1) is

$$y = C\Delta^{p-1}Pf(x) + C_1\Delta^{-p-2}Pf_1(x).$$

Let $x^2\Delta^2 y + m\Delta y - p(p+1)y = 0$, ($p$ a positive integer)...(2).

Here again make $y = \Delta^{p-1}u$, and we have

$$x^2\Delta^{p+1}u + m\Delta^p u - p(p+1)\Delta^{p-1}u = 0;$$

Or, by putting for $x^2\Delta^{p+1}u$ its value given by (a),

$$\Delta^{p+1}(x-p-1)^2 u - (p+1)\Delta^p(2x-2p-1)u + m\Delta^p u = 0;$$

or $\Delta(x-p-1)^2 u - \{(p+1)2x - (p+1)(2p+1) - m\} u = 0$;

and $\qquad (x-p)^2 u_{x+1} - \{x^2 - p(p+1) - m\} u_x = 0.$

Make $\dfrac{x^2 - p(p+1) - m}{(x-p)^2} = f(x+1)$; and we find, as before,

$$u_x - f(x)u_{x-1} = 0,\ u_x = CPf(x).$$

Again make $y = \Delta^{-p-2}u$, and (2) will become

$$x^2\Delta^{-p}u + m\Delta^{-p-1}u - p(p+1)\Delta^{-p-2}u = 0.$$

By the same steps as before, we have

$$\Delta^{-p}(x+p)^2 u + p\Delta^{-p-1}(2x+2p+1)u + m\Delta^{-p-1}u = 0,$$

$$\Delta(x+p)^2 u + (2px + 2p^2 + p + m)u = 0,$$

$$(x+p+1)^2 u_{x+1} - \{x^2 - p(p+1) - m\} u_x = 0.$$

And if $\dfrac{x^2 - p(p+1) - m}{(x+p+1)^2} = f_1(x+1)$; then $u_x - f_1(x)u_{x-1} = 0$,

and $u_x = C_1Pf_1(x)$. Therefore the complete integral of (2) is

$$y = C\Delta^{p-1}Pf(x) + C_1\Delta^{-p-2}Pf_1(x).$$

Let $\qquad \Delta^2 y + mx\Delta y + pmy = 0 \ldots\ldots\ldots\ldots (3).$

Make $y = \Delta^{p-1}u$; then we have successively

$$\Delta^{p+1}u + mx\Delta^p u + pm\Delta^{p-1}u = 0,$$

$$\Delta^{p+1}u + m\Delta^p(x-p)u = 0,$$

$$\Delta u + m(x-p)u = \Delta^{-p}0;$$

which we know how to integrate, and which will give the complete integral of the proposed at once. We cannot in this example obtain two particular integrals separately. The supernumerary arbitraries contained in

$$\Delta^{-p}0 = \Delta^{-p+1}a = \Delta^{-p+2}(ax+b) = \&c.,$$

must be determined by substituting the value found for y in the given equation. And the same thing must be done with regard to $C_1\Delta^{-p-2}Pf(x)$ in examples (1) and (2), and in all similar cases.

Let $\qquad x\Delta^2 y + mx\Delta y + pmy = 0 \ldots\ldots\ldots\ldots (4).$

Make $y = \Delta^{p-1}u$; then $x\Delta^{p+1}u + mx\Delta^p u + pm\Delta^{p-1}u = 0$,

and $\Delta^{p+1}(x-p-1)u-(p+1)\Delta^{p}u+m\Delta^{p}(x-p)u=0$,

$$\Delta(x-p-1)u-(1+p+mp-mx)u=\Delta^{-p}0,$$

or $$(x-p)\Delta u-(p+pm-mx)u=\Delta^{-p}0.$$

As we know how to integrate this, I shall not proceed with it further.

Let $(1+x^2)\Delta^2 y+mx\Delta y+p(m-p-1)y=0\ldots.(5)$, where it must be remembered p, as always, is a positive integer. Make $y=\Delta^{p-1}u$, and we have by the same steps as heretofore

$$\Delta^{p+1}u+x^2\Delta^{p+1}u+mx\Delta^{p}u+p(m-p-1)\Delta^{p-1}u=0,$$

$$\Delta^{p+1}u+\Delta^{p+1}(x-p-1)^2u-(p+1)\Delta^{p}(2x-2p-1)u+m\Delta^{p}(x-p)u=0,$$

$$\Delta\{(x-p-1)^2+1\}u-\{(2p-m+2)x-(p+1)(2p+1)+pm\}u=\Delta^{-p}0,$$

or $\{(x-p)^2+1\}u_{x+1}-\{x^2-mx-p(p-m+1)+1\}u_x=\Delta^{-p}0$, which will give the complete integral. We cannot find two particular ones separately.

The next three examples give only particular integrals.

Let $$\Delta^2 y-mx\Delta y+pmy=0\ldots\ldots\ldots\ldots(6).$$

Make $y=\Delta^{-p-1}u$; then $\Delta^{-p+1}u-mx\Delta^{-p}u+pm\Delta^{-p-1}u=0$,

and $$\Delta^{-p+1}u-m\Delta^{-p}(x+p)u=0,$$

whence we obtain either

$$\Delta u-m(x+p)u=0,\quad\text{or}\quad\Delta u-m(x+p)u=\Delta^{p}0,$$

$\Delta^{p}0$ is an actual number; but as the characteristic Δ has reference only to the variable x, we must here make $\Delta^{p}0=0$, and therefore shall only have a particular integral.

Let $$x\Delta^2 y-mx\Delta y+pmy=0\ldots\ldots\ldots\ldots(7).$$

Make $y=\Delta^{-p-1}u$, and the proposed becomes

$$x\Delta^{-p+1}u-mx\Delta^{-p}u+pm\Delta^{-p-1}u=0,$$

Or $\Delta^{-p+1}(x+p-1)u-\Delta^{-p}(mx+pm-p+1)u=0$,

and $\Delta(x+p-1)u-(mx+pm-p+1)u=0.$

Or $(x+p)u_{x+1}-(mx+x+pm)u_x=0$,

which gives a particular integral.

Let $(1+x^2)\Delta^2 y+mx\Delta y-p(m+p-1)y=0\ldots.(8)$.

Make $y=\Delta^{-p-1}u$; then

$$\Delta^{-p+1}u+x^2\Delta^{-p+1}u+mx\Delta^{-p}u-p(m+p-1)\Delta^{-p-1}u=0,$$

and $\Delta^{-p+1}u+\Delta^{-p+1}(x+p-1)^2u$

$$+\Delta^{-p}\{(2p+m-2)x+(p-1)(2p-1)+pm\}u=0,$$

$$\Delta\{(x+p-1)^2+1\}u+\{(2p+m-2)x+(p-1)(2p-1)+pm\}u=0,$$

$$\{(x+p)^2+1\}u_{x+1}-\{x^2-mx-p(p+m-1)-1\}u_x=0.$$

For let the corners of the quadrilateral be denoted by the letters A, B, C, D, and let the side from A to B be cut in two points A′ and B″, while the three other sides are cut in three other pairs of points, which may be called B′ and C″, C′ and D″, and D′ and A″ respectively. Then, if the arcs from A′ to C′ and from B′ to D′ be commedial portions of one common great circle, or of a first transversal arc, the arcs from A′ to B′ and from D′ to C′ will be ***symbolically equal arcs***, in the sense of the preceding article; and therefore, in the notation of that article, we may now write the equation

$$\frown B'A' = \frown C'D' \ldots\ldots\ldots\ldots (133).$$

In like manner the conditions, that the four portions of the sides of the quadrilateral shall be commedial with the sides themselves, give the four other equations of the same kind,

$$\left.\begin{array}{l}\frown A'A = \frown BB''; \quad \frown B'B = \frown CC'';\\ \frown C'C = \frown DD''; \quad \frown D'D = \frown AA''.\end{array}\right\} \ldots (134).$$

Hence, by alteration and inversion, we find that the five successive sides

$$\frown AB'', \frown D'A, \frown C'D', \frown CC', \frown C''C,$$

of the spherical hexagon B″AD′C′CC″ are respectively and symbolically equal to the five successive diagonals

$$\frown A'B, \frown DA'', \frown B'A', \frown D''D, \frown BB',$$

of the other hexagon BA″A′DB′D″; and therefore, by the theorem of the two hexagons, the sixth side of the former figure must be symbolically equal to the sixth diagonal of the latter; that is, we may write the symbolical equation,

$$\frown B''C'' = \frown A''D'' \ldots\ldots\ldots\ldots (135).$$

But this expresses a relation equivalent to the following, that the two arcs from A″ to C″ and from B″ to D″ are commedial portions of one common great circle, or second transversal arc, which was the thing to be proved.

Reciprocally, the associative principle of geometrical multiplication, in so far as it relates to the directions of straight lines in space, may be expressed by the assertion that the symbolical equation between arcs (135) is a consequence of the five other equations of the same kind (133) and (134); this principle of symbolical geometry may therefore be so interpreted as to coincide with the foregoing *theorem of the two conjugate transversals* of a spherical quadrilateral, instead of the theorem of the two spherical hexagons. It is easy to see that to a given quadrilateral correspond infinitely many such pairs of conjugate transversal arcs; and those readers

who are familiar with the theory of *spherical conics** will recognise in these conjugate transversals, A′B′C′D′, A″B″C″D″, the two *cyclic arcs* of such a conic, circumscribed about the proposed quadrilateral ABCD; but it suits better the plan of this communication on symbolical geometry to pass at present to another view of the subject.

It may however be noticed here, that in the first of the two hexagons already mentioned, *any two pairs of opposite sides intercept commedial portions on either of the two sides remaining;* and that the associative principle asserts that *if* a spherical hexagon have *five* of its sides thus *cut commedially*, the *sixth* side also will be cut in the same way. Or, because the two sets of alternate diagonals of the second hexagon are sides of two triangles, which have for their corners the alternate corners of this hexagon, we may in another way eliminate this second hexagon, and may express the same principle of spherical geometry by saying, that *if one set of alternate sides of a* (first) *spherical hexagon,* taken in their order (as first, third, and fifth), *be respectively and symbolically equal to the three successive sides of a triangle,* then the *other set of alternate sides of the same hexagon will be* in like manner *symbolically equal to the sides of another triangle.* This last interpretation of the associative principle is even more immediately suggested than any other, by the forms of the

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* The plane of the first side of the quadrilateral, or the plane of OAB, if O denote the centre of the sphere, is cut by the plane of the first transversal arc in the radius A′O, and by the plane of the second transversal arc in the radius B″O. Thus the four plane faces of the tetrahedral angle, of which the four edges are the four radii from O to the four corners A, B, C, D of the quadrilateral, are cut by any secant plane parallel to the plane of the first transversal arc in four indefinite straight lines, which are respectively parallel to the four other radii A′O, B′O, C′O, D′O of the sphere; and consequently, in virtue of the equation (133), between the arcs which these last radii include, these four new lines in one common secant plane have the angular relation required for their being the (prolonged) sides of a (plane) quadrilateral inscribed in a circle; therefore the four edges of the same tetrahedral angle are cut by the same secant plane in points which are on the circumference of a circle; therefore they are edges or sides of a cone which has this circle for its base, and has its vertex at the centre of the sphere. But the intersection of such a cone with such a concentric sphere is called a *spherical conic;* a plane through its vertex, parallel to its circular base, is called a *cyclic plane;* and the intersection of this latter plane with the sphere has received the designation of a *cyclic arc*. Therefore the first transversal arc A′B′C′D′ is (as asserted in the text) a cyclic arc of a spherical conic circumscribed about the quadrilateral ABCD: and by a reasoning of exactly the same kind it may be proved, that the second transversal A″B″C″D″ is another cyclic arc of the same conic, or that its plane is a second cyclic plane, being parallel to the plane of another (or *subcontrary*) circular section.

equations (131) (132); in the notation of the present article, *the two triangles* are BA'B' and A''DD'', which may be considered as having their *bases* A'B' and A''D'' *on the two cyclic arcs* above alluded to, while their *vertical angles* at B and D may be said to be *angles in the same segment* (or in alternate segments) *of the spherical conic:* since, by (134), the two arcual sides BA', BB' of the one angle intersect respectively the two sides DA'', DD'' of the other angle, in the points A and C, which points of intersection, as well as the vertices B and D, are corners of the quadrilateral inscribed in that spherical conic.

Symbolical Addition of Arcs upon a Sphere; Associative and Non-commutative Properties of such Addition.

19. The foregoing geometrical interpretations of the associative principle or property of the multiplication of geometrical fractions, may assist us in forming and applying the conception of the symbolical addition of arcs of great circles upon a sphere, and in establishing and interpreting an analogous principle or property of such symbolical addition.

As it has been already proposed in the third article of this paper, and also in the works of other writers on subjects connected with the present, to adopt, for the *addition of straight lines having direction,* a rule expressed by the formula

$$\mathrm{CB} + \mathrm{BA} = \mathrm{CA} \;\ldots\ldots\ldots\; (7),$$

in whatever manner the three points ABC may be situated or related to each other; so it seems natural to adopt now, for the analogous *addition of arcs upon a sphere,* when directions as well as lengths are attended to, the corresponding formula,

$$\frown\mathrm{CB} + \frown\mathrm{BA} = \frown\mathrm{CA} \;\ldots\ldots\; (136).$$

Admitting this latter formula as *the definition of the effect of the sign + when inserted between two such symbols of arcs,* and granting also that it is permitted, in any such formula, to substitute for any arcual symbol another which is *equal* thereto, we shall have, by the two first and two last equations (134) respectively, the two following other equations,

$$\left.\begin{array}{l} \frown\mathrm{B''C''} = \frown\mathrm{AA'} + \frown\mathrm{B'C} \\ \frown\mathrm{A''D''} = \frown\mathrm{AD'} + \frown\mathrm{C'C} \end{array}\right\} \ldots\; (137).$$

The two sums in these second members will therefore be symbolically equal if we have the equation

$$\frown\mathrm{A'D'} = \frown\mathrm{B'C'} \;\ldots\ldots\ldots\; (138),$$

because (135) has been seen to follow from (133) and (134). But by (136) and (138), we have the expression

$$\frown B'C = \frown A'D' + \frown C'C \;\ldots\ldots (139);$$

consequently the associative principle of multiplication, considered in several recent articles, when combined with the *formula of arcual addition* (136), conducts to the following formula,

$$\frown AA' + (\frown A'D' + \frown C'C) = (\frown AA' + \frown A'D') + \frown C'C \ldots (140),$$

or, as it may be more concisely written,

$$\frown''' + (\frown'' + \frown') = (\frown''' + \frown'') + \frown' \ldots\ldots (141):$$

which in its form agrees with ordinary algebra, and may be said to express the *associative principle of the symbolical addition of arcs;* since the three arcs added in (140) or (141) may be any three arcs of great circles upon one common spheric surface. It is remarkable that so much geometrical meaning should be contained in so simple and elementary a form; for this form (141), which is *apparently an algebraic truism,* and has been here deduced from the associative principle of multiplication of geometrical fractions, may reciprocally be substituted for it, and therefore includes in its interpretation, *if we adopt the symbolical definition* (136) of the effect of + between two symbols of arcs, all those theorems respecting spherical great circles, triangles, quadrilaterals, hexagons, and conics, which have been deduced or mentioned as geometrical results of the associative principle in the two foregoing articles. And this encouragement to adopt the foregoing very simple definition (136) of the meaning of a symbol such as $\frown'' + \frown'$, is the more worthy of attention, because the *same definition* conducts to a *departure from the ordinary rules of symbolical addition* in *another* important point; since, when combined with the *definition of symbolical equality between arcs* assigned in the 17th article, it shews that *addition of arcs is in general a non-commutative operation.* For if we conceive two arcs of different great circles on one sphere, from A to B and from C to D, to bisect each other in a point E, we shall then have the two symbolical equations

$$\frown AE = \frown EB, \quad \frown CE = \frown ED \;\ldots\ldots (142);$$

and therefore, whereas by (136),

$$\frown AE + \frown ED = \frown AD \;\ldots\ldots (143),$$

the result of the addition of the same two arcs, taken in a different order, will be

$$\frown ED + \frown AE = \frown CB \;\ldots\ldots (144).$$

And although the two *sum-arcs*, $\frown$AD and $\frown$CB, thus obtained, connecting two opposite pairs of extremities of the two commedial arcs $\frown$AB and $\frown$CD, are *equally long*, yet they are in general *parts of different great circles*, and therefore *not symbolically equal* in the sense of the 17th article. This result, which may at first sight seem a paradox, illustrates and is intimately connected with the analogous result obtained in the 13th article, respecting the general non-commutativeness of geometrical multiplication; for we shall find that there exists a species of *logarithmic connexion* between arcs situated in different great circles on a sphere and fractional factors belonging to different planes, which is analogous to, and includes as a limiting case, the known connexion between ordinary imaginary logarithms and angles in a single plane. It may be here remarked, that with the same definition (136) *in any symbolical addition of three successive arcs, the two partial sum-arcs,*

$$\frown'' + \frown' \text{ and } \frown''' + \frown'' \;\ldots\ldots\; (145),$$

are portions of the cyclic arcs of a certain spherical conic, circumscribed about a quadrilateral which has

$$\frown', \frown'', \frown''', \text{ and } \frown''' + \frown'' + \frown' \;\ldots.\; (146),$$

that is, *the three proposed summand-arcs and their total sum-arc, for portions of its four sides*, or of those sides prolonged; as will appear by supposing that the three summands, $\frown'$, $\frown''$, $\frown'''$, coincide respectively with the arcs $\frown$CC′, $\frown$B′C, $\frown$AA′, in the notation of the preceding article.

[*To be continued.*]

ON THE THEORY OF INVOLUTION IN GEOMETRY.

By Arthur Cayley.

When three conics have the same points of intersection, any transversal intersects the system in six points, which are said to be in involution. It appears natural to apply the term to the conics themselves; and then it is easy to generalize the notion of involution so as to apply it to functions of any number of variables. Thus, if $U, V\ldots$ be homogeneous functions of the same order of any number of variables $x, y\ldots$. A function Θ, which is a linear function of $U, V\ldots$, is said to be in involution with these functions. More generally Θ may be said to be in involution with any

system of factors of these functions: or if $U, V \ldots$ be given functions of $x, y, z \ldots$, homogeneous of the degrees $m, n \ldots$, and $u, v \ldots$ arbitrary homogeneous functions of the degrees $r-m$, $r-n \ldots$; then, if $\Theta = uU + vV + \ldots$, Θ is a function of the degree r, which is in involution with $U, V \ldots$. The question which immediately arises, is to find the degree of generality of Θ, or the number of arbitrary constants which it contains. And this is a question which, from the variety and interest of its geometrical interpretations, has very frequently been treated of by geometers, though never, I believe, in quite so general a form, (the number r has almost always had particular values given to it, except in a short paper of my own, on the particular case of two curves, in the *Journal*, III. 211).* There is also an analytical application of the theory, of considerable interest, to the problem of elimination between any number of equations containing the same number of variables. Suppose, for instance, two equations, $U = 0$, $V = 0$, when U, V are homogeneous functions of x, y of the degrees m, n respectively. To eliminate the variables it is sufficient to multiply the first equation by $x^{n-1}, x^{n-2}y \ldots, y^{n-1}$, and the second by $x^{m-1} \ldots, y^{m-1}$, and from the equations so obtained to eliminate linearly the $(m+n)$ quantities $x^{m+n-1}, x^{m+n-2}y \ldots, y^{m+n-1}$. But in the case of a greater number of equations it is not at first obvious how many new equations should be obtained; and when a number apparently sufficiently great have been found, it may happen that the equations so obtained are not independent, and that

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* The first suggestion of the problem is contained in a memoir of Euler's —"Sur une contradiction apparente dans la doctrine des lignes courbes." *Mem. de Berlin*, tom. IV. p. 219, 1748. It is noticed also in Cramer's *Introduction a l'analyse des lignes courbes*. The following memoirs also have been published on the subject. Plücker, "Recherches sur les courbes algebraiques de tous les degrès," *Gerg. Ann.* tom. XIX. p. 97; "Recherches sur les surfaces algebraiques de tous les degrès," p. 129. (A great number of memoirs on particular applications of the theory are contained in Gergonne.) "Jacobi de relationibus quæ locum habere debent inter puncta intersectionis duarum curvarum vel trium superficierum dati ordinis, simul cum enodatione paradoxi Algebraici."—*Crelle*, tom. XV. Plücker, "Theorèmes generaux concernant les equations d'un degrè quelconque entre un nombre quelconque d' inconnues."—*Crelle*, tom. XVI. (But this last must be read with caution, as several of the theorems are incorrect, or at least stated without the proper limitations.) And the *Einleitende Betrachtungen*, in Plücker's "Theorie der Algebraischen Curven." The following memoirs of Hesse, containing developments relative to the case of three surfaces of the second order, may likewise be mentioned, "De curvis et superficiebus secundi gradus," *Crelle*, tom. XX. p. 285; and "Ueber die lineare Construction des achten Schnitt-punctes dreier Oberflächen zweiter Ordnung, wenn Sieben Schnitt-puncte derselben gegeben sind," *Crelle*, tom. XXVI. p. 147.

the elimination cannot be performed. But in shewing the connexion that exists between these different equations, the theory of involution explains in what manner a system is to be formed, which includes all the really independent equations, and gives the means of detecting the extraneous factors which appear in the result of the linear elimination of the different terms of these; but I do not see at present any mode of obtaining the final result at once in its reduced form free from any extraneous factors.

Let X, Y, ... be given homogeneous functions of the same degree of any number of variables, and suppose

$$\Theta = \alpha X + \beta Y + \ldots,$$

$\alpha, \beta \ldots$ being constants, and the number of terms in the series being g; Θ contains therefore g arbitrary constants. If however, by giving to $\alpha, \beta \ldots$ particular values $\alpha_1, \beta_1, \ldots$, or $\alpha_2, \beta_2, \ldots$, and representing by $\Theta_1, \Theta_2 \ldots$ the corresponding values of Θ, we have *identically*

$$\Theta_1 = 0,\ \ \Theta_2 = 0, \ldots \ (h \text{ equations});$$

then the constants in Θ group themselves together into a smaller number $g - h$ of arbitrary constants. This supposes, however, that the equations (2) are linearly independent, if there are a certain number (k) of equations

$$\Phi_1 = 0, \quad \Phi_2 = 0 \ldots,$$

(where $\Phi_1, \Phi_2, \ldots$ are linear functions of $\Theta_1, \Theta_2 \ldots$) which are identically satisfied, independently of the equations (2), then the equations (2) are equivalent to $h - k$ equations, and the function Θ contains $g - (h - k)$ or $g - h + k$, arbitrary constants. Similarly if the functions Φ are not independent; so that the number of arbitrary constants really contained in Θ is always

$$N = g - h + k - \&c. \ldots\ldots$$

Consider now the case of a function Θ, homogeneous of the rth degree in the variables $x, y \ldots \{(\theta + 1) \text{ in number}\}$. Let $U, V \ldots$ be functions of the degrees $m, n \ldots$, and suppose

$$\Theta = uU + vV + \ldots$$

where $u, v \ldots$ are arbitrary functions of the degrees $r - m$, $r - n, \ldots$ [r is supposed throughout greater than $m, n \ldots$]. Suppose for shortness that the number of terms in the complete function of θ variables, and of the order ρ, or the quotient $\dfrac{[\rho + \theta]^\theta}{[\theta]^\theta}$ is represented by $[\rho, \theta]$.

Then the function Θ contains apparently a number

$$([r-m,\theta]+[r-n,\theta]+\ldots)$$

of arbitrary constants.

But since we should have identically $\Theta=0$ by assuming $u=LV$, $v=-LU$, $w=0$, &c... (L the general function of the order $r-m-n$), or $u=MW$, $v=0$, $w=-MU$ (M the general function of the order $r-m-p$) &c., the number N must be diminished by

$$[r-m-n,\theta]+[r-m-p,\theta]+[r-n-p,\theta]+\ldots.$$

But the equations just obtained are themselves not linearly independent, and in consequence of this the number of arbitrary constants has to be increased by $[r-m-n-p,\theta]+\ldots$ and so on. So that finally the whole number of arbitrary constants in the function θ is

$$\begin{aligned} N=[r-m,\theta]+[r-n,\theta]+[r-p,\theta]+\ldots\\ -[r-m-n,\theta]-[r-m-p,\theta]-[r-n-p,\theta]-\ldots\\ +[r-m-n-p,\theta]+\ldots\ldots\ldots\ldots(\text{A}). \end{aligned}$$

This however supposes that all the numbers $r-m$, $r-n$.. $r-m-n$,.... are positive: whenever this is not the case for any one of them, the corresponding term is obviously to be omitted. With this convention the equation (A) gives always the correct number of arbitrary constants in Θ: it will be convenient to represent it in the abbreviated form

$$N=\{r:m,n,p,\ldots:\theta\}.$$

An expression analogous to this, for the particular case of $r=m$, but incorrect on account of the omission of all the terms after the second line, has been given by M. Plücker (*Crelle*, tom. p.), and even some of his particular formulæ are incorrect. But proceeding to examine some particular cases: if $r>m+n+p+..-\theta-1$, then in the expression (A) either no terms are to be omitted, or else the terms to be omitted reduce themselves to zero, so that N is given by this formula continued to its last term. It will be subsequently shewn that in this case

$$\{r:m,n,p\ldots:\theta\}=[r,\theta]-mnp\ldots$$

Or in the case of two or three variables, we have the theorem, "If a curve or surface of the order r be determined to pass through the mn points of intersection of two curves of the orders m and n, or the mnp points of intersection of three surfaces of the orders m, n, p; then if $r>m+n-3$, or $r>m+n+p-4$, the curve or surface contains precisely the same number of arbitrary constants as if the mn or mnp points were perfectly arbitrary."

This is natural enough; the peculiarity is in the case where $r \not> m + n - 3$, or $r \not> m + n + p - 4$. For instance, for two curves, $r \not> m + n - 3$, we have

$$\{r : m, n : 2\} = [r - m, 2] + [r - n, 2]$$
$$= [r, 2] - mn + [r - m - n, 2].$$

Or the new curve contains $\frac{1}{2}[m + n - r - 1]^2$ more arbitrary constants than it would do if the mn points, through which it was made to pass, had been perfectly arbitrary; a result given before in the *Journal.*

In the case of surfaces, if $r \not> m + n + p - 4$. Then assuming $r > m + n - 4$, $m + p - 4$, or $n + p - 4$, we have

$$\{r : m, n, p : 3\} = [r - m, 3] + [r - n, 3] + [r - p, 3]$$
$$- [r - m - n, 3] - [r - m - p, 3] - [r - n - p, 3]$$
$$= [r, 3] - mnp - [r - m - n - p, 3].$$

Or the surface contains $\frac{1}{6}[m + n + p - r - 1]^3$ more arbitrary constants than it would do if the mnp points, through which it was made to pass, had been perfectly arbitrary. Similarly, in the case where r is not greater than one or more of the quantities $m + n - 4$, $m + p - 4$, $n + p - 4$. Thus in particular, if r be not greater than the least of these quantities

$$\{r : m, n, p : 3\} = [r, 3] - mnp + [r - n - p, 3] + [r - m - p, 3]$$
$$+ [r - m - n, 3] - [r - m - n - p, 3].$$

Or the surface contains

$$\tfrac{1}{6}[m + n + p - r - 1]^3 + \tfrac{1}{6}[n + p - r - 1]^3 + \tfrac{1}{6}[m + p - r - 1]^3$$
$$+ \tfrac{1}{6}[m + n - r - 1]^3$$

more arbitrary constants than it would otherwise have done. Again, for a surface of the r^{th} order, subjected to pass through the curve of intersection of two surfaces of the orders m, n,

$$\{r : m, n, 3\} = [r - m, 3] + [r - n, 3] - [r - m - n, 3].$$

In which the last term, or $\frac{1}{6}[m + n - r - 1]^3$, is to be omitted when $r \not> m + n - 4$.

The function of the r^{th} order, which is satisfied by the systems of values which satisfy the equations of the orders $m, n..$ contains, we have seen, $[r, m, n, p..\theta]$ arbitrary constants; hence it may be determined so as to pass through this number, diminished by unity, of arbitrary points. But the equation being determined in general by the condition of being satisfied by $[r, \theta] - 1$ systems of variables, it will be completely determined if, in addition to the above number

of arbitrary systems, we suppose it to be satisfied by a number $N = [r, \theta] - \{r; m, n, p..:\theta\}$ of systems satisfying the equations above. Hence the theorem

"The equation of the r^{th} order which is satisfied by a number $N = [r, \theta] - \{r; m, n, p..:\theta\}$ of systems satisfying the equations of the orders $m, n, p..$ is satisfied by any systems whatever which satisfy these equations."

In particular—"The surface of the r^{th} order which passes through $[r, \theta] - \{r: m, n:\theta\}$ of points in the curve of intersection of two surfaces of the orders m, n—or through $[r, \theta] - [r; m, n, p:\theta\}$ of the mnp points of intersection of three surfaces of the orders m, n, p—passes through the curve of intersection, or through the mnp points of intersection."

Thus a surface of the second order which passes through eight points of the curve of intersection of two surfaces of the second order passes through this curve; and any surface of the second order which passes through seven of the points of intersection of three surfaces of the second order passes through the eighth point. (The first theorem obviously fails if the eight points have the relation in question, *i.e.* if they are the eight points of intersection of three surfaces of the second order.)

Again—"The curve of the r^{th} order which passes through $[r, \theta] - \{r: m, n:\theta\}$ of the points of intersection of two curves of the orders mn, passes through the remaining points of intersection." *e.g.* Any curve of the third order which passes through eight of the points of intersection of two curves of the third order, passes also through the ninth point.

Consider next the following question, which has been treated of by Jacobi in the memoir already quoted. "To find the number of relations which must exist between $K(\theta+1)$ variables, forming K systems, each of which satisfies simultaneously equations of the orders $m, n, p..$ respectively; the number ϕ of these equations being anything less than θ; or ϕ being equal to θ, provided at the same time $K = mnp..$

Suppose $m \nless n, n \nless p..$ and write

$$[m, \theta] - \{m: m, n, p..:\theta\} = N,$$
$$[n, \theta] - \{n: n, p....:\theta\} = N',$$
&c.

Imagine the equations of the orders $n, p..$ given. Any function of the m^{th} order which is satisfied by N of the systems of values which satisfy the given equations, and any

particular equation of the m^{th} order, is satisfied by the remaining $K-N$ systems of values. Hence assuming N systems, satisfying the equations of the orders $n, p \ldots .$ but otherwise arbitrary, the remaining systems must satisfy these equations, and a completely determinate equation of the m^{th} order; *i.e.* there must be ϕ relations between the variables of each system, and consequently $\phi(K-N)$ relations in all. Similarly, if the equations of the orders $p . .$ were given, N' systems of variables might be assumed satisfying these equations, but otherwise arbitrary; the remaining $N-N'$ systems satisfy $(\phi - 1)$ determinate equations, or the number of relations between the variables is $(\phi-1)(N-N') . .$; continuing in the same manner the total number of relations between the variables is

$$\phi(K-N)+(\phi-1)(N-N')+(\phi-2)(N'-N'')+\ldots .$$

in which however any term $(\phi-1)(N-N')$ or $(\phi-2)(N-N')$ ....&c., which becomes negative, *must be omitted.* It is obvious that we may write more simply

$$N=[m, \theta]-1-\{m ; n, p . . \theta\},$$

$$N'=[n, \theta]-1-\{n ; p \ldots . \theta\}, \text{ \&c.}$$

In particular, to find the relations which must exist between the coordinates of mn points in order that they may be the points of intersection of two curves of the orders m, n respectively: here $K=mn$, $N=\frac{1}{2}[m+2]^2-\frac{1}{2}[m-n+2]^2=\frac{1}{2}(2mn-n^2+3n)$, $N'=\frac{1}{2}(n^2+3n+2)$, so that $N-N'=m(m-n)-1$ which becomes negative when $m=n$; hence in general the required number of conditions is $mn-3n+1$, but when $m=n$, the number in question becomes $(n-1)(n-2)$.

Passing to the case of surfaces; to determine the number of relations which must exist between the coordinates of mnp points, in order that they may be the points of intersection of surfaces of the orders m, n, p respectively. The number required is

$$3(mnp-N)+2(N-N')+(N'-N''),$$

where

$$N=[m, 3]-1-[m-n, 3]-[m-p, 3]+[m-n-p, 3]$$

(this last term to be omitted when $m < n+p-3$),

$$N'=[n, 3]-1-[n-p, 3],$$

$$N''=[p . 3]-1.$$

If, for instance, $m > n+p-3$, so as to retain the term $[m-n-p, 3]$, and $n > p$, so as to retain the term $N'-N''$, the number becomes, after all reductions,

$$2mnp+np^2-4np-2p^2-\tfrac{1}{3}(p-1)(p-2)(p-3),$$

a formula given by Jacobi. If, however, $n = p$, this number must be augmented by unity. Again, for $m < n + p - 3$, the required number is

$$2mnp + np^2 - 4np - 2p^2 - \tfrac{1}{3}(p-1)(p-2)(p-3)$$
$$- \tfrac{1}{6}(n+p-m-1)(n+p-m-2)(n+p-m-3),$$

which however must be augmented by unity if $m = n$ or $n = p$, and by 3 if $m = n = p$. But without entering into further details about this part of the subject, which has been sufficiently illustrated by the examples that have been given, I pass on to notice the application of the above theory to the problem of elimination. Imagine $(\theta + 1)$ equations between the $(\theta + 1)$ variables, the first sides of these being, as before, rational and integral homogeneous functions of the variables of the orders $m, n, p \ldots$ respectively. Writing $m + n + p \ldots - \theta = r$, and multiplying the first equation by all the terms of the form $x^\alpha y^\beta \ldots$ of the degree $r - m$, the second equation by all the terms of the same form, of the degree $r - n$, and so on, there result a certain number of equations, containing all the terms $x^\alpha y^\beta \ldots$ of the degree r. But these equations are not independent; and the reasoning in the former part of the present paper shews that the number of independent equations is given by the symbol $\{r : m, n, p \ldots : \theta\}$; the number of terms $x^\alpha y^\beta \ldots$ is evidently $[r, \theta]$; and it will be shewn immediately that for the actual value of r,

$$[r, \theta] - \{r ; m, n, p \ldots : \theta\} = 0 \ldots\ldots\ldots (\mathrm{B}).$$

So that the number of quantities to be linearly eliminated is precisely equal to the number of equations, or the elimination is always possible. I may mention also that, supposing the coefficients of all the equations to be of the order unity, the order of the result, free from extraneous factors, may be shewn to be

$$[r-m,\theta] + \ldots - 2\{[r-m-n,\theta] + \ldots\} + 3\{[r-m-n-p,\theta] + \ldots\}$$
$$- \&\text{c.} = mn \ldots + mp \ldots + np \ldots\ldots\ldots (C),$$

(the equality of which will be presently proved) a result which agrees with that deduced from the theory of symmetrical functions; but I am not in possession of any mode of directly obtaining the final result in this its most simplified form. My method, which it is not necessary to explain here more particularly, leads me to the formation of a set of functions.

$$P, Q, \ldots\ldots X, Y, Z,$$

θ in number, such that Z divides Y, this quotient divides X, and so on until we have a certain quotient which divides P

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and this quotient equated to zero is the result of the elimination freed from extraneous factors. It only remains to demonstrate the formulæ (A), (B), and (C). Suppose in general that (k) denotes the sum of all the terms of the form $m^a n^b$.., which can be formed with a given combination of k letters out of the ϕ letters m, n, p . . . And let Σ (k) denote the sum of all the series (k) obtained by taking all the possible different combinations of k letters. It is evident that Σ (k) is a multiple of (ϕ), [(ϕ) denoting of course the sum of all the terms $m^a n^b$.., m, n.. being any letters whatever out of the series m, n, p. .]. Let g be the number of exponents $a, b, \ldots$, then (ϕ) contains $[\phi]^g$ terms, also (k) contains $[k]^g$ terms, and the number of terms such as (k) in the sum Σ (k) is $[\phi]^{\phi-k} \div [\phi - k]^{\phi-k}$. Whence evidently

$$\Sigma\,(k) = \frac{[\phi - g]^{\phi-k}}{[\phi - k]^{\phi-k}}\,(\phi).$$

Or, what comes to the same thing,

$$\Sigma\,(\phi - k) = \frac{[\phi - g]^k}{[k]^k}\,(\phi).$$

Let A be an indeterminate coefficient, σ a summatory sign referring to different systems of exponents; then

$$\Sigma\sigma A\,(\phi - k) = \sigma\,\frac{[\phi - g]^k}{[k]^k}\,A\,(\phi).$$

Or, giving to k the values $1, 2 \ldots \phi$, multiplying each equation by an arbitrary coefficient, and adding, putting also for shortness $\sigma A\,(\phi - k) = U_{\phi-k}$, we have

$$a_\phi U_\phi + a_{\phi-1}\Sigma U_{\phi-1} + \ldots = \sigma\left(a_\phi + a_{\phi-1}\,.\,\frac{[\phi - g]^1}{[1]^1} + \ldots\right)A\,(\phi);$$

whence in particular,

$$U_\phi - \Sigma U_{\phi-1} + \ldots = \sigma\,\{0^{\phi-g}A\,(\phi)\},$$

$$\Sigma U_{\phi-1} - 2\Sigma U_{\phi-2} + \ldots = \sigma\,\{(\phi - g)\;0^{\phi-g-1}A.(\phi)\},$$

which are still equations of considerable generality. If now $\phi = \theta$ and U_θ is a function of $m + n + p + \ldots$ of the order θ, the quantity $\sigma\,\{0^{\theta-g}A\,(\theta)\}$ reduces itself to the single term of U_θ which contains the product $mnp \ldots$ Hence, if

$$U_\theta = [a + m + n + p \ldots, \theta]$$

in which afterwards $a = r - m - n - p - \ldots$ we have the formula (A). Again, if $\phi = \theta + 1$, and $U_{\theta+1}$ a function of $m + n + p \ldots$ of the order θ, the sum $\sigma\,\{0^{\theta+1-g}A\,.\,(\phi)\}$ vanishes; whence writing $U_{\theta+1} = [m + n + p \ldots - 0, \theta]$, we have the formula ($B$).

Similarly, if in the second formula $\phi = \theta + 1$, and $U_{\theta+1}$ is a function of $m + n + p \ldots$ of the degree θ,

$$\sigma\{(\theta + 1 - g)\, 0^{\theta - g} A\, (\theta + 1)\},$$

reduces itself to the term which contains $mn. \ldots + np. \ldots + mp \ldots$; whence, if $U_{\theta+1} = [m + n + p + \ldots - \theta, \theta]$, we have the formula ($C$).

[*To be continued.*]

ON A MECHANICAL REPRESENTATION OF ELECTRIC, MAGNETIC, AND GALVANIC FORCES.

By William Thomson.

Mr. Faraday, in the eleventh series of his *Experimental Researches in Electricity*, has set forth a theory of Electro-statical Induction, which suggests the idea that there may be a problem in the theory of elastic solids corresponding to every problem connected with the distribution of electricity on conductors, or with the forces of attraction and repulsion exercised by electrified bodies. The clue to a similar representation of magnetic and galvanic forces is afforded by Mr. Faraday's recent discovery of the affection with reference to polarized light, of transparent solids subjected to magnetic or electromagnetic forces. I have thus been led to find three distinct particular solutions of the equations of equilibrium of an elastic solid, of which one expresses a state of distortion such that the absolute displacement of a particle, in any part of the solid, represents the resultant attraction at this point, produced by an electrified body; another gives a state of the solid in which each element has a certain resultant angular displacement, representing in magnitude and direction the force at this point, produced by a magnetic body; and the third represents in a similar manner the force produced by any portion of a galvanic wire; the directions of the forces in the latter cases being given by the axes of the resultant rotations impressed upon the elements of the solid.

The general equations of equilibrium of an elastic solid have been investigated by Mr. Stokes,* without the assumption of any relation between the "cubical compressibility" and the elasticity, with reference to variations of form which are not accompanied by change of volume. If we denote by

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* In a paper "On the Friction of Fluids in Motion, and the Equilibrium and Motion of Elastic Solids," read at the Cambridge Philosophical Society April 14, 1845. See *Trans.*, vol. VIII. Part 3.

[redacted]

α, β, γ the projections on three rectangular axes of coordinates, of the infinitely small displacement of a point (x, y, z) of the solid, it follows from Mr. Stokes' results that the equations of equilibrium, when the body is acted on by no forces except at its bounding surfaces, may be written as follows:

$$\left.\begin{aligned} -\frac{dp}{dx}+\frac{d^2\alpha}{dx^2}+\frac{d^2\alpha}{dy^2}+\frac{d^2\alpha}{dz^2}&=0\\ -\frac{dp}{dy}+\frac{d^2\beta}{dx^2}+\frac{d^2\beta}{dy^2}+\frac{d^2\beta}{dz^2}&=0\\ -\frac{dp}{dz}+\frac{d^2\gamma}{dx^2}+\frac{d^2\gamma}{dy^2}+\frac{d^2\gamma}{dz^2}&=0\\ p=-k\left(\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}\right)& \end{aligned}\right\}\ldots\ldots\ldots(1).$$

In the ideal limiting case in which the solid is incompressible, k will have an infinite value, and we shall have the relation

$$\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}=0\ \ldots\ldots\ldots\ldots\ (2).$$

Hence equations (1) and (2) express the conditions of the interior equilibrium of an incompressible elastic solid. These equations are to be employed for the representation of the forces in the several physical problems considered in this paper.

Now equations (1) merely shew that the expression

$$\nabla^2\alpha.dx+\nabla^2\beta.dy+\nabla^2\gamma.dz\ \ldots\ldots\ldots\ldots(a),$$

(in which ∇^2 denotes the operation $\dfrac{d^2}{dx^2}+\dfrac{d^2}{dy^2}+\dfrac{d^2}{dz^2}$), must be a complete differential, and therefore any expressions for α, β, γ subject to this condition, which satisfy (2), will represent an interior state of the body which can be produced by the action of forces at its bounding surface or surfaces.

We may obtain a particular solution by assuming $\alpha dx + \beta dy + \gamma dz$ to be a complete differential. Again, if we suppose this expression not to be a complete differential, we may assume

$$\left(\frac{d\beta}{dz}-\frac{d\gamma}{dy}\right)dx+\left(\frac{d\gamma}{dx}-\frac{d\alpha}{dz}\right)dy+\left(\frac{d\alpha}{dy}-\frac{d\beta}{dx}\right)dz\ldots(c)$$

to be a complete differential and find another solution; or lastly we may obtain a particular solution by means of a third supposition, according to which neither of these expressions is a complete differential. These three solutions I shall now proceed to consider, with reference to the representation of Electrical, Magnetic, and Galvanic forces.

I.—*Electrical Forces.*

Let $$r^2 = x^2 + y^2 + z^2,$$

and assume $$\alpha dx + \beta dy + \gamma dz = -d\left(\frac{1}{r}\right).$$

Then, since $$\frac{d^2}{dx^2}\frac{1}{r} + \frac{d^2}{dy^2}\frac{1}{r} + \frac{d^2}{dz^2}\frac{1}{r} = 0,$$

equation (2) is satisfied, and the coefficients of the differentials in (a) vanish; so that all the conditions of equilibrium are satisfied. Now $\frac{1}{r}$ is the potential at (x, y, z), due to a unit of electricity at the origin, and

$$\alpha = \frac{x}{r^3}, \quad \beta = \frac{y}{r^3}, \quad \gamma = \frac{z}{r^3} \quad \ldots\ldots\ldots\ldots (\text{I.})$$

are the components of the force exerted at the point (xyz).

II.—*Magnetic Forces.*

Let

$$\left(\frac{d\beta}{dz} - \frac{d\gamma}{dy}\right)dx + \left(\frac{d\gamma}{dx} - \frac{d\alpha}{dz}\right)dy + \left(\frac{d\alpha}{dy} - \frac{d\beta}{dx}\right)dz = d\,\frac{lx + my + nz}{r^3}.$$

This equation is satisfied by

$$\alpha = \frac{mz - ny}{r^3}, \quad \beta = \frac{nx - lz}{r^3}, \quad \gamma = \frac{ly - mx}{r^3} \quad \ldots . (3),$$

which also satisfy equation (2), and make the coefficients of the differentials in (a) vanish. Hence, displacements expressed in this way may be produced by externally applied forces. Now

$$\frac{lx + my + nz}{r^3}$$

is the potential due to a small magnet, of which the 'moment' is unity, placed at the origin, with its axis of polarization in the direction $l:m:n$. The components X, Y, Z of the force which this magnet exerts upon an ideal unit of magnetism (one end of a thin uniformly magnetized bar) at the point x, y, z being the differential coefficients of this expression, we have

$$X = \frac{d\beta}{dz} - \frac{d\gamma}{dy}, \quad Y = \frac{d\gamma}{dx} - \frac{d\alpha}{dz}, \quad Z = \frac{d\alpha}{dy} - \frac{d\beta}{dx} \ldots (\text{II.})$$

The halves of the expressions $\frac{d\beta}{dz} - \frac{d\gamma}{dy}$, &c. indicate the

components round lines parallel to the axes, of the infinitely small rotation which an element of the solid receives, besides its change of form, when $\alpha dx + \beta dy + \gamma dz$ is not a complete differential. This rotation therefore represents the resultant magnetic force, in direction and magnitude.

III.—*Galvanic Forces.*

Let $\nabla^2\alpha.dx + \nabla^2\beta.dy + \nabla^2\gamma.dz = -d\,\dfrac{lx+my+nz}{r^3}$,

which is true if

$$\left.\begin{aligned}
\alpha &= \tfrac{1}{2}\frac{d}{dx}\frac{lx+my+nz}{r} - \frac{l}{r}\\
\beta &= \tfrac{1}{2}\frac{d}{dy}\frac{lx+my+nz}{r} - \frac{m}{r}\\
\gamma &= \tfrac{1}{2}\frac{d}{dz}\frac{lx+my+nz}{r} - \frac{n}{r}
\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (4).$$

It is readily verified that these expressions also satisfy equation (2), and hence they represent an interior state of the body which may be produced by externally applied forces. Now, we find by means of these equations

$$\left.\begin{aligned}
\frac{d\beta}{dz} - \frac{d\gamma}{dy} &= \frac{mz - ny}{r^3}\\
\frac{d\gamma}{dx} - \frac{d\alpha}{dz} &= \frac{nx - lz}{r^3}\\
\frac{d\alpha}{dy} - \frac{d\beta}{dx} &= \frac{ly - mx}{r^3}
\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (\text{III.})$$

which are the expressions for the components of the force an infinitely small element of a galvanic current, in the direction l, m, n, at the origin, produces on a unit of magnetism at the point (x, y, z); the intensity of the current, multiplied by the length of the element, being unity. Thus we conclude that the rotation of any element of the solid, in the state expressed by (4), represents in direction and magnitude, the force of an element of a galvanic wire.

I should exceed my present limits were I to enter into a special examination of the states of a solid body representing various problems in electricity, magnetism, and galvanism, which must therefore be reserved for a future paper.

Glasgow College, Nov. 28, 1846.

ON THE DEGREE OF A SURFACE RECIPROCAL TO A GIVEN ONE.

By the Rev. George Salmon, M.A., Fellow of Trinity College, Dublin.

1. Of all the additions which modern investigation has made to the ancient geometry, none seems more important than the method of reciprocal polars, by which our knowledge of extension is at once doubled, and we are enabled from any known property of curves or surfaces at once to deduce another correlative one. Nor is it only in the multiplication of isolated theorems that this method has been useful; it has thrown new light on some important points in the general theory of curves. For instance, it is to this method that we owe an accurate investigation of the simple question, how many tangents can be drawn from a given point to a curve of the m^{th} degree? It had previously been answered, "In general $m(m-1)$." The theory of reciprocal curves made it evident that the number could not always be so great. For since the degree of the reciprocal to any curve is equal to the number of tangents that can be drawn from any point to this curve, and therefore in general $m.(m-1)$, if the degree of the reciprocal of this reciprocal were to be determined by the same rule, it would be $\{m.(m-1)\}\{m.(m-1)-1\}$, instead of m, as it plainly ought to be. It became then an interesting question, "What are the circumstances in which the number of tangents which can be drawn from a point to a curve of the m^{th} degree is less than $m.(m-1)$?"

2. The reply to this question was found to be, "When the given curve has multiple points." A little consideration shews that this result might have been anticipated; for the problem "to draw a tangent from a point to a curve," when expressed analytically, becomes "to draw a line such that two values of the radius vector may be equal;" and since this condition is satisfied by the lines drawn from the given point to the multiple points of the curve, we must deduct these lines from the $m.m-1$ solutions, of which the general question admits. It was found then that the degree of the reciprocal curve is diminished by two for every double point on the curve; by three for every cusp (that is, a double point at which the two tangents coincide); by six for every triple point, and so on.

3. As I am not aware that the corresponding question as to reciprocal surfaces has been before investigated, I purpose in the present paper to enquire, what is the degree of the surface reciprocal to one of the m^{th} degree; and to consider

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how this degree is diminished if the given surface have multiple points or lines.

4. The degree of the reciprocal surface is plainly the same as the number of tangent planes which can be drawn to the surface through a given line: now we know that all the points of contact of tangent lines passing through a given point lie on a surface of the $n-1^{st}$ degree, which we call the $(n-1)^{st}$ polar surface of that point.

In order, then, to find the points of contact of planes passing through a fixed line, we have only to take the polar surfaces of any two points on this line. The intersections of these surfaces with the given one are the points of contact required; and since three surfaces respectively of the k^{th}, l^{th}, and m^{th} degrees intersect each other in $k.l.m$ points, the number of intersections in the present case will be $m.(m-1)^2$. This therefore is the degree of the surface reciprocal to one of the m^{th} degree.

5. There is another method by which we might have determined the degree of the reciprocal surface, the result of which does not at first sight appear the same as the preceding.

The degree of the reciprocal surface is the same as that of any plane section of it; but any plane section of the reciprocal surface is reciprocal to a tangent cone of the given surface. Now the degree of the cone touching a surface of the m^{th} degree is $m.m-1$, therefore (as the reciprocals of cones follow the same rules as curves) if the cone have no multiple sides, its reciprocal will be of the degree $m.(m-1)\{m.(m-1)-1\}$. We appear, then, to have arrived at a result contradictory to that of the preceding section.

6. I proceed to remove the apparent contradiction by establishing the following theorems. (1) "Every cone touching a surface of the m^{th} degree must in general have $\dfrac{m.(m-1)^2.m-2}{2}$ double sides, real or imaginary." (2) "Of these double sides, $m.m-1.m-2$ are cuspidal lines, and consequently there are only $\dfrac{m.m-1.m-2.m-3}{2}$ ordinary double lines." Assuming for a moment the truth of these theorems, we see that the degree of the curve reciprocal to this cone will be

$$m.m-1.\left\{m^2-m-1-2.\frac{m-2.m-3}{2}-3.(m-2)\right\}=m.(m-1)^2,$$

the same result at which we had arrived by the other method.

7. I proceed to prove the theorems just enunciated. It is evident that any side of the tangent cone will be a double line if it touch the surface in two different points: it is necessary, therefore, to find "How many lines can be drawn through a given point which will have double contact with a given surface?" Suppose, for greater simplicity, the point at an infinite distance, and that we enquire how many lines parallel to a given one, to the axis of z for instance, will touch the surface twice. Let $U=0$ be the equation of the surface; $\frac{dU}{dz}=0$ will be the equation of the surface passing through the points of contact of tangents parallel to the axis of z; and if we eliminate z between these equations, we shall have the equation of the cylinder enveloping the surface. But if any side of that cylinder touch the surface twice, the two equations $U=0$, $\frac{dU}{dz}=0$, which, considered as functions of z, have for every side a root in common, must for the x and y of this side have two common roots. Let us suppose then that we have solved the algebraical problem to find the two conditions necessary that these two equations considered as functions of z, should have a pair of common roots. Both of these will be functions of x and y, and will therefore represent cylinders: one of them will be the cylinder circumscribing the surface; and the intersections of this cylinder with the other will be the double sides required. All that is necessary for our present purpose is to find the degree of the second condition in x and y. Now I find that if we had two algebraical equations, one of the m^{th} the other of the n^{th} degree, the conditions that they should have two common roots will be the result of elimination between them and another equation which can be reduced to the $(m-1).(n-1)^{\text{st}}$ degree. This second equation will be in the present case of the degree $m-1.m-2$, and the cylinder which it represents will meet the circumscribing cylinder in $\frac{m.(m-1)^2.m-2}{2}$ lines. I divide by 2 because each intersection counts for two, since from the nature of the question each is a double side of the circumscribing cylinder. The first theorem is therefore established.

8. Before proving the second, I must digress a little to state another theorem which is an immediate consequence of

the algebraical principle employed in the last section, and which is of importance in some extensions of the present investigation. "If two surfaces intersect, the projection from any point on any plane of their curve of intersection must in general have $\frac{m\,.\,n\,.(m-1)\,.(n-1)}{2}$ double points." For it is evident that the projection will have a double point whenever one of the projecting lines passes through two distinct points of the curve of intersection, and by the algebraical principle just mentioned these points will be found by combining with the equations of the given surfaces, another of the degree $m-1\,.\,n-1$.*

9. This property admits of a simple geometrical proof in the case where the two surfaces are of the second degree. That is to say, "The projection on any plane of the intersection of two surfaces of the second degree will in general be a curve of the fourth degree having two double points." To see this we only want to know how many lines can be drawn through a given point which will meet two surfaces of the second degree in the same pairs of points: now the point of harmonic section of such a line must be the same for both surfaces, and consequently it must meet in the same point the polar planes of the given point with regard to the two surfaces. Hence we have the following construction: "Take the polar planes of the point with regard to the two surfaces; join their intersection to the given point; the joining plane will meet the surfaces in two conics whose two common chords will pass through the given point, and these chords will meet any plane in two points which are double points on the projection of the curve of intersection."

10. I return to the proof of the second theorem stated in § 6. We know that the tangent plane at any point of a surface meets the surface in a curve of which that point is a double point (since every line drawn through that point in the tangent plane meets the surface in two indefinitely near points); and since every double point has two tangents, we learn that at any point of a surface there can be drawn two tangent lines, each of which will meet the surface in *three* indefinitely near points. If, then, from any point on one of these tangent lines we draw the enveloping cone, it will

* From this property it appears that the developable surface circumscribing two surfaces whose reciprocals are respectively of the kth and lth degrees will be of the degree $kl\,(k+l-2)$, and hence that the developable circumscribing two surfaces of the second degree will in general be of the eighth degree. I have verified this for the case of two concentric ellipsoids, but the equation is rather too long to give here.

easily appear that this line will be a cuspidal side of that cone, and the question concerning cuspidal lines reduces itself to this: Through a given point how many tangent lines can be drawn to meet the surface in three indefinitely near points?

Suppose, as before, we substitute for a cone a cylinder parallel to the axis of z, the points of such contact will be found by combining the three equations

$$U = 0, \quad \frac{dU}{dz} = 0, \quad \frac{d^2U}{dz^2} = 0,$$

which will evidently only be satisfied for $m.(m-1).(m-2)$ points.

11. Having thus shewn that the degree of the reciprocal surface is in general $m.(m-1)^2$, I proceed to examine how this degree is affected by the existence of multiple points or lines on the original surface. The case of multiple *points* presents little difficulty, as it may be treated by the same methods as for plane curves. We find, for example, by the same reasoning as for plane curves, that for every double point, nodal or conjugate, on the surface, the degree of the reciprocal surface will be diminished by 2. It may be useful to give a few illustrations of the truth of this assertion.

12. Perhaps the first case which would naturally occur to any one desirous to test the truth of such a theorem, is Fresnel's wave surface. We know that it is of the fourth degree, that its reciprocal is a similar surface, and we are accustomed to say that it has four double points. If these were all, it would appear to contradict the theory just laid down, since the reciprocal of a biquadratic surface is in general of the 36th degree, and the four double points would only reduce this number by 8.

The difficulty is solved by recollecting that we must take into account not only the real but the imaginary double points of the surface. Now the very same arguments which shew that the wave surface has four real double points in *one* of its three principal planes, prove that it has four imaginary double points in each of the other two. Moreover, the form of the equation shews that it has four other imaginary double points in the plane at infinity. Hence the wave surface has in all 16 double points, 4 real and 12 imaginary, and therefore the degree of its reciprocal $= 36 - 32 = 4$.

13. I take as another example the surface of the third degree having 4, (its maximum number) of double points.

Its equation must be of the form $A^{-1} + B^{-1} + C^{-1} + D^{-1} = 0$, $A = 0$, &c. being the equations of the four sides of the pyramid formed by the double points. This belongs to the class of surfaces $A^m + B^m + C^m + D^m = 0$, whose reciprocal is of the same form, the new m being equal to $\dfrac{m}{m-1}$. In the present case the reciprocal is of the form $A^{\frac{1}{2}} + B^{\frac{1}{2}} + C^{\frac{1}{2}} + D^{\frac{1}{2}} = 0$, a surface of the fourth degree as we expected.

14. We found in curves that though the degree of the reciprocal is only reduced by two for an ordinary double point, nodal or conjugate, yet if the tangents at it coincide, the degree will be reduced by *three*. We should expect to find an analogous result for surfaces, and accordingly I find that such is the case when the tangent cone at the double point reduces itself to two planes, real or imaginary. In this case the two surfaces of the $n - 1^{st}$ degree, whose intersections with the original we employed to determine the degree of the reciprocal, will not only pass through the double point, but will also both touch the line of intersection of the two planes; hence it appears that the degree of the reciprocal will be diminished by 3.

15. An instance of such points we have in the surface of the third degree $A.B.C = D^3$, $A = 0$, &c. being the equations of planes. Here the three points ABD, ACD, BCD are double points, and the tangent cone at any reduces to two of the planes $A = 0$, $B = 0$, $C = 0$. But the reciprocal of this surface is another of the same form, reducing from the twelfth degree to the third, on account of the three double points just mentioned.

16. Again, if the two planes coincide, both the surfaces of the $n - 1^{st}$ degree must touch this plane, and it is not difficult to see that the degree of the reciprocal surface will be reduced by 6.

Multiple points of higher degrees present no difficulty.

17. The case of multiple *lines*, however, involves much more complicated considerations. Suppose a surface to have a double line, the two polar surfaces of the $n - 1^{st}$ degree will each pass through it, and the question becomes "In how many other points will three surfaces intersect which each pass through a given curve, that curve being a double line on one of them?"

Let us commence by the simpler question: "Three surfaces respectively of the m^{th}, n^{th}, and p^{th} degrees pass through a

given right line, to how many points of intersection is this line equivalent, or in how many other points do they intersect?" The direct solution of this question is attended with some difficulty, which however we can evade by the following process.

18. We know that in general the number of points in which two curves intersect is wholly independent of the existence of multiple points on either of them (provided indeed that one of the points of intersection be not a multiple point), and that it is invariably true that a curve of the m^{th} degree cuts a curve of the n^{th} degree in mn points, real or imaginary, even when one or both curves degenerate into compound curves of lower dimensions. Following out this observation, we infer that if we could in any one case determine the number of points in which three such surfaces as we are considering intersect, we should be safe in asserting that they would always intersect in the same number of points. Let us suppose, then, that the first surface consists of a plane passing through the given line and of a surface of the $(m-1)^{\text{st}}$ dimensions. This surface will meet the other two in $m-1 \,.\, np$ points, of which $n-1$ are on the given line, and therefore the number of other points is $(m-1).(np-1)$, and the plane meets the two surfaces in two curves of the $n-1^{\text{st}}$ and $p-1^{\text{st}}$ degrees, which intersect in $(n-1).(p-1)$ points. The total number, therefore, of points of intersection is

$$(m-1).(np-1)+(n-1).(p-1) = mnp - m - n - p + 2.$$

We find, then, that the common line reduces the number of points of intersection by $m+n+p-2$.

19. Let us advance now to the question: "A surface of the m^{th} degree has a double right line; two others of the degrees n and p pass through this line: in how many points not on the line will they intersect?"

Let the surface of the m^{th} degree consist of two of the degrees m' and m'' each passing through the line: from the last section we learn that the number of points of intersection is reduced by

$$m'+n+p-2+m''+n+p-2, \text{ or since } m'+m''=m,\ m+2n+2p-4.$$

Let $n=p=m-1$, and the number of points of intersection is diminished by $5m-8$. At first sight this would appear to be the number by which the degree of the reciprocal of a surface is reduced when it has a double right line. There are however some remarkable points on the double line

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which affect the degree of the reciprocal, and which we have not yet taken into consideration.

20. If a surface have a double line of any kind, in general at any point of it two planes can be drawn tangent to the surface; but there will always be a determinate number of points (which I call cuspidal points), at which the two tangent planes coincide, and for each of these points the degree of the reciprocal will be further diminished by one. Take the case where the double line is a right line (suppose the axis of z), the equation of the surface will be

$$Ay^2 + Byx + Cx^2 = 0,$$

A, B, C being any functions of the variables.

The tangent planes at any point are determined by the equation $A'y^2 + B'xy + C'x^2 = 0$, where $A'B'C'$ are the values which ABC take for that point, and it is evident that these planes will coincide at the points where the axis meets the surface $B^2 = 4AC$. This surface being of the $2m - 4^{th}$ degree, we are to add this number to the number $5m - 8$ already determined, and we find that if a surface have a double right line, the degree of its reciprocal is diminished $7m - 12$. Hence the reciprocal of a surface of the third degree which has a double line, is of the third degree, since this double line is necessarily a right line.

21. If the surface have a triple right line, proceeding by a precisely similar method, we find that the degree of the reciprocal surface is diminished by $20m - 48$.

And in general, that if a surface have a multiple right line of the degree r, we find, by the same method, that the reciprocal is diminished by $(r - 1).(3r + 1)\,m - 2r.(r^2 - 1)$. Hence if a surface have a multiple line of the degree $m - 1$ (which must be a right line, since no plane can cut it in more points than one), the degree of the reciprocal will be the same as that of the given surface.

22. Let us now suppose the double line to be the curve of intersection of a surface of the k^{th} with one of the l^{th} degree.

First let us consider if three surfaces pass through such a curve, in how many other points will they intersect. Take the case where the first surface is one of the k^{th} and one of $(m - k)^{th}$; the second, one of l^{th} and of $n - l^{th}$ degree; the third, one of the p^{th}; the number of points of intersection will be

$$k.(n - l).(p - l) + l.(m - k).(p - k) + p.(m - k).(n - l),$$

therefore the general number will in this case be diminished

by $kl.\{m+n+p-(k+l)\}$. Now if one of the surfaces have this curve for a double line (suppose, as before, it to be made up of two surfaces each passing through the curve), the number of points of intersection will be diminished by

$$kl\{m'+n+p-(k+l)\}+kl\{m''+n+p-(k+l)\}$$
$$=kl.\{m+2n+2p-2(k+l)\};$$

let $n=p=m-1$, and diminution is $kl.\{5m-2.(k+l)-4\}$.

23. We must add to this the number of cuspidal points on the double line.

Let the equation of the surface be $AU^2+BUV+CV^2$, where U is of the k^{th} degree and V of the l^{th} degree. The cuspidal points are the points of intersection of $U=0$, $V=0$, and $B^2=4AC$, and are therefore in number $kl\{2m-2(k+l)\}$. Hence the degree of reciprocal is diminished by

$$kl.\{7m-4(k+l)-4\}.$$

24. We can verify this formula for the case in which the surface of the m^{th} degree consists of two, one of the k^{th} and another of the l^{th} degree. Put $m=k+l$ in the preceding, and the reciprocal is to be diminished by $kl.\{3(k+l)-4\}$; but this is precisely the difference between $(k+l).(k+l-1)^2$ and $\{k.(k-1)^2+l.(l-1)^2\}$.

25. In general let a surface of the m^{th} degree have a line of the r^{th} degree of multiplicity, said line being the intersection of a surface of the k^{th} with one of the l^{th} degree, then the degree of reciprocal will be reduced by

$$kl\{(r-1).(3r+1)m-r^2.(r-1).(k+l)-2r.(r-1)\}.$$

To verify this formula, suppose r surfaces, each of the k^{th} degree, to pass through the same curve, we must make in the above $k=l$ and $m=rk$, and the formula becomes

$$r(r^2-1)k^3-2r(r-1)k^2,$$

but this is the difference between $rk.(rk-1)^2$ and $rk.(k-1)^2$.

General as the above formula appears, it does not include some interesting cases which I am compelled to omit for the present.*

Trinity College, Dublin, Nov. 25, 1846.

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* I take this opportunity to correct a statement made by Mr. Townsend in the last number of the *Journal* (p. 36), where he ascribes to me a theorem concerning surfaces of the second degree, to the discovery of which I have no claim. It was given in the year 1836, by Professor MacCullagh, who introduced the study of these surfaces into this University, and from whom three or four years afterwards I obtained my first knowledge of the subject.

NOTE ON THE PARABOLIC POINTS OF SURFACES.

By the Rev. George Salmon.

"The *parabolic* points on a surface of the n^{th} degree lie on the intersection of the given surface with another of the $4(n-2)$ degree." This is most easily demonstrated from the analogy which these points bear to points of inflexion on plane curves. The number of points of inflexion on a plane curve of the n^{th} degree may be found as follows, from the consideration of the well-known curves called the polar curves of any point with regard to the given curve. Let the equation of the curve be $U = 0 = u_n + \rho u_{n-1} + \rho^2 u_{n-2} + \&c.$, introducing the factor ρ to make the equation homogeneous: then the equations of the successive polar curves of the origin will be $\frac{dU}{d\rho} = 0$, $\frac{d^2u}{d\rho^2} = 0$, &c. And to come to those with which we are immediately concerned, the equation of the polar line will be $u_1 + n\rho u_0 = 0$, and of the polar conic will be

$$u_2 + (n-1)\rho u_1 + \frac{n(n-1)}{1.2}\rho^2 u_0 = 0.$$

Suppose, now, the origin to be a point of inflexion, then $u_0 = 0$ and u_1 is a factor in u_2. The last equation therefore is divisible by u_1, and hence the polar conic of a point of inflexion resolves itself into two right lines. Now if we were to find the locus of all the points whose polar conics break up into two right lines, the intersections of this locus with the original curve must include all the points of inflexion. But the equation of the polar conic of any point is

$$y^2\frac{d^2u}{dy^2} + 2xy\frac{d^2u}{dx^2dy^2} + x^2\frac{d^2u}{dx^2} + 2y\rho\frac{d^2u}{d\rho dy^2} + 2x\rho\frac{d^2u}{dx^2d\rho} + \rho^2\frac{d^2u}{d\rho^2} = 0.$$

Applying to this equation the ordinary criterion that a conic should resolve into two right lines, we obtain an equation of the third degree in $\frac{d^2u}{dx^2}$ &c., and consequently in the $3(n-2)$ degree in x and y. The number of points of inflexion is therefore in general $3.n.(n-2)$. But if the curve have multiple points, these points also fulfil the conditions of the question, and will therefore lie on the above-mentioned locus. The number of points of inflexion will be diminished accordingly: by six, I find, for every double point; and by eight, if the tangents at that double point coincide.

It is very easy to extend these considerations to the case of surfaces. Suppose a point on the surface to be the origin,

and the tangent plane the plane of xy, the part of the equation below the third degree will be

$$Ay^2 + Bxy + Cx^2 + z(Dy + Ex + Fz + G) = 0.$$

The equation $Ay^2 + Bxy + Cx^2 = 0$ determines the directions of the tangents at the origin to the intersection of the surface with the tangent plane; and when these directions coincide, the origin is a parabolic point. If the equation be

$$(ay + \beta x)^2 + z(Dy + Ex + Fz + G) = 0,$$

it is evident that the polar of the second degree will be a cone whose vertex is the intersection of the three planes $z = 0$, $ay + \beta x = 0$, $Dy + Ex + Fz + G = 0$. Hence to find the parabolic points we have only to seek the locus of a point whose polar of the second degree shall be a cone; and the intersection of this locus with the given surface will determine the points in question. But the condition that the general equation of the second degree should represent a cone, is of the fourth degree with regard to the coefficients; hence the locus required will be of the $4.n - 2$ degree.

Trinity College, Dublin, Dec. 14, 1846.

INVESTIGATION OF THE VALUE OF $\int_0^\infty \frac{\sin x dx}{x}$.*

By Francis W. Newman, Professor of Latin in University College, London.

Put $A_n = \int_0^{n\pi} \frac{\sin x dx}{x}$. Then, observing that

$$\int_{n\pi}^{(n+1)\pi} \frac{\sin x dx}{x} = \int_0^\pi \frac{\cos n\pi \sin x' dx'}{n\pi + x'},$$

which vanishes when n = an infinite integer, since the denominator is then infinite; it follows, *a fortiori*, that

$$\int_{n\pi}^{n\pi+\mu} \frac{\sin x dx}{x}$$

vanishes when n = an infinite integer and μ is between 0 and π: consequently we find the value of A from A_n by supposing $n = \infty$, without any difference in the result, whether n be integer or fractional.

[* Before I received this article, Mr. Cayley had communicated to me a paper containing the application of the method here given to the evaluation of various integrals, both single and double; but his results have not yet been published. The solution given in the article published at present was given by Mr. Cayley as an example.—Ed.]

Now $\int_0^{n\pi} = \int_0^{\pi} + \int_{\pi}^{2\pi} + \int_{2\pi}^{3\pi} + \ldots + \int_{(n-1)\pi}^{n\pi}$.

Hence we have

$$A_n = \int_0^{\pi} \frac{\sin y_1 dy_1}{y_1} + \int_{\pi}^{2\pi} \frac{\sin y_2 dy_2}{y_2} + \ldots + \int_{(n-1)\pi}^{n\pi} \frac{\sin y_n dy_n}{y_n};$$

since, when the limits are fixed, we may arbitrarily change x into y.

Let
$$y_1 = \pi - x; \quad y_2 = \pi + x;$$
$$y_3 = 3\pi - x; \quad y_4 = 3\pi + x;$$
$$y_5 = 5\pi - x; \quad y_6 = 5\pi + x; \ \&c.\ldots$$

Then when $y_1, y_3, y_5, y_7 \ldots$ are at their lower limits, x is π; but at their upper limits, $x = 0$. The cases are reversed for $y_2, y_4, y_6, y_8 \ldots$ Observing, then, that $\int_{\pi}^{0} - du = \int_0^{\pi} du$, we change every term of the series into an integral, in which x varies between the same limits 0 and π; so that

$$A_n = \int_0^{\pi} \left\{ \frac{\sin x dx}{\pi - x} - \frac{\sin x dx}{\pi + x} + \frac{\sin x dx}{3\pi - x} - \frac{\sin x dx}{3\pi + x} + \frac{\sin x dx}{5\pi - x} - \&c. \ldots \text{to } n \text{ terms} \right\}.$$

Since the denominators increase, the fractions diminish; and as they are alternately positive and negative, the sum will approximate to a single real and finite limit, when $n = \infty$, by a well-known law of infinite series. Hence, making $n = \infty$, we get

$$A = \int_0^{\pi} \left\{ \frac{1}{\pi - x} - \frac{1}{\pi + x} + \frac{1}{3\pi - x} - \frac{1}{3\pi + x} + \&c.\ \&c. \right\} \sin x dx.$$

But by an easy result of the equation

$$\cos\frac{x}{2} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\left(1 - \frac{x^2}{25\pi^2}\right) \&c.\ \&c.$$

it is well known that the series within brackets $= \frac{1}{2}\tan\frac{x}{2}$;

$$\therefore \quad A = \int_0^{\pi} \tfrac{1}{2}\tan\frac{x}{2}\sin x dx = \int_0^{\pi} \sin^2\frac{x}{2} dx = \int_0^{\pi} \frac{1 - \cos x}{2} dx$$

$$= \left(\frac{x}{2} - \frac{\sin x}{2}\right) \text{ within proper limits; or } A = \frac{\pi}{2}.$$

P.S. I ventured to send this Article to the Editor, at the request of a mathematician, who had doubted of the truth of the *result*, in consequence of the defectiveness of the current proof.—F. N.

ON LOGARITHMIC INTEGRALS OF THE SECOND ORDER.

By FRANCIS W. NEWMAN.

§. I.

1. THE general formula $\int F_1x \,.\, \log F_2x \,.\, dx$, where F_1F_2 denote rational functions, contains a variety of integrals, all of which, it will be shewn, can be reduced to *three*.

By the common method of finding $\int F_1x \,.\, dx$, we perceive that there is some rational function F_3 which fulfils the equation

$$F_1x = \frac{d}{dx} F_3x + \Sigma \frac{A}{x - e} + \Sigma \frac{px + q}{(x - \mu)^2 + \nu^2}.$$

Also, if F_2x be reduced to the form of a single algebraic fraction, it may be denoted by $F'x \div F''x$, where F' and F'' are each *integer*. Consequently we may write

$$\log F_2x = \Sigma . A_1 \log (ax + b) + \Sigma . A_2 \log (a'x^2 + b'x + c').$$

It immediately follows that $\int F_1x \,.\, \log F_2x dx$ is separable into the two forms $\int F_1x . \log (ax+b)\, dx$ and $\int F_1x . \log (a'x^2+b'x+c')\, dx$. In the former, introduce the preceding value of F_1x, and we obtain for the integral

$$\log (ax + b) \,.\, F_3x - \int \frac{F_3x . adx}{ax + b}$$

$$+ \Sigma . A \int \frac{\log (ax + b)}{x - e} dx + \Sigma \int \log (ax + b) \,.\, \frac{(px + q)\, dx}{(x - \mu)^2 + \nu^2}.$$

Of the three integrals which here appear, the first is rational. In the second assume $ax + b = mx'$; $\therefore a(x - e) = mx' - b - ae$. Assume farther, $m = b + ae$; then

$$\int \frac{\log (ax + b)\, dx}{x - e} = \log m \,.\int \frac{dx}{x - e} + \int \frac{\log x' . dx'}{x' - 1},$$

provided that m, or $(b + ae)$, is positive. If otherwise, put $x = e + m'x''$, and $am' = -(b + ae)$;

$$\therefore \int \frac{\log (ax + b)\, dx}{x - e} = \log (ax + b) \,.\, \log \frac{x - e}{m'} - \int \frac{\log x'' . dx''}{x'' - 1}.$$

In either case we arrive at the elementary form

$$L(x) = \int_1 \frac{\log x . dx}{x - 1} \; \ldots\ldots\ldots\ldots \; (1),$$

which Spence has tabulated. As for the integral

$$\int \log (ax + b) \,.\, \frac{(px + q)\, dx}{(x - \mu)^2 + \nu^2},$$

the same assumption, $ax + b = mx'$, if we give to m a suitable constant value, produces two general forms which may be denoted by

$$\int \frac{\log x\,dx}{X} \text{ and } \tfrac{1}{2} \int \log x\, d \log X; \text{ if } X = x^2 - 2x \cos a + 1.$$

Let ω be an arc such that $\tan \omega = x \sin a \div (1 - x \cos a)$, or, what is the same, $x = \sin \omega \div \sin (\omega + a)$: then

$$d\omega = \frac{\sin a . dx}{X}; \text{ and } \sin a . \int \frac{\log x\,dx}{X}$$

$$= \int \log \sin \omega d\omega - \int \log \sin (\omega + a)\, d\omega.$$

Suppose $\natural$ to be a symbol for a new function, such that

$$\natural(\omega) = -\int_0 \log \sin \omega . d\omega \;..........(2);$$

then $$\sin a . \int_0 \frac{\log x\,dx}{X} = \natural(\omega + a) - \natural\omega - \natural a \;....\; (2).$$

No similar reduction occurs, by which we can exterminate the arbitrary constant from the next integral; and we must be satisfied with writing

$$\Lambda(x, a) \text{ for } \int_0 \frac{\log x . (x - \cos a)\, dx}{x^2 - 2x \cos a + 1} \text{ or } \tfrac{1}{2} \int lx . dlX ... (3).$$

It will be sometimes convenient to put

$$\lambda(x, a) \text{ for } \tfrac{1}{2} \int_0 \log (x^2 - 2x \cos a + 1) . \frac{dx}{x} \;....(4),$$

which is a supplemental function to Λ, and so related that

$$\Lambda(x, a) + \lambda(x, a) = \tfrac{1}{2} \log x . \log X.$$

We may write Λx, λx when no change of a is contemplated.

2. We have now to go back to $\int F_1 x . \log (ax^2 + bx + c)\, dx$. By substituting as before for F_1, we reduce the integral to

$$F_2 x . \log (ax^2 + bx + c) - \int F_2 x . \frac{(2ax + b)\, dx}{ax^2 + bx + c}$$

$$+ \Sigma A \int \frac{\log (ax^2 + bx + c)}{x - e}\, dx + \Sigma \int \log (ax^2 + bx + c) . \frac{(px + q)\, dx}{(x - \mu)^2 + \nu^2}.$$

Of the three integrals remaining, the first is rational. The second is readily reduced to the form λ, by making $(x - e) = mx'$. The third, by making $x - \mu = mx'$, and determining m aright, produces the two new forms

$$X_1 = \int \log X . \frac{n dx}{x^2 + n^2}; \quad X_2 = \int \log X . \frac{x dx}{x^2 + n^2};$$

each of which has two arbitrary constants, a and n. But fortunately we can reduce X_1 to ϟ, and X_2 to L or λ. First, for X_1, put $x = n \tan \omega$, $n = \tan \nu$; $\dfrac{ndx}{x^2 + n^2} = d\omega$.

$$\begin{aligned} X &= 1 - 2n \tan \omega \cos a + n^2 \tan^2 \omega \\ &= (\cos^2 \omega - 2n \sin \omega \cos \omega \cos a + n^2 \sin^2 \omega) \div \cos^2 \omega \\ &= \{(1 + n^2) - 2n \sin 2\omega \cos a + (1 - n^2) . \cos 2\omega\} \div 2 \cos^2 \omega \\ &= (1 - \sin 2\nu \sin 2\omega \cos a + \cos 2\nu . \cos 2\omega) \div 2 \cos^2 \nu . \cos^2 \omega. \end{aligned}$$

Let μ, β be taken such that $\left.\begin{aligned} \sin \mu \sin \beta &= \sin 2\nu \cos a \\ \sin \mu \cos \beta &= \cos 2\nu \end{aligned}\right\}$;

$\therefore$ $\cos \mu = \sin 2\nu \sin a$, and $\tan \beta = \tan 2\nu . \cos a$.

Also $X = \{1 + \sin \mu (\cos 2\omega \cos \beta - \sin 2\omega \sin \beta)\} \div 2 \cos^2 \nu . \cos^2 \omega$,

$= (1 + \sin \mu \cos \theta) \div 2 \cos^2 \nu \cos^2 \omega$; if $\theta = 2\omega + \beta$:

whence $X_1 = \int \log X . d\omega = \frac{1}{2} \int \log (1 + \sin \mu \cos \theta) \, d\theta$

$$- \omega \log (2 \cos^2 \nu) - 2\text{ϟ}(\tfrac{1}{2}\pi - \omega) \ldots . (5):$$

in which the remaining integral has but one arbitrary constant.

Farther, let $m = \tan \frac{1}{2}\mu$, or $\sin \mu = 2m \div (1 + m^2) = 2m \cos^2 \frac{1}{2}\mu$;

$\therefore$ $\log (1 + \sin \mu \cos \theta) = \log (1 + 2m \cos \theta + m^2) + 2 \log \cos \frac{1}{2}\mu$.

Assume η such that $\tan \eta = \sin \theta \div (m + \cos \theta)$,

or $m = \sin (\theta - \eta) \div \sin \eta$,

$$\begin{aligned} \therefore\ 1 + 2m \cos \theta + m^2 &= \sin^2 \theta + (m + \cos \theta)^2 \\ &= \sin^2 \theta + \left(\frac{\sin \theta}{\tan \eta}\right)^2 = \left(\frac{\sin \theta}{\sin \eta}\right)^2. \end{aligned}$$

whence $\frac{1}{2} \int \log (1 + \sin \mu \cos \theta) \, d\theta$

$$\begin{aligned} &= \int \{\log \sin \theta - \log \sin \eta + \log \cos \tfrac{1}{2}\mu\} \, d\theta \\ &= - \text{ϟ}\theta - \int \log \sin \eta . d\theta + \theta . \log . \cos \tfrac{1}{2}\mu. \end{aligned}$$

Now $\int \log \sin \eta . d\theta$

$$\begin{aligned} &= \int \log . \frac{\sin (\theta - \eta)}{m} . d\theta = \int \log \frac{\sin (\theta - \eta)}{m} . \{d(\theta - \eta) + d\eta\} \\ &= - \text{ϟ}(\theta - \eta) - (\theta - \eta) \log m + \int \log \sin \eta . d\eta \\ &= - \text{ϟ}(\theta - \eta) - (\theta - \eta) \log \tan \tfrac{1}{2}\mu - \text{ϟ}\eta. \end{aligned}$$

$$\left.\begin{aligned} &\text{Therefore } \tfrac{1}{2} \int \log (1 + \sin \mu \cos \theta) \, d\theta \\ &= \text{ϟ}(\theta - \eta) + \text{ϟ}\eta - \text{ϟ}\theta + \theta \log \sin \tfrac{1}{2}\mu - \eta \log \tan \tfrac{1}{2}\mu \end{aligned}\right\} \ldots (6),$$

which is a general formula, provided that $\tan \eta = \dfrac{\sin \theta}{\tan \frac{1}{2}\mu + \cos \theta}$;

and completes the reduction of X_1 to the function ϟ.

3. The integral X_2 remains. Using for X the same transformation as before, let us write 2θ in place of θ, so that now $2\theta = 2\omega + \beta$. We have, moreover,

$$\frac{xdx}{x^2 + n^2} = \tfrac{1}{2} d \log (n^2 + x^2) = \tfrac{1}{2} d \log \sec^2 \omega = \tan \omega d\omega$$

$$\text{and also} = -\tfrac{1}{2} d \log (2 \cos^2 \nu \,.\, \cos^2 \omega).$$

Whence

$$X_2 = \int \{\log (1 + \sin \mu \cos 2\theta) - \log (2 \cos^2 \nu \,.\, \cos^2 \omega)\} \frac{xdx}{x^2 + n^2}$$

$$= \int \log (1 + \sin \mu \cos 2\theta) \tan \omega d\omega + \tfrac{1}{4} \log^2 .\, (2 \cos^2 \nu \,.\, \cos^2 \nu).$$

Put $b = \tan \frac{1}{2}\beta$; $t = \tan \theta$, $d\omega = d\theta = \dfrac{dt}{1 + t^2}$:

$$\cos 2\theta = \frac{1 - t^2}{1 + t^2}; \ \tan \omega = \tan \left(\theta - \frac{b}{2}\right) = \frac{t - b}{1 + bt};$$

$$\text{and } \tan \omega . d\omega = \frac{t - b}{1 + bt} \cdot \frac{dt}{1 + t^2} = \frac{tdt}{1 + t^2} - \frac{bdt}{1 + bt}.$$

Hence the integral which remains, becomes

$$\left\{\int \log \{1 + t^2 + \sin \mu \,.\, (1 - t^2)\} - \log (1 + t^2)\right\} . \left(\frac{tdt}{1 + t^2} - \frac{bdt}{1 + bt}\right).$$

Write

$$T_1 = \int \log \{1 + \sin \mu + (1 - \sin \mu) \,.\, t^2\} \,.\, \tfrac{1}{2} d \log (1 + t^2),$$
$$T_2 = \int \log (1 + t^2) \,.\, \tfrac{1}{2} d \log (1 + t^2) = \tfrac{1}{4} \log^2 (1 + t^2) = \log^2 \cos \theta,$$
$$T_3 = \int \log \{1 + \sin \mu + (1 - \sin \mu) \,.\, t^2\} \ d \log (1 + bt),$$
$$T_4 = \int \log (1 + t^2) \, d \log (1 + bt).$$

Then $X_2 = T_1 - T_2 - T_3 + T_4 + \frac{1}{4} \log^2 .\, (2 \cos^2 \nu \,.\, \cos^2 \omega)$.

To find T_1, let $1 + t^2 = mv$, and $m = 2 \sin \mu \div (1 - \sin \mu)$;

$$\therefore \quad T_1 = \tfrac{1}{2} \int l \, \{2 \sin \mu \,.\, (1 + v) \, dl \, (mv)$$
$$= \tfrac{1}{2} l \, (2 \sin \mu) \, l \, (mv) + \tfrac{1}{2} L \, (1 + v);$$

where $mv = \sec^2 \theta$,

$$1 + v = \frac{1 + t^2 + \sin \mu \, (1 - t^2)}{2 \sin \mu} = \frac{\sec^2 \theta \, (1 + \sin \mu \cos 2\theta)}{2 \sin \mu};$$

so that $T_1 = -\log (2 \sin \mu) \log \cos \theta + \frac{1}{2} L \dfrac{1 + \sin \mu \cos 2\theta}{\sin \mu \, (1 + \cos 2\theta)}$.

For T_3, put $1 + bt = kz$, $1 + \sin \mu + (1 - \sin \mu) \, t^2$

$= b^{-2} . \{1 + b^2 - (1 - b^2) \sin \mu - 2kz \, (1 - \sin \mu) + k^2 z^2 \, (1 - \sin \mu)\}$.

Take k such that $k^2 (1 - \sin \mu) = 1 + b^2 - (1 - b^2) \sin \mu$;

or $= (1 + b^2) \{1 - \cos \beta \sin \mu\} = \sec^2 \frac{1}{2}\beta \, (1 - \cos 2\nu)$;

$$\therefore \quad k = \sec\frac{\beta}{2}\sqrt{\frac{1-\cos 2\nu}{1-\sin\mu}}.$$

$$\text{Also let } \cos\gamma = k^{-1} = \cos\frac{\beta}{2}\sqrt{\frac{1-\sin\mu}{1-\cos 2\nu}},$$

and observe that $b^{-2}k^2(1-\sin\mu) = (\sin\tfrac{1}{2}\beta)^{-2}.(1-\cos 2\nu)$,

also $kz = 1 + bt = 1 + \tan\tfrac{1}{2}\beta\tan\theta = \dfrac{\cos(\theta-\tfrac{1}{2}\beta)}{\cos\tfrac{1}{2}\beta\cos\theta} \propto \dfrac{\cos\omega}{\cos\theta}$.

Hence

$$T_3 = \int\log\{(\sin\tfrac{1}{2}\beta)^{-2}.(1-\cos 2\nu).(1-2z\cos\gamma+z^2)\}\,d\log(kz)$$

$$= \log\{(\sin\tfrac{1}{2}\beta)^{-2}.(1-\cos 2\nu)\}\log\frac{\cos\omega}{\cos\theta} + 2\lambda(z,\gamma).$$

From this we may deduce T_4 by momentarily supposing $\mu = 0$, which makes $\cos 2\nu = 0$; so that, writing $k'y$ for kz, we get $k' = \sec\tfrac{1}{2}\beta$, and γ changes into $\tfrac{1}{2}\beta$. Also $y = \dfrac{\cos\omega}{\cos\theta}$.

$$\therefore \quad T_4 = -\log\sin^2\tfrac{1}{2}\beta\log\frac{\cos\omega}{\cos\theta} + 2\lambda(y,\tfrac{1}{2}\beta),$$

and $-T_3 + T_4$

$$= -\log(1-\cos 2\nu)\log\frac{\cos\omega}{\cos\theta} + 2\lambda(y,\tfrac{1}{2}\beta) - 2\lambda(z,\gamma);$$

in which we may deduce z, γ from $y, \tfrac{1}{2}\beta$ by writing

$$c^2 = \frac{1-\sin\mu}{1-\cos 2\nu}, \quad z = cy, \quad \cos\gamma = c.\cos\tfrac{1}{2}\beta.$$

Combining all the results, we have to observe that (neglecting constants)

$$\tfrac{1}{4}l^2(2\cos^2\nu\cos^2\omega) - l(2\sin\mu)\,l\cos\theta - l^2\cos\theta$$
$$- l(1-\cos 2\nu)(l\cos\omega - l\cos\theta)$$
$$= \log^2(n\sec\omega) - \log^2\left(\cos\theta.\frac{\sqrt{\sin\mu}}{\sin\nu}\right).$$

Whence, finally,

$$\left.\begin{aligned} X_2 &= \log^2(n\sec\omega) - \log^2\left(\cos\theta.\frac{\sqrt{\sin\mu}}{\sin\nu}\right)\\ &+ \tfrac{1}{2}L\frac{1+\sin\mu\cos 2\theta}{\sin\mu(1+\cos 2\theta)} + 2\lambda\left(y,\frac{\beta}{2}\right) - 2\lambda(z,\gamma)\end{aligned}\right\}\ldots(7).$$

Observe that $ny^{-1} = n\cos\tfrac{1}{2}\beta - x\sin\tfrac{1}{2}\beta$; and the quantity under L may also be denoted by $\dfrac{y^2X\cos^2\nu}{\sin\mu}$. The result thus obtained admits likewise of other forms, by means of the

properties of λ and Λ; but all that is here aimed at, is to shew the possibility of the reduction.

It is easy to verify our result, in the case of $a = \frac{1}{2}\pi$. On the whole it has appeared that the integral $\int F_1 x \log F_2 x dx$ contains only three elementary forms, which we have denoted by L, $\daleth$, Λ. It is proposed to call these Logarithmic Integrals *of the Second Order.*

4. Before leaving the integrals X_1, X_2, it may be well to examine the special cases of $n = 1$, and of $x = \infty$. First, to find X_1 when $x = \infty$.

Put $X' = \int_0 \tan^{-1}\frac{x}{n}\,.\,d\log X = \log X\,.\tan^{-1}\frac{x}{n} - X_1$:

$$\therefore\ \frac{dX'}{dn} = -\log X\,.\,\frac{x}{n^2+x^2} - \frac{dX_1}{dn};$$

$$\text{and when } x = \infty,\ \frac{dX'}{dn} = -\frac{dX_1}{dn}.$$

Again, $\dfrac{dX'}{dn} = \displaystyle\int_0 \frac{-x}{n^2+x^2}\,.\,d\log X$; which we assume

$$= \int_0 \left\{\frac{2px+2q}{x^2+n^2} + \frac{2r(x-\cos a)+2s\sin a}{x^2-2x\cos a+1}\right\} dx;$$

and by common methods we find that if $N = n^4 + 2n^2\cos 2a + 1$,

$$p = -r = \frac{\cos a.(n^2+1)}{N};\ q = -\frac{n(n^2+\cos 2a)}{N};\ s = \frac{(n^2-1)\sin a}{N}.$$

Also $\dfrac{dX'}{dn} = -p\log(n^2) + p\log\dfrac{x^2+n^2}{X}$

$$+ 2q\tan^{-1}\frac{x}{n} + 2s\,.\tan^{-1}.\frac{x\sin a}{1-x\cos a}.$$

Let $x = \infty$; then integrating for n, observing that $\dfrac{dX'}{dn} = -\dfrac{dX_1}{dn}$,

$$-X_1 = -\int_0 2\log n\,.pdn + \pi\int_0 qdn + 2(\pi - a)\int_0 sdn;$$

observing that as X_1 vanishes with n, no function of x is to be added. Now

$$2pdn = \frac{2\cos a\,(n^2+1)\,dn}{n^4+2n^2\cos 2a+1} = \frac{\cos a\,.dn}{n^2-2n\sin a+1} + \frac{\cos a\,dn}{n^2+2n\sin a+1};$$

$\therefore$ if $\tan\rho = \dfrac{n\cos a}{1-n\sin a}$, and $\tan\sigma = \dfrac{n\cos a}{1+n\sin a}$

$$\int_0 2\log n.pdn = \int_0 \frac{\cos a\,.\log n\,.\,dn}{n^2-2n\sin a+1} + \int_0 \frac{\cos a\,.\log n\,.\,dn}{n^2+2n\sin a+1}$$

$$= [\daleth\{\rho+(\tfrac{1}{2}\pi-a)\} - \daleth\rho - \daleth(\tfrac{1}{2}\pi-a)] + [\daleth\{\sigma+(\tfrac{1}{2}\pi+a)\} - \daleth\sigma - \daleth(\tfrac{1}{2}\pi+a)].$$

It will in a following section appear that

$$\text{ϟ}(\tfrac{1}{2}\pi - a) + \text{ϟ}(\tfrac{1}{2}\pi + a) = \text{ϟ}\pi.$$

Again, $\int_0 q\,dn = -\frac{1}{4}\log N$; $\int_0 2s\,dn = \frac{1}{2}\log . \dfrac{n^2 - 2n\sin a + 1}{n^2 + 2n\sin a + 1}$.

As before, take $\tan\beta = \tan 2\nu \cos a$, and $\cos\mu = \sin 2\nu \sin a$; add to this, $\tan\beta' = \cos 2\nu \tan a$; $\therefore \rho + \sigma = \beta$, $\rho - \sigma = a - \beta'$; from which we easily find ρ and σ. Also

$$\int_0 2s\,dn = \tfrac{1}{2}\log . \frac{1 - \sin 2\nu \sin a}{1 + \sin 2\nu \sin a} = \log\tan\tfrac{1}{2}\mu,$$

$$\text{and } N = (n^2+1)^2 . (1 - \sin^2 2\nu . \sin^2 a) = \sec^4\nu . \sin^2\mu:$$

so that finally,
$$\left.\begin{aligned}&\int_0^\infty \log(1 - 2n\tan\omega\cos a + n^2\tan^2\omega)\,d\omega\\ &= \tfrac{1}{2}\pi\log(\sec^2\nu . \sin\mu) - (\pi - a)\log\tan\tfrac{1}{2}\mu\\ &\qquad + \text{ϟ}(\tfrac{1}{2}\pi + \rho - a) - \text{ϟ}(\tfrac{1}{2}\pi - \sigma - a) - \text{ϟ}\rho - \text{ϟ}\sigma\end{aligned}\right\} \ldots(8).$$

A similar process applies to X_2 when $x = \infty$; and by help of the property (to be hereafter proved) that, when $x = \infty$,

$$\{2\Lambda(x, a) - \log^2 x\} = \tfrac{2}{3}\pi^2 - 2\pi a + a^2;$$

yields
$$\left.\begin{aligned}&\{\log^2 x - X_2\} \text{ when } x = \infty,\\ &= \tfrac{2}{3}2\pi^2 - \pi a - \tfrac{1}{2}\pi\beta - (\pi - a)\beta' + \tfrac{1}{2}\Lambda(n^2, \pi - 2a)\end{aligned}\right\}\ldots(9).$$

Lastly:

When $n=1$, $\dfrac{dX_2}{da} = \displaystyle\int_0 \frac{2x\sin a}{X} . \frac{x\,dx}{1+x^2} = \tan a . \int_0 \left\{\frac{x}{X} - \frac{x}{1+x^2}\right\} dx$

$$= \tan^{-1} . \frac{x\sin a}{1 - x\cos a} + \frac{\tan a}{2}\log\frac{X}{1+x^2}.$$

Let $\cos a = h$, and observe that

$$\frac{d\lambda(x, a)}{da} = \int_0 \frac{\sin a . dx}{X} = \tan^{-1} . \frac{x\sin a}{1 - x\cos a};$$

$$\therefore\ X_2 = f(x) + \lambda(x, a) - \tfrac{1}{2}\int_0 \log\left\{1 - \frac{2xh}{1+x^2}\right\}\frac{dh}{h}.$$

To find the arbitrary f, let $a = \frac{1}{2}\pi$, $h = 0$, $\therefore X_2 = \frac{1}{4}\log^2(1+x^2)$,

$$\text{and } \lambda(x, a) = \tfrac{1}{2}\int_0 \log(1+x^2)\frac{dx}{x} = \tfrac{1}{4}L(1+x^2);$$

$$\left.\begin{aligned}&\therefore f(x) = \tfrac{1}{4}\log^2(1+x^2) - \tfrac{1}{4}L(1+x^2)\\ &\text{and } X_2 = f(x) + \lambda(x, a) - \tfrac{1}{2}L\left(\frac{X}{1+x^2}\right)\end{aligned}\right\}\ldots .\ (10),$$

which is a simpler expression than would arise from putting $n = 1$ in equation (7).

§ II.—*On Spence's Integral* $\cdot \int_1 \frac{\log x \, dx}{x - 1}$.

5. Spence has tabulated this integral, on the assumption that x is positive; and this suffices in practice. Yet it embarrasses us in generalizing concerning the integrals which are partially reducible to L, not to be at liberty to suppose x negative. Supposing $\log x$ to have arisen out of integration, and to be $= \int \frac{dx}{x}$, no imaginary quantity results from regarding x as negative: in fact, we may look on *log* x as a short mode of writing $\frac{1}{2} \log x^2$; then, in passing through 0, x produces no discontinuity in L.

The following are the chief properties of L, which are easily verified:

$$Lx + L(-x) = \tfrac{1}{2} L(x^2) + \tfrac{3}{2} L0,$$
$$L(\pm x) + L(1 \mp x) = \log x \, . \log (1 \mp x) + L0,$$
$$Lx + Lx^{-1} = \tfrac{1}{2} \log^2 x \quad (x \text{ positive}),$$
$$L(1 + x) + L(1 - x) = \tfrac{1}{2} L(1 - x^2),$$
$$L(1 + x) + L(1 + x^{-1}) = \tfrac{1}{2} \log^2 x + C;$$

where $C = 2L2$, if x is positive; but $C = 2L0$, if x is negative. This is proved by making $x = 1$ in the former case, and $x = -1$ in the latter. The discontinuity is occasioned by $L(1 + x^{-1})$ becoming infinite, when x is passing through 0. So, if we wish to make x negative in the third formula, we must add $2L(-1)$ or $-\frac{1}{2}\pi^2$ on the right-hand side. Farther, we have

$$-L0 = 2L2 = \tfrac{1}{6}\pi^2, \quad L(-1) = -3L2 = -\tfrac{1}{4}\pi^2.$$

When $(x - 1)$ is infinitesimal,

$$Lx = x - 1, \quad \text{and } \tfrac{1}{4}\pi^2 + L(-x) = \left(\frac{x-1}{2}\right)^2.$$

When x is large,

$$L(-x+1) = 2L0 + \tfrac{1}{2} \log^2 x + 1^{-2}x^{-1} + 2^{-2}x^{-2} + 3^{-2}x^{-3} + 4^{-2}x^{-4} + \&c....$$

If we desire to know $L(-x)$ numerically, we may either calculate it by the last formula, or (when x is not large) deduce it by the first or second of the equations from Spence's Table.

In future I shall always employ $\log x$ as a mere representation of $\int \frac{dx}{x}$ or $\frac{1}{2} \log (x^2)$; and it will only be necessary,

[redacted]

in correcting integrals, to observe whether the arbitrary constant is altered by supposing the quantity under *log* to pass from positive to negative.

§. III.—*On the integral,* $-\int_0 \log \sin x\, dx$.

6. Since $\log \sin x$ and $\log \sin (-x)$ are by hypothesis the same, or to speak otherwise, since $\text{ϟ}(x) = -\tfrac{1}{2}\int_0 \log \sin^2 x\, dx$,

$$\therefore \quad \text{ϟ}(-x) = -\text{ϟ}x \ldots\ldots\ldots\ldots\ldots (11).$$

Also $\text{ϟ}(n\pi \pm x) = \mp \int \log \sin (n\pi \pm x)\, dx = \mp \int \log \sin x\, dx$,

$$\text{or} \quad \text{ϟ}(n\pi \pm x) = \text{ϟ}(n\pi) \pm \text{ϟ}x.$$

Make n successively $1, 2, 3, \ldots$ and $x = \pi$ $\Big\}$ and we find $\text{ϟ}(n\pi) = n\text{ϟ}\pi$.

Hence it readily follows that

$$\left.\begin{array}{r} \text{ϟ}(n\pi \pm x) = n\text{ϟ}\pi \pm \text{ϟ}x \\ \text{ϟ}(\pi - x) = \text{ϟ}\pi - \text{ϟ}x\,; \quad 2\text{ϟ}\tfrac{1}{2}\pi = \text{ϟ}\pi \end{array}\right\} \ldots\ldots (12).$$

These equations indicate, that to tabulate ϟ from $x = 0$ to $x = \tfrac{1}{2}\pi$ will suffice.

7. To find $\text{ϟ}\pi$.

Since $-\log(2 \sin x)$

$$= \cos 2x + 2^{-1} \cos 4x + 3^{-1} \cos 6x + 4^{-1} \cos 8x + \&c.,$$

therefore $\text{ϟ}x = x \log 2$

$$+ \tfrac{1}{2}\left\{1^{-2} \sin 2x + 2^{-2} \sin 4x + 3^{-2} \sin 6x + \&c. \ldots\right\} \ldots (13).$$

Hence $\quad \text{ϟ}\pi = \pi \log 2 = 2{\cdot}177586\ 0933046.$

Also $\quad \text{ϟ}\tfrac{1}{4}\pi = \tfrac{1}{4}\text{ϟ}\pi + \tfrac{1}{2}\left\{1^{-2} - 3^{-2} + 5^{-2} - 7^{-2} + 9^{-2} - \&c.\right\}.$

8. Since $\sin 2x = 2 \sin x \sin(\tfrac{1}{2}\pi - x)$, take logs. and integrate;

$$\therefore \quad \tfrac{1}{2}\text{ϟ}(2x) = (\tfrac{1}{2}\pi - x)\log 2 + \text{ϟ}x - \text{ϟ}(\tfrac{1}{2}\pi - x) \ldots (14).$$

We may generalize this theorem. Since

$$\sin nx = 2^{n-1}.\sin x.\sin\left(\frac{\pi}{n} + x\right).\sin\left(\frac{2\pi}{n} + x\right)\ldots\sin\left(\frac{n-1}{n}\pi + x\right),$$

take the logarithms, as before, and integrate;

$$\therefore \quad \frac{1}{n}\text{ϟ}(nx) = C - (n-1)\,x \log 2 + \text{ϟ}x + \text{ϟ}\left(\frac{\pi}{n} + x\right) + \ldots$$

$$+ \text{ϟ}\left(\frac{n-1}{n}\pi + x\right):$$

To find C, make $x = 0$;

$$\therefore \quad -C = \text{ϟ}\frac{\pi}{n} + \text{ϟ}\frac{2\pi}{n} + \ldots + \text{ϟ}\frac{n-1}{n}\pi.$$

In inverted order,

$$-C = \gimel\frac{n-1}{n}\pi + \gimel\frac{n-2}{n}\pi + \ldots + \gimel\frac{\pi}{n}.$$

Add these together, observing that

$$\gimel\left(\frac{r\pi}{n}\right) + \gimel\left(\frac{n-r}{n}\pi\right) = \gimel\pi;$$

$$\therefore\ -2C = (n-1)\gimel\pi = (n-1)\pi\log 2,$$

whence $\frac{1}{n}\gimel(nx) = -(n-1)(\frac{1}{2}\pi + x)\,l2$

$$+\gimel x + \gimel\left(\frac{\pi}{n} + x\right) + \ldots + \gimel\left(\frac{n-1}{n}\pi + x\right) \ldots\ldots (15).$$

If we change x to $-x$, remembering (11),

$$\frac{1}{n}\gimel(nx) = +(n-1)(\tfrac{1}{2}\pi - x)\,l2 + \gimel x - \gimel\left(\frac{\pi}{n} - x\right) - \&c.\ldots$$
$$- \gimel\left(\frac{n-1}{n}\pi - x\right),$$

which contains (14) as a particular case.

From either of them, by help of (12), putting $n = 3$ and $n = 5$,

$$\left.\begin{aligned} \tfrac{1}{3}\gimel(3x) &= -2xl2 + \gimel x + \gimel(60^\circ + x) - \gimel(60^\circ - x) \\ \tfrac{1}{5}\gimel(5x) &= -4xl2 + \gimel x + \gimel(36^\circ + x) - \gimel(36^\circ - x) \\ &\quad + \gimel(72^\circ + x) - \gimel(72^\circ - x) \end{aligned}\right\}\ldots(16).$$

If in (14) we make $x = 30^\circ$, and in the former equation of (16) make $x = 15^\circ$, we get, by help of (14),

$$\left.\begin{aligned} \tfrac{3}{2}\gimel 60^\circ &= \gimel 30^\circ + \tfrac{1}{3}\gimel\pi \\ \tfrac{4}{3}\gimel 45^\circ &= 2\gimel 15^\circ - \tfrac{1}{2}\gimel 30^\circ + \tfrac{1}{4}\gimel\pi \end{aligned}\right\}\ldots\ldots(17).$$

9. By help of equation (14), if a table of $\gimel$ has been computed from $x = 0$ to $x = 45^\circ$, we can continue it to $x = 90^\circ$.

Generally, if the table be given from $x = 0$ to $x = a$, we can work by a double process, as follows. First, suppose $2x$ to vary from 0 to a, in which case $\gimel x$ and $\gimel(2x)$ being known, we determine $\gimel(90^\circ - x)$ by equation (14): thus $\gimel x$ becomes known from $x = 90^\circ$ to $x = 90^\circ - \frac{1}{2}a$.

Next, let $2x$ vary within the last-named limits, *supposing a to be not less than* 45°, and x may lie within the limits $x = 0$, $x = a$; thus $\gimel(2x)\ \gimel(x)$ are again known; and we deduce $\gimel(\frac{1}{2}\pi - x)$; that is, we find $\gimel x$ from $x = \frac{1}{4}\pi$ to $x = \frac{1}{4}(\pi + a)$. Let $a_2 = \frac{1}{4}(\pi + a)$; and the process may be repeated, writing a_2 for a; then we fill the table as *high* as $x = \frac{1}{4}(\pi + a_2) = a_3$, and

as *low* as $x = \frac{1}{2}(\pi - a_2)$. Again, let $a_4 = \frac{1}{4}(\pi + a_3)$; and, by a third process, we rise as high as $x = a_4$ and come down as low as $x = \frac{1}{2}(\pi - a_3)$; and so on.

Now $a_3 = 4^{-1}\pi + 4^{-2}(\pi + a)$; $a_4 = 4^{-1}\pi + 4^{-2}\pi + 4^{-3}(\pi + a)$; &c.

Ultimately $a_\infty = \pi\,\{4^{-1} + 4^{-2} + 4^{-3} + 4^{-4} + \ldots\} = \frac{1}{3}\pi$,

and $$\tfrac{1}{2}(\pi - a_\infty) = \tfrac{1}{2}(\pi - \tfrac{1}{3}\pi) = \tfrac{1}{3}\pi.$$

Thus the opposite series meet, and the table is filled.

In practice, if x in the table passes from degree to degree, the steps will be as follows. Given the table up to $x = 45°$.

First; let $x = 1°, 2° \,..\, 22°$; and, by (14), fill the table from $x = 89°$ to $x = 68°$.

Next; let $x = 34°, 35° \,..\, 44°$; and fill from 56° to 46°.

Thirdly; let $x = 23°, 24° \,..\, 28°$; and fill from 67° to 62°.

Fourthly; let $x = 31°, 32°, 33°$; and fill from 59° to 57°.

Fifthly; let $x = 29°$, and we get $\text{ƽ}\,61°$.

Finally; $\text{ƽ}\,60°$ is found from $\text{ƽ}\,30°$ by (17).

10. If we combine the use of (14) with the former of equations (16), we can fill the whole table by starting from the limit $x = 30°$: and although errors might accumulate in so long a process, equations (15), (16) give us such easy modes of verification, that this perhaps is not to be feared. From $x = 0$ to $x = 30°$, $\text{ƽ}x$ may be found with two or three decimal figures more than are wanted in the higher parts of the table, which will obviate this difficulty.

To give conciseness to the following explanation, write y for x in the former of equations (16), then we have

$$(a)\quad \begin{cases} \frac{1}{2}\text{ƽ}\,(2x) = (\frac{1}{2}\pi - x)\,l2 + \text{ƽ}x - \text{ƽ}\,(90° - x), \\ \frac{1}{3}\text{ƽ}\,(3y) = -\,2yl2 + \text{ƽ}y + \text{ƽ}\,(60° + y) - \text{ƽ}\,(60° - y). \end{cases}$$

Suppose $\text{ƽ}x$ to have been found as high as $x = 30°$. Find

$$\text{ƽ}\,60° \text{ and } \text{ƽ}\,45° \text{ by (17).}$$

Put $x = 1°, 2°, \,..\, 15°$, and find $\text{ƽ}x'$ from $x' = 89°$ to $x' = 75°$. In equations (a) make $x = 18°$, $y = 24°$;

$$\therefore\quad \left.\begin{aligned} \tfrac{1}{2}\text{ƽ}\,36° + \text{ƽ}\,72° &= \tfrac{2}{5}\text{ƽ}\pi + \text{ƽ}\,18° \\ \text{ƽ}\,36° + \tfrac{1}{3}\text{ƽ}\,72° &= \text{ƽ}\,24° + \text{ƽ}\,84° - \tfrac{4}{15}\text{ƽ}\pi \end{aligned}\right\}.$$

Since the right-hand members of these two equations are known, we can solve for $\text{ƽ}\,36°$ and $\text{ƽ}\,72°$.

Put $y = 29°, 28°, \ldots .25°$; then $\text{ƽ}y$, $\text{ƽ}\,(3y)$ and $\text{ƽ}\,(60° + y)$ being known, we can deduce $\text{ƽ}(60° - y)$; *i.e.* we find $\text{ƽ}y'$ from $y' = 31°$ to $y' = 35°$.

, and we find $\digamma 40^\circ$: make $x=36^\circ$, and we find $\digamma 54^\circ$.
..........$\digamma 63^\circ$: .... $y=21^\circ$,..........$\digamma 39^\circ$.
o,..........$\digamma 42^\circ$: .... $x=21^\circ$,..........$\digamma 69^\circ$.
$y=23^\circ$,..........$\digamma 37^\circ$: .... $y=9^\circ$,..........$\digamma 51^\circ$.
$y=17^\circ$,..........$\digamma 43^\circ$: .... $y=12^\circ$,..........$\digamma 48^\circ$.
$y=16^\circ$,..........$\digamma 44^\circ$: .... $x=24^\circ$,..........$\digamma 66^\circ$.
$y=22^\circ$,..........$\digamma 38^\circ$: .... $x=33^\circ$,.......... $\digamma 57^\circ$.
$y=19^\circ$,..........$\digamma 41^\circ$.

he table is filled as high as $x = 45^\circ$; and the gaps in r portion of it may be completed by the former od.

1. To expand $\digamma x$ in converging series, when x does not ed 30°.

rst, put $\sin x = y$,

$$\therefore \quad \digamma x = -x \log y + \int \sin^{-1} y \, . \, y^{-1} dy \, .$$

nd $\sin^{-1} y$ and integrate. There results

$$- x \log x + 1^{-2} \sin x + \tfrac{1}{2} \, . \, 3^{-2} \sin^3 x + \frac{1.3}{2.4} \, . \, 5^{-2} \sin^5 x + \&c. \ldots (18).$$

Thus, in particular, if $x = \frac{1}{6}\pi$,

$$\digamma 30^\circ = \tfrac{1}{6} \digamma \pi + 1^{-2} . \, 2^{-1} + \tfrac{1}{2} . \, 3^{-2} . \, 2^{-3} + \frac{1.3}{2.4} 5^{-2} . \, 2^{-5} + \&c. \ldots$$

Next, let $S_n = 1^{-n} + 2^{-n} + 3^{-n} + \&c. \ldots$ a known sum; and $x = \pi\omega$;

$$\therefore \quad \log \sin (\pi\omega) = \log (\pi\omega) - S_2 \frac{\omega^2}{1} - S_4 \frac{\omega^4}{2} - S_6 \frac{\omega^6}{3} - \&c. \ldots$$

Integrate:

$$\frac{1}{\pi} \digamma (\pi\omega) = \omega \left\{ 1 - \log \pi\omega \right\} + S_2 \, . \, \frac{\omega^3}{1.3} + S_4 \, . \, \frac{\omega^5}{2.5} + S_6 \, . \, \frac{\omega^7}{3.7} + \&c. \ldots\ldots (19).$$

To increase the convergence, add to the penultimate series before integration:

$$- \log (1 - \omega^2) = \omega^2 + \tfrac{1}{2} \omega^4 + \tfrac{1}{3} \omega^6 + \ldots . \&c.$$

$$\therefore \quad - \log \sin (\pi\omega) + \log (1 - \omega^2)$$

$$= - \log (\pi\omega) + (S_2 - 1) \frac{\omega^2}{1} + (S_4 - 1) \tfrac{1}{2} \omega^4 + (S_6 - 1) \tfrac{1}{3} \omega^6 + \&c.$$

whence

$$\left. \begin{aligned} \frac{1}{\pi} \digamma (\pi\omega) = \omega \{ 3 - \log \pi\omega - \log (1-\omega) - \log (1+\omega) \} - \log \frac{1+\omega}{1-\omega} \\ + (S_2 - 1) \frac{\omega^3}{1.3} + (S_4 - 1) \frac{\omega^5}{2.5} + (S_6 - 1) \frac{\omega^7}{3.7} + \&c. \ldots \end{aligned} \right\} \ldots (20),$$

and if ω is less than $\frac{1}{6}$, each term of the series is less than 144^{th} of that which precedes it.

If the coefficients are formed into a table, and the series be adapted (if necessary) to the common logarithms, it will enable us to compute $\daleth x$ from $x = 0$ to $x = 30°$ with much ease. The most troublesome part of the calculation, when many decimal places are required, is the multiplying by π, (or by π . hyp. log. 10, as the case may need.)

12. We may modify the process so as to obtain a somewhat simpler series, thus: Since

$$\daleth x = -x \log \sin x + \int_0 x \cot x \, dx,$$

$$\text{also} \quad \cot x = \frac{1}{x} - \frac{2x}{\pi^2 - x^2} - \frac{2x}{4\pi^2 - x^2} - \frac{2x}{9\pi^2 - x^2} - \&c.,$$

$$\text{and} \int_0 \frac{-2x^2\,dx}{r^2\pi^2 - x^2} = 2x - r\pi \,.\, \log \frac{r\pi + x}{r\pi - x}; \quad \text{let } x = \pi\omega:$$

$$\left.\begin{aligned} \therefore \; \frac{1}{\pi}\daleth(\pi\omega) &= \omega\,(1 - l \sin . \pi\omega) \\ &+ \left(2\omega - l \,.\, \frac{1+\omega}{1-\omega}\right) + \left(2\omega - 2l\,\frac{2+\omega}{2-\omega}\right) + \ldots. \\ &\ldots.+ \left(2\omega - r \log . \frac{r+\omega}{r-\omega}\right) + \ldots. \end{aligned}\right\} \ldots (21).$$

Take n terms of this series, so that $r = (n-1)$ in the last, and let R equal the remainder. Put N_m for $n^{-m} + (n+1)^{-m} + (n+2)^{-m} + \&c.\,.\,.$ and observe that

$$2\omega - r \log \frac{r+\omega}{r-\omega} = -\,2\omega \left\{\frac{\omega^2}{3r^2} + \frac{\omega^4}{5r^4} + \frac{\omega^6}{7r^6} + \&c.\right\};$$

$$\therefore \; \tfrac{1}{2}R = -\,N_2\,\tfrac{1}{3}\omega^3 - N_4\,\tfrac{1}{5}\omega^5 - N_6 \,.\, \tfrac{1}{7}\omega^7 - \&c. \ldots (22).$$

If we take the most obvious case of $n = 2$, $N_m = S_m - 1$;

$$\left.\begin{aligned} &\therefore \; \frac{1}{\pi}\daleth(\pi\omega) = 3\omega - \omega \log \sin . \pi\omega - \log \frac{1+\omega}{1-\omega} \\ &-(S_2 - 1)\,\tfrac{1}{3}2\omega^3 - (S_4 - 1)\,\tfrac{1}{5}2\omega^5 - (S_6 - 1)\,\tfrac{1}{7}2\omega^7 - \&c. \end{aligned}\right\} \ldots (23).$$

If between the two formulas (20) and (23) we eliminate the term which contains $(S_2 - 1)$, we get

$$\frac{3}{\pi}\daleth(\pi\omega) = \omega\,\{9 - 2\log \pi\omega - \log \sin \pi\omega - 2 \log . (1 - \omega^2)\}$$

$$-\,3 \log \frac{1+\omega}{1-\omega} - (S_4 - 1)(1 - \tfrac{1}{2})\,\tfrac{1}{5}2\omega^5 - (S_6 - 1)(1 - \tfrac{1}{3}) . \tfrac{1}{7}2\omega^7 - \&c.\,.\,.$$

which may have some advantage when $(S_4 - 1) . \frac{1}{5}\omega^5$ is small enough to omit.

xpand ϟ$(90° - x)$ when x is small.

$\int$log cos $x\,dx =$ ϟ$\tfrac{1}{2}\pi + x \log\cos x + \int_0 \tan x\,.\,x dx$.

z, and assume

$$\frac{z\tan^{-1}z}{1+z^2} = A_1 z^2 + A_2 z^4 + A_3 z^6 - \&c.\,;$$

$$\therefore\ \tan^{-1}z = \left.\begin{array}{l} A_1 z - A_2 z^3 + A_3 z^5 - A_4 z^7 + \&c.\,.\,. \\ \quad + A_1 z^3 - A_2 z^5 + A_3 z^7 - \&c.\,.\,. \end{array}\right\}$$

ence $A_1 = 1,\ A_2 = 1 + \tfrac{1}{3},\ A_3 = 1 + \tfrac{1}{3} + \tfrac{1}{7}$, &c. . .

$$\int_0 \tan x\,.\,x dx = \int_0 \frac{z\tan^{-1}z\,.\,dz}{1+z^2} = A_1\tfrac{1}{3}z^3 - A_2\,.\,\tfrac{1}{5}z^5 + A_3\tfrac{1}{7}z^7 - \&c.\,.\,.$$

$$= \{\tfrac{1}{3}z^3 - \tfrac{1}{5}z^5 + \tfrac{1}{7}z^7 - \&c.\,.\,.\} - \tfrac{1}{3}\{\tfrac{1}{5}z^5 - \tfrac{1}{7}z^7 + \tfrac{1}{9}z^9 - \&c.\}$$
$$+ \tfrac{1}{5}\{\tfrac{1}{7}z^7 - \tfrac{1}{9}z^9 + \&c.\,.\,.\} - \tfrac{1}{7}\ \&c.\ \&c.\,.\,.\,.\,.$$

us henceforth use $\phi_n x$ for $\int_0 \tan^{n-1} x\,.\,dx$,

$$\text{or}\quad \phi_n x = \frac{\tan^n x}{n} - \frac{\tan^{n+2}x}{n+2} + \frac{\tan^{n+4}x}{n+4} - \&c.\,.\,.\,.\,.$$

hat $\phi_1 x = x$; $\phi_2 x = \log\sec x$; and $\phi_{n+2} x = \dfrac{\tan^n x}{n} - \phi_n x$:

and we finally obtain

$$\text{ϟ}(\tfrac{1}{2}\pi - x) = \text{ϟ}\tfrac{1}{2}\pi + x\log\cos x$$
$$+ \phi_3 x - \tfrac{1}{3}\phi_5 x + \tfrac{1}{5}\phi_7 x - \tfrac{1}{7}\phi_9 x + \&c.\,.\,.\,.\,(24),$$

When x is $< 10°$, $\phi_7 x$ will not affect the sixth decimal.

To obtain a more converging series, let $v = 1 - \cos x$.

$$\int_0 \tan x\,.\,x dx = \int - x\,.\,d\log\cos x = \int_0 \frac{\cos^{-1}(1-v)\,.\,dv}{1-v}\,.$$

$$\text{Now } \cos^{-1}(1-v) = \sqrt{(2v)}\left\{1 + \frac{1}{2.3}\,.\,(\tfrac{1}{2}v) + \frac{1.3}{2.4.5}\,.\,(\tfrac{1}{2}v)^2 + \&c.\right\}.$$

$$\text{Let, then, } \frac{\cos^{-1}(1-v)}{1-v} = \sqrt{(2v)}\,\{B_1 + B_2 v + B_3 v^2 + \&c.\,.\,.\,.\}$$

and we get $B_1 = 1$, $B_2 = B_1 + \tfrac{1}{2}.\tfrac{1}{3}2^{-1}$; $B_3 = B_2 + \dfrac{1.3}{2.4}\,.\,\tfrac{1}{5}2^{-2}$; &c. . .

$$\text{whence } B_\infty = 1 + \tfrac{1}{2}.\tfrac{1}{3}2^{-1} + \frac{1.3}{2.4}\,.\,\tfrac{1}{5}2^{-2} + \frac{1.3.5}{2.4.6}\,.\,\tfrac{1}{7}2^{-3} + \&c.\,.\,.$$

$$= \frac{1}{\sqrt{2}}\cos^{-1}.\,\{1 - 1\} = \frac{\pi}{2\sqrt{2}}\,.$$

To increase therefore the convergence, put $C_n = \dfrac{\pi}{2\sqrt{2}} - B_n$;

so that $\qquad C_{n+1} - C_n = B_n - B_{n+1}$;

$$\therefore\ \frac{\cos^{-1}(1-v)}{1-v}=\frac{\pi}{2\sqrt{2}}\,.\,\sqrt{(2v)}\,(1+v+v^2+v^3+\,.\,.)$$
$$-\sqrt{(2v)}\,(C_1+C_2v+C_3v^2+C_4v^3+\ldots.)$$
$$=\tfrac{1}{2}\pi\,.\,\frac{\sqrt{v}}{1-v}-\sqrt{2}\,(C_1v^{\frac{1}{2}}+C_2v^{\frac{3}{2}}+C_3v^{\frac{5}{2}}+\&c.\ldots.),$$

whence $\displaystyle\int_0\frac{\cos^{-1}(1-v)}{1-v}\,dv=\tfrac{1}{2}\pi\log\frac{1+\sqrt{v}}{1-\sqrt{v}}-\pi\sqrt{v}$

$$-2\sqrt{2}\,\{C_1\tfrac{1}{3}v^{\frac{3}{2}}+C_2\tfrac{1}{5}v^{\frac{5}{2}}+\&c.\ldots.\}.$$

Put $C_0=\dfrac{\pi}{2\sqrt{2}}$ for uniformity;

$$\therefore\ C_1=C_0-1;\ C_2=C_1-\tfrac{1}{2}\,.\,\tfrac{1}{3}2^{-1};\ C_3=C_2-\frac{1.3}{2.4}\,.\,\tfrac{1}{5}2^{-2};\ \&c.\ldots$$

and C_0, C_1, C_2. . may be easily tabulated. Finally, observing that $\sqrt{(2v)}=2\sin\frac{1}{2}x$, and modifying the fraction under *log*

$$\daleth(\tfrac{1}{2}\pi-x)=\daleth\tfrac{1}{2}\pi+x\log\cos x+\tfrac{1}{2}\pi\log\frac{\tan\frac{1}{4}(\frac{1}{2}\pi+x)}{\tan\frac{1}{4}(\frac{1}{2}\pi-x)}$$
$$-4\sin\tfrac{1}{2}x\,\{C_0+C_1\,.\,\tfrac{1}{3}v+C_2\,.\,\tfrac{1}{5}v^2+C_3\,.\,\tfrac{1}{7}v^3+\&c.\,.\}\,.\,.(25)$$

which converges well even when x is as large as 60°.

14. In constructing a table of $\daleth x$ we need to find $\Delta\daleth x$, or $\daleth(x+h)-\daleth x$.

Taylor's theorem may of course be used, but the law of the terms is cumbrous. As a substitute for it, let us recal equation (2), which gave

$$\sin a\,.\int_0\frac{\log x\,dx}{X}=\daleth(\omega+a)-\daleth\omega-\daleth a,\ \text{where}\tan\omega=\frac{x\sin a}{1-x\cos a}.$$

whence

$$\omega=\tan^{-1}\left(\frac{x\sin a}{1-x\cos a}\right)=x\sin a+\tfrac{1}{2}x^2\sin 2a+\tfrac{1}{3}x^3\sin 3a+\,.\,.$$

Also $\displaystyle\sin a\,.\int_0\frac{\log x\,dx}{X}=\int_0\log x\,.\,d\omega=\omega\log x-\int_0\omega\,.\,\frac{dx}{x};$

$$\therefore\ \daleth(\omega+a)-\daleth a=\daleth\omega+\omega\log x$$
$$-1^{-2}x\sin a-2^{-2}x^2\sin 2a-3^{-2}x^3\sin 3a-\&c.$$

To conform to the usual notation, write x, h for a, ω, and then we must put y for x. We hereby find that

If y stands for $\dfrac{\sin h}{\sin(x+h)}$,

$$\therefore\ \Delta\daleth x=\daleth h+h\log y$$
$$-1^{-2}y\sin x-2^{-2}y^2\sin 2x-3^{-2}y^3\sin 3x-\&c.\,.\,.(26).$$

compared with h, and $\daleth h$ is known, y will be to give a good convergence, and the law of the ple and convenient. Nevertheless, methods of iation, such as the following, will be often better.

st, let $m_1, m_2, m_3 ..$ satisfy the equation

$$x \div \log(1+x) = 1 + m_1 x + m_2 x^2 + m_3 x^3 + \ldots.$$

$$\Delta \int Fx\, dx = h\{Fx + m_1 \Delta Fx + m_2 \Delta^2 Fx + \&c.\ldots.\}$$

by slightly modifying the process, it is easy to shew that have also

$$dx = h\{F(x+h) - m_1 \Delta Fx$$
$$+ m_2 \Delta^2 F(x-h) - m_3 \Delta^3 F(x-2h) + \&c.\},$$

re F is any function whatever. Here we assume

$$Fx = -\log \sin x:$$

$$m_1 = \tfrac{1}{2},\quad m_2 = -\tfrac{1}{12},\quad m_3 = \tfrac{1}{24};$$

$$= -h\{l \sin x + \tfrac{1}{2}\Delta l \sin x - \tfrac{1}{12}\Delta^2 l \sin x + \tfrac{1}{24}\Delta^3 l \sin x\}\ldots(27),$$

$$x = -h\{l \sin(x+h) - \tfrac{1}{2}\Delta l \sin x$$
$$- \tfrac{1}{12}\Delta^2 l \sin(x-h) - \tfrac{1}{24}\Delta^3 l \sin(x-2h)\}\quad .(28).$$

ke the sum:

$$\therefore\ 2\Delta \daleth x = -h\,[l \sin(x+h) - \tfrac{1}{12}\{\Delta^2 l \sin x + \Delta^2 l\,.\sin(x-h)\}$$
$$+ l \sin x + \tfrac{1}{24}\{\Delta^3 l \sin x - \Delta^3 l \sin(x-2h)\}];$$

or, when the last term is negligible,

$$\Delta \daleth x = -\tfrac{1}{2}h\,[l \sin(x+h) + l \sin x - \tfrac{1}{12}\{\Delta^2 l \sin x$$
$$+ \Delta^2 l \sin.(x-h)\}]\ldots(29).$$

But perhaps certain series of Legendre's are better still.

Let $M_1, M_2, M_3 \ldots$ be determined by the equation

$$\tfrac{1}{2}x \div \sin^{-1}(\tfrac{1}{2}x) = 1 + M_1 x^2 + M_2 x^4 + M_3 x^6 + \ldots.$$

then $M_1 = \dfrac{1}{24},\quad M_2 = -\dfrac{17}{24^2.\,10},\quad M_3 = \dfrac{367}{8^2.5.6.7.8.9};$

and we have

$$\Delta \int Fx\,dx = h\{F(x+\tfrac{1}{2}h) + M_1 \Delta^2 F(x-\tfrac{1}{2}h) + M_2 \Delta^4 F(x-\tfrac{1}{2}3h) + \&c\ldots\}$$

also $= h\{F(x+\tfrac{1}{2}h) + M_1 \Delta^2 F(x+\tfrac{1}{2}3h) + M_2 \Delta^4 F(x+\tfrac{1}{2}5h) + \&c\ldots\}$

which here give

$$\left.\begin{array}{l} \Delta \daleth x = -h\{l \sin(x+\tfrac{1}{2}h) \\ \quad + \tfrac{1}{24}\Delta^2 l \sin(x-\tfrac{1}{2}h) - \tfrac{17}{5760}\Delta^4 l \sin(x-\tfrac{1}{2}3h) + \&c.\} \\ \text{also} = -h\{l \sin(x+\tfrac{1}{2}h) \\ \quad + \tfrac{1}{24}\Delta^2 l \sin(x+\tfrac{1}{2}3h) - \tfrac{17}{5760}\Delta^4 l \sin(x+\tfrac{1}{2}5h) + \&c.\} \end{array}\right\}\ldots(30),$$

which are easy to us, because we have tables of *log sin.*

Again, let N_1, N_2, N_3. . . .be such that

$$1 - N_1x^2 + N_2x^4 - N_3x^6 + \&c.\ldots = (1 + M_1x^2 + M_2x^4 + \&c.\ldots)^2;$$

$$\therefore \Delta^2 \int Fx dx = h^2 \{F'(x+h) + N_1\Delta^2 F'x + N_2\Delta^4 F'(x-h) + \&c.\}$$

$$\text{also} = h^2 \{F'(x+h + N_1\Delta^2 F'(x+2h) + N_2\Delta^4 F'(x+3h) + \&c.\},$$

$$\text{where } N_1 = \frac{1}{12}, \quad N_2 = -\frac{1}{240}, \quad N_3 = \frac{31}{4.5.6.7.8.9}.$$

Put $Fx = -\log \sin x$, $F'x = -\cot x$.

$$\left.\begin{aligned} \therefore \Delta^2 ϟ x &= -h^2 \{\cot(x+h) \\ &\qquad + \tfrac{1}{12}\Delta^2 \cot x - \tfrac{1}{240}\Delta^4 \cot(x-h) + \&c.\} \\ \text{also} \quad &= -h^2 \{\cot(x+h) \\ &\quad + \tfrac{1}{12}\Delta^2 \cot(x+2h) - \tfrac{1}{240}\Delta^4 \cot(x+3h) + \&c.\} \end{aligned}\right\} \ldots (31).$$

§. IV.—*Applications of* ϟ.

15. (1) To find $\Theta = -\int_0 \log(\sin^2\theta - \sin^2 a)\, d\theta$,
or $= -\int_0 \log(\cos^2 a - \cos^2\theta)\, d\theta$.

Observe that $\sin^2\theta - \sin^2 a = \sin(\theta + a) . \sin(\theta - a)$.

$$\therefore \quad \Theta = ϟ(\theta + a) + ϟ(\theta - a), \quad \text{or} = ϟ(a+\theta) - ϟ(a - \theta).$$

(2) To find $\Theta = -\int_0 \log(\cos\theta - \cos a)\, d\theta$.

Since $\cos\theta - \cos a = 2(\cos^2 \frac{1}{2}\theta - \cos^2 \frac{1}{2}a)$,

$$\Theta = -\theta \log 2 + 2ϟ\left(\frac{a+\theta}{2}\right) - 2ϟ\left(\frac{a-\theta}{2}\right).$$

(3) If $\Theta = \int_0 \log(1 + \sec\mu \cos\theta)\, d\theta$,
since $\log(1 + \sec\mu\cos\theta) = \log\{\cos\mu - \cos(\pi - \theta)\} - \log\cos\mu$,

$$\therefore \Theta = -\theta\log(2\cos\mu) + 2ϟ\left(\frac{\mu + \pi - \theta}{2}\right) - 2ϟ\left(\frac{\mu - \pi + \theta}{2}\right).$$

(4) If $\Theta = \int_0 \log(1 + \sin\mu\cos\theta)\, d\theta$, put $\tan\eta = \dfrac{\sin\theta}{\tan\frac{1}{2}\mu + \cos\theta}$,
and we had by equation (6)

$$\tfrac{1}{2}\Theta = ϟ(\theta - \eta) + ϟ\eta - ϟ\theta + \theta\log\sin\tfrac{1}{2}\mu - \eta\log\tan\tfrac{1}{2}\mu.$$

Thus $\int \log(a \pm b\cos\theta)\, d\theta$ can always be found by ϟ.

(5) If $\Theta = \int_0 \log(\tan\theta + \tan a)\, d\theta$;

$$\text{since } \tan\theta + \tan a = \frac{\sin(\theta + a)}{\cos\theta\cos a},$$

$$\Theta = ϟ\tfrac{1}{2}\pi + ϟ a - \theta\log\cos a - ϟ(\theta + a) - ϟ(\tfrac{1}{2}\pi - \theta).$$

d $S = 1^{-2}x - 3^{-2}x^3 + 5^{-2}x^5 - 7^{-2}x^7 +$ &c.. . . . In the h gave rise to equation (26), put $a = \frac{1}{2}\pi$; and observing that $\daleth(\omega + \frac{1}{2}\pi) - \daleth\frac{1}{2}\pi = \daleth\frac{1}{2}\pi$. - $\omega)$, we get

$$S = \omega \log x + \daleth\omega + \daleth(\tfrac{1}{2}\pi - \omega) - \tfrac{1}{2}\daleth\pi.$$

series S received from Spence a special discussion.

) If $\Theta = \int \log(1 - 2n \cos a \tan\theta + n^2\tan^2\theta)\,d\theta$, we reduce o No. (4), as in the process for finding X. See equa- 5).

If $\Theta = \int_0 \log(\tan^2\theta + \tan^2\beta)\,d\theta$, make $n = \cot\beta$;

$\therefore\quad \Theta = 2\theta\log\tan\beta + \int_0 \log(1 + n^2\tan^2\theta)\,d\theta.$

last falls under Ex. (7), as a particular case, when $\frac{1}{2}\pi$.

If $\Omega = \int_0 \log(1 - 2r\cos a \sin\omega + r^2\sin^2\omega)\,d\omega$, this also be reduced to $\daleth$, by the following process.

ppose r positive, $x = \tan\frac{1}{2}\omega$, or $\sin\omega = \dfrac{2x}{1+x^2}$: and $x^2 = \sec^2\frac{1}{2}\omega$;

$$\therefore \Omega = \int_0 \log\{1 - 4rx\cos a + (4r^2+2)x^2 - 4rx^3\cos a + x^4\}\,d\omega$$
$$+ 8\daleth\left(\frac{\pi - \omega}{2}\right).$$

Assume the quantity under *log* to be

$$= (1 - 2nx\cos\gamma + n^2x^2)(1 - 2n^{-1}x\cos\gamma + n^{-2}x^2);$$

$\therefore (n + n^{-1})\cos\gamma = 2r\cos a$, and $4r^2 + 2 = 4\cos^2\gamma + n^2 + n^{-2}$.

Let $\tan\nu = n$, $\tan\rho = r$; $\therefore 4r^2 + 4 = 4\cos^2\gamma + (n + n^{-1})^2$;

or $\cos\gamma = r\cos a \sin 2\nu$, and $2r\,\mathrm{cosec}\,2\rho = \cos^2\gamma + \mathrm{cosec}^2\,2\nu$.

Eliminate γ, and solve for cosec 2ν. The result is, that if we take $\sin 2\zeta = \cos a \sin 2\rho$, and select that root of ζ which makes $\pm\sin\zeta$ least, we have

$$\sin 2\nu = \frac{\cos\rho}{\cos\zeta},\ \text{and}\ \cos\gamma = \frac{\sin\zeta}{\cos\rho}.$$

Hence, having found ν and γ,

$$\left.\begin{aligned} \text{let}\ \ \Omega' &= \int_0 \log(1 - 2n\cos\gamma\tan\tfrac{1}{2}\omega + n^2\tan^2\tfrac{1}{2}\omega)\,\tfrac{1}{2}d\omega \\ \Omega'' &= \int_0 \log(1 - 2n^{-1}\cos\gamma\tan\tfrac{1}{2}\omega + n^{-2}\tan^2\tfrac{1}{2}\omega)\,\tfrac{1}{2}d\omega \end{aligned}\right\}$$

where we pass from Ω' to Ω'' by changing ν to $(\frac{1}{2}\pi - \nu)$; then Ω becomes $= 2\Omega' + 2\Omega'' + 8\daleth\left(\dfrac{\pi - \omega}{2}\right)$.

As in the process of Art. (2), make $\tan\beta = \tan 2\nu \cos\gamma$, $\cos\mu = \sin 2\nu \sin\gamma$, and $\theta = \omega + \beta$; observing that γ and $\frac{1}{2}\omega$ replace a and ω. When ν becomes $(\frac{1}{2}\pi - \nu)$, β becomes $-\beta$ and μ is unchanged. Let $\theta' = \omega - \beta$; then

$$\Omega' = \tfrac{1}{2}\int \log(1 + \sin\mu\cos\theta)\, d\theta - \tfrac{1}{2}\omega\log(2\cos^2\nu) - 2ち\left(\frac{\pi-\omega}{2}\right):$$

whence $$\Omega = \int\log(1+\sin\mu\cos\theta)\,d\theta + \int\log(1+\sin\mu\cos\theta')\,d\theta' - 2\omega\log\sin 2\nu \quad ..(32),$$

which reduces the case to No. (4).

(10) When F is rational, $\int \log Fx \cdot \frac{dx}{\sqrt{(p^2-x^2)}}$ is reducible to the sum or difference of integrals such as

$$\int\log(a+bx)\frac{dx}{\sqrt{(p^2-x^2)}} \text{ and } \int\log(1-2x\cos a + x^2)\frac{dx}{\sqrt{(r^2-x^2)}}.$$

The former falls under No. (3) or No. (4), if $x = p\cos\theta$. The latter is the case of No. (9), if $x = r\cos\omega$. Thus $\int\log Fx \cdot \frac{dx}{\sqrt{(p^2-x^2)}}$ is wholly reducible to ち.

(11) To find $\int\frac{x^m \log x\, dx}{(a+bx^2)^{n+\frac{1}{2}}}$, when m and n are integers, positive or negative.

Represent the integral by $V_{m,n}$ and let $U_{m,n} = \frac{x^m\log x}{(a+bx^2)^{n-\frac{1}{2}}}$. Differentiate $U_{m,n}$ and integrate back again; and we obtain

$$U_{m,n} - \int\frac{x^{m-1}\,dx}{(a+bx^2)^{n-\frac{1}{2}}} = mV_{m-1,n-1} - (2n-1)\,bV_{m+1,n} \;\ldots\ldots (1),^*$$
$$= maV_{m-1,n} - (2n-m-1)\,bV_{m+1,n} \;..\; (2),^*$$
$$= (2n-1)aV_{m-1,n} - (2n-m-1)bV_{m-1,n-1} ..(3).^*$$

When $m = 0$, the first gives $V_{1,n}$ but fails to reduce $V_{-1,n-1}$ to $V_{1,n}$: also the second then merges in the first. Yet by the third, $V_{-1,n-1}$ can be reduced to $V_{-1,n}$. When $m = 2n-1$, the second and third coincide, and give $V_{2n-2,n}$, but fail to reduce $V_{2n-2,n-1}$ to $V_{2n-2,n}$.

There being no other cases of failure, and $V_{2n-2,n-1}$ being reducible by the first formula to $V_{\pm 4,\pm 2}$, $V_{\pm 2,\pm 1}$ and finally to $V_{0,0}$; it is evident that $V_{m,n}$ is universally reducible to one of the three, $V_{1,0}$, $V_{0,0}$, $V_{-1,0}$, of which the first (being included under $V_{1,n}$) can be found by common methods. This is the case of m being positive and odd. If m is negative and odd, $V_{m,n}$ is reduced to $V_{-1,0}$; if m is even, it is reduced to $V_{0,0}$.

et $x = y^{-1}$; $\therefore \int \frac{\log x\, dx}{x\sqrt{(a+bx^2)}} = \int \frac{\log y\, dy}{\sqrt{(ay^2+b)}}$; which is
e form as $V_{0.0}$; which alone remains. When $a = 1$
$= -1$, put $x = \sin \omega$, $\therefore V_{0.0} = -\text{ϟ}\omega$. When $a = -1$
$b = 1$, let $y = x + \sqrt{(x^2 - 1)}$, $\frac{dx}{\sqrt{(x^2-1)}} = \frac{dy}{y}$; $\log x = \log$
$y^2) - \log (2y)$;

$$\therefore \quad V_{0.0} = \int \log (1 + y^2) \frac{dy}{y} - \int \log (2y) \frac{dy}{y}$$

$$= \tfrac{1}{2} L (1 + y^2) - \tfrac{1}{2} \log^2 (2y) + \text{const.}$$

n $\quad a = 1$ and $b = 1$, let $y = x + \sqrt{(x^2 + 1)}$:

$V_{0.0} = \log y \,.\, \log (y^2 - 1) - \tfrac{1}{2} L (y^2) - \tfrac{1}{2} \log^2 (2y) + \text{const.}$

recapitulate then: $V_{m,n}$ is always reducible to ϟ or L; in particular, $\int \frac{x^{2m-1} \log x\, dx}{(a + bx^2)^{n+\frac{1}{2}}}$ can be found by circular and logarithms;

$\int \frac{x^{2m} \log x\, dx}{(1 - x^2)^{n+\frac{1}{2}}}$ is reducible to ϟ; $\int \frac{x^{2m} \log x\, dx}{(x^2 \pm 1)^{n+\frac{1}{2}}}$ to L;

$$\int \frac{x^{-2m+1} \log x\, dx}{(x^2 - 1)^{n+\frac{1}{2}}} \text{ to ϟ}; \quad \int \frac{x^{-2m+1} \log x\, dx}{(1 \pm x^2)^{n+\frac{1}{2}}} \text{ to } L;$$

where m and n are integers, m positive and n either positive or negative.

§. V.—*On the Higher Transcendents derivable from* ϟ.

16. Spence has imagined the integrals L^3, L^4, L^5. . deduced from L^2 or L, by the law $L^n (1 + x) = \int L^{n-1} (1 + x)\, x^{-1}\, dx$; and has exhibited various fundamental properties of L^n. Put

$$x = e^{2\omega}, \quad \therefore e^{2\omega} + 1 = (e^{\omega} + e^{-\omega})\, e^{\omega},$$

or $\log (x + 1) = \log (\varepsilon^{\omega} + \varepsilon^{-\omega}) + \omega$, and $x^{-1} dx = 2d\omega$;

$$\therefore L (1 + x) = \int \log (\varepsilon^{\omega} + \varepsilon^{-\omega})\, 2d\omega + \omega^2.$$

When ω changes to $\omega \sqrt{(-1)}$, the integral here becomes $2\sqrt{(-1)} \,.\, \int \log (2 \cos \omega)\, d\omega$, which exhibits the relation which exists between L and ϟ by imaginaries.

17. Since $d\omega \propto x^{-1} dx$, we may imagine a series of functions $\text{ϟ}^3, \text{ϟ}^4, \text{ϟ}^5$. . analogous to L^3, L^4, L^5. ., by the law $\text{ϟ}^3 x = \int_0 \text{ϟ} x\, dx$, and generally $\text{ϟ}^n x = \int_0 \text{ϟ}^{n-1} x\, dx$; and we now regard ϟ as virtually ϟ^2. Write also

$$\lambda_2 x = - \int_0 \log x\, dx, \quad \lambda_n x = \int_0 \lambda_{n-1}\, x\, dx;$$

then $\lambda_2 x = x(1 - \log x)$,

$$\lambda_{n+1} x = \frac{x^n}{1.2\ldots n}\left\{1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots\ldots + \frac{1}{n} - \log x\right\} \ldots (33).$$

Also, as $\text{ϟ}^2 x = \lambda_2 x + H_1 \dfrac{x^3}{1.3} + H_2 \dfrac{x^5}{2.5} + H_3 . \dfrac{x^7}{3.7} + \&\text{c}\ldots\ldots$

if H_n stands for $\pi^{-2n} S_{2n}$; (see equation 19.)

$$\therefore \text{ϟ}^n x = \lambda_n x + \frac{2H_1 x^{n+1}}{2.3\ldots(n+1)} + \frac{2H_2 x^{n+3}}{4.5\ldots(n+3)} + \frac{2H_3 x^{n+5}}{6.7\ldots(n+5)} + \&\text{c.} \quad \ldots\ldots(34).$$

18. Since $2\text{ϟ}^2(\tfrac{1}{2}x) = xl2 + 1^{-2}\sin x + 2^{-2}\sin 2x + 3^{-2}\sin 3x + \&\text{c}\ldots$. perpetual integration, with suitable addition of constants, gives

$$\left.\begin{array}{l} 2^{2n-1}.\,\text{ϟ}^{2n}(\tfrac{1}{2}x) = \dfrac{x^{2n-1} l2}{1.2\ldots(2n-1)} \\ + \dfrac{x^{2n-3} S_3}{1.2\ldots(2n-3)} - \dfrac{x^{2n-5} S_5}{1.2\ldots(2n-5)} + \&\text{c}\ldots\ldots \pm \dfrac{x}{1} S_{2n-1} \\ - 1^{-2n}\sin(x - 2n.\tfrac{1}{2})\pi - 2^{-2n}\sin(2x - 2n\,\tfrac{1}{2}\pi) \\ \qquad - 3^{-2n}\sin(3x - 2n\,\tfrac{1}{2}\pi) - \&\text{c.} \end{array}\right\} \ldots (35).$$

$$\left.\begin{array}{l} 2^{2n}\,\text{ϟ}^{2n+1}(\tfrac{1}{2}x) = \dfrac{x^{2n} l2}{1.2\,.\,.2n} \\ + \dfrac{x^{2n-2} S_3}{1.2\,.\,.(2n-2)} - \dfrac{x^{2n-4} S_5}{1.2\,.\,.(2n-4)} + \&\text{c}\ldots \mp S_{2n+1} \\ - 1^{-2n-1}\sin\{x - (2n+1).\tfrac{1}{2}\pi\} \\ \qquad - 2^{-2n-1}\sin\{2x - (2n+1)\,\tfrac{1}{2}\pi\} - \&\text{c}\ldots\ldots \end{array}\right\} \ldots (36).$$

And if in these we put $x = 2\pi$, we get

$$\text{ϟ}^3\pi = \frac{\pi^2 l2}{1.2} : \quad \text{ϟ}^4\pi = \frac{\pi^3 l2}{1.2.3} + 2^{-2}.\frac{\pi}{1} S_3 :$$

$$\text{ϟ}^5\pi = \frac{\pi^4 l2}{1.2.3.4} + 2^{-2}.\frac{\pi^2}{1.2}.S_3 :$$

$$\text{ϟ}^6\pi = \frac{\pi^5 l2}{1.2\,.\,.5} + 2^{-2}.\frac{\pi^3}{1.2.3} S_3 - 2^{-4}.\frac{\pi}{1} S_5 :$$

$$\text{ϟ}^7\pi = \frac{\pi^6 l2}{1.2\,.\,.6} + 2^{-2}.\frac{\pi^4}{1.2.3.4} S_3 - 2^{-4}.\frac{\pi^2}{1.2}.S_5 :$$

The law is evident. After the two first terms, the signs are alternate. Thus $\text{ϟ}^n\pi$ is known.

19. If in equations (35), (36) we make $x = \pi$; and with reference to the latter, observe that

$$1^{-m} - 2^{-m} + 3^{-m} - 4^{-m} + \&c.\ldots = (1 - 2^{-m+1}) S_m ;$$

we obtain

$$2^{2n-1} \text{ל}^{2n} (\tfrac{1}{2}\pi) = \frac{\pi^{2n-1} l2}{1.2..(2n-1)}$$

$$+ \frac{\pi^{2n-3} S_3}{1.2..(2n-3)} - \frac{\pi^{2n-5} S_5}{1.2..(2n-5)} + \ldots\ldots \pm \frac{\pi}{1} S_{2n-1} \ldots (37):$$

$$2^{2n} \text{ל}^{2n+1} (\tfrac{1}{2}\pi) = \frac{\pi^{2n} l2}{1.2..2n} + \frac{\pi^{2n-2} S_3}{1.2..(2n-2)} - \&c.\ldots$$

$$\mp (2 - 2^{-2n}) S_{2n+1} \ldots (38):$$

by which $\text{ל}^n (\frac{1}{2}\pi)$ is known.

20. If we perpetually integrate $\text{ל}^2 x + \text{ל}^2 (\pi - x) = \text{ל}^2 \pi$, we get

$$\text{ל}^n x + (-1)^n . \text{ל}^n (\pi - x) = \frac{x^{n-2} \text{ל}^2 \pi}{1.2..(n-2)} - \frac{x^{n-3} \text{ל}^3 \pi}{1.2..(n-3)} + \ldots\ldots$$

$$+ (-1)^n . \text{ל}^n \pi \ldots (39),$$

which reduces $\text{ל}^n x$ to $\text{ל}^n (\pi - x)$.

21. Perpetually integrate

$$\tfrac{1}{2} \text{ל}^2 (2x) = \text{ל}^2 x - \text{ל}^2 (\tfrac{1}{2}\pi - x) + \text{ל}^2 (\tfrac{1}{2}\pi) - xl2 ;$$

$$\therefore 2^{-n+1} \text{ל}^n (2x) = \text{ל}^n x + (-1)^{n-1} \text{ל}^n (\tfrac{1}{2}\pi - x)$$

$$- \frac{x^{n-1} l2}{1.2..(n-1)} + \frac{x^{n-2} \text{ל}^2 \frac{1}{2}\pi}{1.2..(n-2)} - \frac{x^{n-3} \text{ל}^3 \frac{1}{2}\pi}{1.2..(n-3)} \ldots \pm \text{ל}^n \tfrac{1}{2}\pi . .(40),$$

which may be used exactly as equation (14) in Art. 9.

Make $x = \frac{1}{2}\pi$, and multiply by 2^{n-1};

$$\therefore \text{ל}^n \pi = 2^{n-1} \{1 + (-1)^{n-1}\} \text{ל}^n \tfrac{1}{2}\pi - \frac{\pi^{n-1} l2}{1.2..(n-1)}$$

$$+ \frac{2\pi^{n-2} \text{ל}^2 \frac{1}{2}\pi}{1.2..(n-2)} - \frac{2^2 . \pi^{n-3} \text{ל}^3 \frac{1}{2}\pi}{1.2..(n-3)} + \ldots\ldots \pm 2^{n-1} \text{ל}^n \tfrac{1}{2}\pi \ldots (41).$$

22. To complete the view of $\text{ל}^n x$, we ought to embrace the cases of $x < 0$ and $x > \pi$.

It is obvious that

$$\text{ל}^{2n} (-x) = - \text{ל}^{2n} x ; \text{ but } \text{ל}^{2n+1} (-x) = \text{ל}^{2n+1} x \ldots\ldots (42).$$

Also, by perpetual integration of $\text{ל}^2 (n\pi + x) = \text{ל}^2 (n\pi) + \text{ל}^2 x$,

$$\text{ל}^m (n\pi + x) = \text{ל}^m (n\pi) + \frac{x}{1} \text{ל}^{m-1} (n\pi) + \frac{x^2}{1.2} \text{ל}^{m-2} (n\pi) + \ldots$$

$$+ \frac{x^{m-2}}{1.2 \ldots (m-2)} . \text{ל}^2 (m\pi) + \text{ל}^m x \ldots\ldots (43).$$

But we farther want to express $\varsigma^m(n\pi)$ by means of $\varsigma^m\pi$, $\varsigma^{m-1}\pi, \dots \varsigma^2\pi$, for which we begin with ς^3 and proceed to ς^4, ς^5 in succession.

For $\varsigma^3(n\pi)$, let $x = \pi$, and $n = 1, 2, 3, 4 \dots$.

$$\therefore\ \varsigma^3(2\pi) = \varsigma^3(\pi + \pi) = \varsigma^3\pi + \frac{\pi}{1}\varsigma^2\pi + \varsigma^3\pi = 2\varsigma^3\pi + \frac{\pi}{1}\varsigma^2\pi,$$

$$\varsigma^3(3\pi) = \varsigma^3(2\pi + \pi) = \varsigma^3(2\pi) + \frac{\pi}{1}\varsigma^2(2\pi) + \varsigma^3\pi$$

$$= 3\varsigma^3\pi + 3\,\frac{\pi}{1}\varsigma^2\pi,$$

$$\varsigma^3(4\pi) = \varsigma^3(3\pi) + \frac{\pi}{1}\varsigma^2(3\pi) + \varsigma^3\pi = 4\varsigma^3\pi + 6\,\frac{\pi}{1}\varsigma^2\pi.$$

Generally, $\quad \varsigma^3(n\pi) = n\varsigma^3\pi + n\,.\,\frac{n-1}{2}\,.\,\frac{\pi}{1}\varsigma^2\pi.$

For $\varsigma^4(n\pi)$ proceed by the same steps:

$$\varsigma^4(2\pi) = 2\varsigma^4\pi + \frac{\pi}{1}\varsigma^3\pi + \frac{\pi^2}{1.2}\varsigma^2\pi:$$

$$\varsigma^4(3\pi) = 3\varsigma^4\pi + 3\,\frac{\pi}{1}\varsigma^3\pi + 5\,\frac{\pi^2}{1.2}\varsigma^2\pi.$$

Generally, $\varsigma^4(n\pi) = n\varsigma^4\pi + \Sigma n\,.\,\frac{\pi}{1}\varsigma^3\pi + \Sigma n^2\,.\,\frac{\pi^2}{1.2}\,.\,\varsigma^2\pi.$

It is easy to see that we may assume

$$\varsigma^m(n\pi) = n\varsigma^m\pi + \psi_1 n\,.\,\frac{\pi}{1}\varsigma^{m-1}\pi + \psi_2 n\,.\,\frac{\pi^2}{1.2}\varsigma^{m-2}\pi + \dots$$

to $(m-1)$ terms,

and that the functions ψ do not contain (m). To determine their law, put $x = \pi$ in equation (43), so as to obtain $\varsigma^m\{(n+1)\,.\,\pi\}$. Also write $(n+1)$ for n in the assumed series, and compare the results. This gives

$$\Delta\psi_1 n = n, \qquad \Delta\psi_2 n = n + 2\psi_1 n,$$

$$\Delta\psi_3 n = n + 3\psi_1 n + 3\psi_2 n, \qquad \Delta\psi_4 n = n + 4\psi_1 n + 6\psi_2 n + 4\psi_3 n,$$

$$\Delta\psi_5 n = n + 5\psi_1 n + 10\psi_2 n + 10\psi_3 n + 5\psi_4 n,$$

where the law is obvious.

Write N_r for $n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}\dots\frac{n-r+1}{r}$, and integrate the above;

$$\therefore\ \psi_1 n = \Sigma n = N_2, \quad \psi_2 n = N_2 + 2N_3,$$

$$\psi_3 n = N_2 + 3N_3 + 3\{N_3 + 2N_4\}$$

$$= N_2 + 6N_3 + 6N_4,$$

$$\psi_4 n = N_2 + 4N_3 + 6\{N_3 + 2N_4\} + 4\{N_3 + 6N_4 + 6N_5\}$$
$$= N_2 + 14N_3 + 36N_4 + 24N_5.$$

Generally, it is easy to satisfy ourselves that

$$\psi_r n = \Delta 0^r \,.\, N_2 + \Delta^2 0^r \,.\, N_3 + \Delta^3 0^r \,.\, N_4 + \ldots\ldots \text{ to } r \text{ terms};$$
$$\therefore \Delta\psi_r n = \Delta 0^r \,.\, N_1 + \Delta^2 0^r \,.\, N_2 + \Delta^3 0^r \,.\, N_3 + \ldots\ldots \text{ to } r \text{ terms},$$
$$= n^r. \qquad \text{Hence } \psi_r n = \Sigma n^r.$$

Finally then we obtain

$$\text{ל}^m (n\pi) = n\,\text{ל}^m \pi + \Sigma n \,.\, \frac{\pi}{1}\,\text{ל}^{m-1}\pi + \Sigma n^2 \,.\, \frac{\pi^2}{1.2} \,.\, \text{ל}^{m-2}\pi + \ldots\ldots$$
$$\ldots\ldots \text{ to } (m-1) \text{ terms} \ldots\ldots (44),$$

where $\qquad \Sigma n^r = 0^r + 1^r + 2^r + \ldots\ldots (n-1)^r.$

[*To be continued.*]

ON THE LAWS OF EQUILIBRIUM AND MOTION OF SOLID AND FLUID BODIES.

By Samuel Haughton.

(*Continued from* Vol. I. p. 173.)

The differential equations of motion of solid bodies are deduced from (11), by writing $X - \dfrac{d^2\xi}{dt^2}$, $Y - \dfrac{d^2\eta}{dt^2}$, $Z - \dfrac{d^2\zeta}{dt^2}$, for X, Y, Z, and consequently are the following:

$$\epsilon\frac{d^2\xi}{dt^2} = \epsilon X + P, \quad \epsilon\frac{d^2\eta}{dt^2} = \epsilon Y + Q, \quad \epsilon\frac{d^2\zeta}{dt^2} = \epsilon Z + R \ldots (12).$$

Let us suppose that no external forces of any kind act upon the body, and endeavour to satisfy the equations of motion by the particular integral for plane waves,

$$\xi = \cos\alpha \,.\, f(\omega), \quad \eta = \cos\beta \,.\, f(\omega), \quad \zeta = \cos\gamma \,.\, f(\omega),$$
$$\omega = lx + my + nz - vt.$$

Substituting these values of ξ, η, ζ in the differential equations

$$\epsilon\frac{d^2\xi}{dt^2} = P, \quad \epsilon\frac{d^2\eta}{dt^2} = Q, \quad \epsilon\frac{d^2\zeta}{dt^2} = R,$$

we shall obtain the following equations of condition among the constants,

$$\left.\begin{aligned} \epsilon v^2 \,.\, \cos\alpha &= p' \cos\alpha + h' \cos\beta + g' \cos\gamma, \\ \epsilon v^2 \,.\, \cos\beta &= q' \cos\beta + f' \cos\gamma + h' \cos\alpha, \\ \epsilon v^2 \,.\, \cos\gamma &= r' \cos\gamma + g' \cos\alpha + f' \cos\beta, \end{aligned}\right\} \ldots (13),$$

where

$$\left.\begin{aligned}
p' &= Al^2 + Nm^2 + Mn^2 + 2\alpha_1 mn + 2\alpha_2 ln + 2\alpha_3 lm,\\
q' &= Bm^2 + Ln^2 + Nl^2 + 2\beta_1 mn + 2\beta_2 ln + 2\beta_3 lm,\\
r' &= Cn^2 + Ml^2 + Lm^2 + 2\gamma_1 mn + 2\gamma_2 ln + 2\gamma_3 lm,\\
f' &= \alpha_1 l^2 + \beta_1 m^2 + \gamma_1 n^2 + 2L\,mn + 2\gamma_3 ln + 2\beta_2 lm,\\
g' &= \alpha_2 l^2 + \beta_2 m^2 + \gamma_2 n^2 + 2\gamma_3 mn + 2M\,ln + 2\alpha_1 lm,\\
h' &= \alpha_3 l^2 + \beta_3 m^2 + \gamma_3 n^2 + 2\beta_2 mn + 2\alpha_1 ln + 2N\,lm,
\end{aligned}\right\} \ldots (14).$$

In the particular integral for plane waves, the direction of the wave (l, m, n) is given; and our object is to determine from the equations of condition among the constants (13), *real* values of $(\alpha, \beta, \gamma, v)$ which denote the direction of the molecular vibration and the velocity of the plane wave. In the present instance this is possible; for it is a well-known property of surfaces of the second degree, that if the equation of the surface be

$$p'x^2 + q'y^2 + r'z^2 + 2f'yz + 2g'xz + 2h'xy = 1 \ldots (15),$$

the equations (13) will determine the directions of the axes of this surface; and, as every surface of the second order has three principal diametral planes, it is evident that there will be three possible directions of molecular vibration, at right angles to each other, for which the direction of the wave plane will be the same, but the velocity of wave propagation will be different; for if (a, b, c) be the axes of the ellipsoid (15), we know (vide *Leroy*, pp. 73, 156) that

$$\epsilon v_{,}^2 = \frac{1}{a^2}, \quad \epsilon v_{,,}^2 = \frac{1}{b^2}, \quad \epsilon v_{,,,}^2 = \frac{1}{c^2},$$

$(v_{,}, v_{,,}, v_{,,,})$ being the three velocities of wave propagation.

Hence we may deduce the following geometrical construction for the directions of molecular vibration corresponding to a given wave plane, and for the velocities of wave propagation.

Construct the six fixed ellipsoids,

$$\left.\begin{aligned}
p &= Ax^2 + Ny^2 + Mz^2 + 2\alpha_1 yz + 2\alpha_2 xz + 2\alpha_3 xy = 1,\\
q &= By^2 + Lz^2 + Nx^2 + 2\beta_1 yz + 2\beta_2 xz + 2\beta_3 xy = 1,\\
r &= Cz^2 + Mx^2 + Ly^2 + 2\gamma_1 yz + 2\gamma_2 xz + 2\gamma_3 xy = 1,\\
f &= \alpha_1 x^2 + \beta_1 y^2 + \gamma_1 z^2 + 2Lyz + 2\gamma_3 xz + 2\beta_2 xy = 1,\\
g &= \alpha_2 x^2 + \beta_2 y^2 + \gamma_2 z^2 + 2\gamma_3 yz + 2Mxz + 2\alpha_1 xy = 1,\\
h &= \alpha_3 x^2 + \beta_3 y^2 + \gamma_3 z^2 + 2\beta_2 yz + 2\alpha_1 xz + 2Nxy = 1,
\end{aligned}\right\} (16),$$

and from their common centre draw a normal to the wave plane; this will pierce the surfaces in six points: let the cor-

responding radii vectores be $(\rho_l, \rho_m, \rho_n, r_l, r_m, r_n)$; with these construct the ellipsoid

$$\frac{x^2}{\rho_l^2}+\frac{y^2}{\rho_m^2}+\frac{z^2}{\rho_n^2}+2\left(\frac{yz}{r_l^2}+\frac{zx}{r_m^2}+\frac{xy}{r_n^2}\right)=1 \ldots\ldots (17).$$

The directions of the axes of this ellipsoid are the directions of the three possible vibrations of molecules for the given wave, and the three velocities of propagation are inversely proportional to the lengths of the axes.

If wave normals were drawn from the common centre of the ellipsoids (16) in every possible direction, and the corresponding ellipsoids (17) constructed for each direction; and if on each normal three intercepts were measured, inversely proportional to the axes of the corresponding ellipsoid, the extremities of these intercepts would form a surface which would be the *surface of wave velocity*, or locus of feet of perpendiculars from centre on tangent planes of wave surface, and the surface formed by producing the radii vectores of the surface of wave velocity, so that the new radii vectores should be the reciprocals of the old radii, would be the *surface of wave slowness*, or the reciprocal polar of the wave surface; and a knowledge of its properties would serve all the purposes of a knowledge of the wave surface itself. These two surfaces of wave slowness and wave velocity may be determined by the following considerations.

The cubic equation, whose roots are the squares of the reciprocals of the axes of the ellipsoid (15), is

$$(p'-s)(q'-s)(r'-s)-f'^2(p'-s)-g'^2(q'-s)-h'^2(r'-s)+2f'g'h'=0.$$

If in this equation we substitute $(x^2+y^2+z^2)$ for s, and $\left(\frac{x}{\rho}, \frac{y}{\rho}, \frac{z}{\rho}\right)$ for l, m, n, we shall have the equation of the surface of wave velocity; and if a radius vector be drawn from the centre, it will pierce this surface in three points, and the lengths of the three radii vectores will measure the three velocities of wave propagation possible for the given direction. This surface, however, being only the locus of feet of perpendiculars on tangent planes to wave surface, is of very little use; but it enables us to find the reciprocal polar of the wave surface. Changing the radii vectores into their reciprocals, we obtain

$$(p'\rho^2-1)(q'\rho^2-1)(r'\rho^2-1)-f'^2\rho^4(p'\rho^2-1)-g'^2\rho^4(q'\rho^2-1)$$
$$-h'^2\rho^4(r'\rho^2-1)+2f'g'h'.\rho^6=0,$$

or $(p-1)(q-1)(r-1)-f^2(p-1)-g^2(q-1)-h^2(r-1)$
$$+\,2fgh = 0 \; \ldots . \; (18).$$

This is the surface of wave slowness of elastic solids, and is evidently of the *sixth* degree, from the values of (p, q, r, f, g, h) (16). It has three sheets corresponding to the three velocities of wave propagation, and determines, not merely the laws of propagation of plane waves in a solid, but also enables us to give a construction for the direction of waves reflected or refracted in passing from one solid to another.

With a point in the surface of separation (supposed plane) as centre, construct the two surfaces of wave slowness for both solids; produce the normal to the incident wave to meet its own surface, and from the point in which it pierces it let fall a perpendicular upon the separating plane: this perpendicular will pierce the surface of wave slowness of the second solid in three points; the lines drawn from these points to the centre are the normals of the three refracted waves, and their lengths are inversely proportional to the wave velocities. The directions of the waves reflected back into the first solid may also easily be found by means of this surface; for we have only to produce the perpendicular backwards to meet the surface of wave slowness of the first solid, and the three lines joining the centre with the three points of intersection will be the normals to the three plane waves reflected back into the first solid. These and all other constructions for the direction of reflected and refracted waves, may be easily proved from the properties of the wave surface and the reciprocal properties of the surface of wave slowness, which, for such purposes as these, answers as well as the wave surface itself. The whole theory of wave surfaces in light or in elastic solids, is only a development of Huygens' construction for uniaxal crystals.

It is important to observe, that the directions of the *wave* and *ray* being both given, (by the radius vector and perpendicular on tangent plane of the surface of wave slowness,) determine completely the direction of molecular vibration in any given case. If the direction of the *wave* only be given, the problem is indeterminate; for three parallel tangent planes may be drawn to the wave surface, each tangent plane being accompanied by its own direction of molecular vibration, and the three directions of vibration being at right angles to each other: but if the direction of the *ray* be also given, there will remain nothing indeterminate, for, the di-

rection of the *ray* being the radius vector of the wave surface, knowing the *ray*, we shall know which of the three tangent planes we must select, and consequently which of the three directions of molecular vibration.

Let us now suppose the solid body to have its molecules so arranged, that at each point they are symmetrically placed around three rectangular planes, so that the molecular forces are in every respect similar in each of the eight regions into which the planes divide the body at each point. It is easily seen that, in this case, the coefficients ($\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, \alpha_3, \beta_3, \gamma_3$) of the function V are all zero; and consequently the function in this case will be reduced to the following,

$$2V_1 = A\left(\frac{d\xi}{dx}\right)^2 + B\left(\frac{d\eta}{dy}\right)^2 + C\left(\frac{d\zeta}{dz}\right)^2 + Lu^2 + Mv^2 + Nw^2$$
$$+ 2\left(L\frac{d\eta}{dy}\frac{d\zeta}{dz} + M\frac{d\xi}{dx}\frac{d\zeta}{dz} + N\frac{d\xi}{dx}\frac{d\eta}{dy}\right);$$

and the differential equations of motion will become

$$\left.\begin{aligned}
\rho\frac{d^2\xi}{dt^2} &= A\frac{d^2\xi}{dx^2} + N\frac{d^2\xi}{dy^2} + M\frac{d^2\xi}{dz^2} + 2\left(N\frac{d^2\eta}{dx\,dy} + M\frac{d^2\zeta}{dx\,dz}\right)\\
\rho\frac{d^2\eta}{dt^2} &= B\frac{d^2\eta}{dy^2} + L\frac{d^2\eta}{dz^2} + N\frac{d^2\eta}{dx^2} + 2\left(L\frac{d^2\zeta}{dy\,dz} + N\frac{d^2\xi}{dx\,dy}\right)\\
\rho\frac{d^2\zeta}{dt^2} &= C\frac{d^2\zeta}{dz^2} + M\frac{d^2\zeta}{dx^2} + L\frac{d^2\zeta}{dy^2} + 2\left(M\frac{d^2\xi}{dx\,dz} + L\frac{d^2\eta}{dy\,dz}\right)
\end{aligned}\right\} \quad \ldots\ldots (19).$$

The equation (18) will be the equation of the surface of wave slowness, if we give to (p, q, r, f, g, h) the following values,

$$\left.\begin{aligned}
p &= Ax^2 + Ny^2 + Mz^2, & f &= 2Lyz,\\
q &= By^2 + Lz^2 + Nx^2, & g &= 2Mxz,\\
r &= Cz^2 + Mx^2 + Ly^2, & h &= 2Nxy,
\end{aligned}\right\} \ldots . (20),$$

and its traces on the three principal planes will be

$$(r-1)\{(p-1)(q-1) - h^2\} = 0, \quad z = 0,$$
$$(q-1)\{(p-1)(r-1) - g^2\} = 0, \quad y = 0,$$
$$(p-1)\{(q-1)(r-1) - f^2\} = 0, \quad x = 0,$$

from which it appears that the traces of the surface of wave slowness are composed of two distinct curves, one an ellipse and the other a curve of the fourth degree.

If the surface of wave slowness possess nodes in its principal planes, they will appear in the form of multiple points in its traces; and it may be shewn that it does possess real nodes in one of the principal planes.

For, its equation may be put under any one of the three forms

$$\phi_1 + z^2\psi_1 = 0, \quad \phi_2 + y^2\psi_2 = 0, \quad \phi_3 + z^2\psi_3 = 0,$$

where the values of ϕ_1, ϕ_2, ϕ_3 are

$$\phi_1 = (r-1)\{(p-1)(q-1) - h^2\},$$
$$\phi_2 = (q-1)\{(p-1)(r-1) - g^2\},$$
$$\phi_3 = (p-1)\{(q-1)(r-1) - f^2\};$$

but if the equation of any surface be

$$u = \phi + z^2\psi = 0,$$

it can be shewn that, if the surface $\phi = 0$ has any singular points in the plane $z = 0$, these are also singular points in the surface $u = 0$; for the conditions for singular points are

$$\frac{du}{dx} = 0, \quad \frac{du}{dy} = 0, \quad \frac{du}{dz} = 0,$$

which, if $z = 0$ become

$$\frac{d\phi}{dx} = 0, \quad \frac{d\phi}{dy} = 0, \quad \frac{d\phi}{dz} = 0,$$

which are the conditions for singular points in the surface $\phi = 0$. Hence, if any singular points of the surface $\phi = 0$ exist in the plane $z = 0$, these will be also singular points in the surface $u = 0$.

Applying this principle to the three forms of the surface of wave slowness just given, it appears that the intersections of the three surfaces

$$r - 1 = 0, \quad (p-1)(q-1) - h^2 = 0, \quad z = 0,$$

will be singular points of the surface of wave slowness in the plane $z = 0$; and similarly for the other principal planes.

As the traces in the principal planes consist of a curve of the second and another of the fourth degree, there will be in general eight points of intersection, real or imaginary, and therefore the surface of wave slowness should have twenty-four singular points; but as it is only the *real* singular points which produce any effect in the physical problem, we must ascertain the number and position of the real singular points.

I shall first prove that the curve of the fourth degree consists of two ovals, lying one inside the other, and not having

any point in common. The equation of this curve in the plane $z = 0$ is

$$(Ax^2 + Ny^2 - 1)(By^2 + Nx^2 - 1) - 4N^2x^2y^2 = 0,$$

and its polar equation is

$$\left(A \cos^2 \alpha + N \sin^2 \alpha - \frac{1}{\rho^2}\right)\left(B \sin^2 \alpha + N \cos^2 \alpha - \frac{1}{\rho^2}\right) - 4N^2 \sin^2 \alpha \cos^2 \alpha = 0,$$

which is a quadratic equation with respect to $\frac{1}{\rho^2}$: the condition necessary for *equal roots* is

$$\{(A + N) \cos^2 \alpha + (B + N) \sin^2 \alpha\}^2$$
$$= 4(A \cos^2 \alpha + N \sin^2 \alpha)(B \sin^2 \alpha + N \cos^2 \alpha) - 16 N^2 \sin^2 \alpha \cos^2 \alpha.$$

This equation of condition must give a *real* value for α; arranging it with respect to $\tan \alpha$, it becomes

$$(B - N)^2 \tan^4 \alpha + \{2(A + N)(B + N) - 4AB + 12N^2\} \tan^2 \alpha + (A - N)^2 = 0;$$

or, assuming $\quad \omega = (A - N)(B - N) - 4N^2,$

$$(B - N)^2 \tan^4 \alpha - 2(\omega - 4N^2) \tan^2 \alpha + (A - N)^2 = 0.$$

Now, in order that this equation should give *real* values for $\tan \alpha$, it must give *real* and *positive* values for $\tan^2 \alpha$; but it can be shewn that its roots, if real, are *negative*; and consequently that no real value exists for $\tan \alpha$.

For, solving the equation with respect to $\tan^2 \alpha$, we obtain

$$(B - N)^2 . \tan^2 \alpha = (\omega - 4N^2) \pm \sqrt{\{(\omega - 4N^2)^2 - (\omega + 4N^2)^2\}};$$

or, $\quad (B - N)^2 . \tan^2 \alpha = (\omega - 4N^2) \pm \sqrt{(-16 N^2 . \omega)}.$

This equation shews that the condition for real values of $\tan^2 \alpha$, is that ω should be either *zero* or *negative*, and that in either case $\tan^2 \alpha$ will be negative, and therefore the two branches of the curve will not have a real point of intersection. The same result is true of the other curves of the fourth degree in the other principal planes.

The curve of the fourth degree, in the plane $z = 0$, consists of two branches, lying one inside the other, each of them cutting at right angles the axes of coordinates, and the semiaxes of one branch are $\frac{1}{\sqrt{A}}, \frac{1}{\sqrt{B}}$; while the semiaxes of the other branch are equal and each $\frac{1}{\sqrt{N}}$; and similarly for the

other coordinate planes. But the equation of the ellipse in the plane $z = 0$, being

$$Ly^2 + Mx^2 - 1 = 0,$$

this curve will cut the oval whose semiaxes are $\frac{1}{\sqrt{N}}$, if N be intermediate in value between L and M; supposing therefore that M is the mean of the three quantities L, M, N; the ellipse in the plane (x, z) will cut the oval whose semiaxes are $\frac{1}{\sqrt{M}}$, and in the other planes it will lie, in one case, completely outside the oval, and in the other completely inside the oval. Hence there are always *at least* four real singular points on the surface of wave slowness. Whether there be more real singular points will depend upon the relative magnitude of A, B, C compared with L, M, N; and if we assume (as seems probable from its being true in homogeneous solids) that A, B, C are greater than L, M, N, then the ovals whose semiaxes are $\frac{1}{\sqrt{A}}, \frac{1}{\sqrt{B}}, \frac{1}{\sqrt{C}}$, will lie completely inside the ellipses and the other ovals whose semiaxes are $\frac{1}{\sqrt{L}}, \frac{1}{\sqrt{M}}, \frac{1}{\sqrt{N}}$. The surface of wave slowness will therefore consist of three sheets; one whose semiaxes are $\frac{1}{\sqrt{A}}, \frac{1}{\sqrt{B}}, \frac{1}{\sqrt{C}}$, isolated, and lying inside the other two sheets; and the other two sheets, having four points in common, like Fresnel's wave surface, and cutting off from the axis of x intercepts equal to $\frac{1}{\sqrt{M}}, \frac{1}{\sqrt{N}}$, from the axis of y, $\frac{1}{\sqrt{L}}, \frac{1}{\sqrt{N}}$; and from the axis of z, $\frac{1}{\sqrt{L}}, \frac{1}{\sqrt{M}}$.

To shew the effect produced by the existence of these nodes in the surface of wave slowness, it is necessary to consider a plane wave in its passage from one solid into another; the construction for the refracted wave is as follows: describe the surfaces of wave slowness (S, Σ) for both solids, having a common centre in the plane which separates the bodies; produce the normal to the incident wave to meet the corresponding sheet of the surface S, and from the point of intersection let fall a perpendicular on the separating plane; this perpendicular will in general pierce the surface Σ in three points; and the corresponding radii vectores will be the normals to the three refracted *waves*, the perpendiculars on the three tangent planes at the points of intersection being the directions of the refracted *rays*. Let us suppose

that the perpendicular pierces the surface Σ in a node, then the line joining the centre with the node will be the normal to the refracted wave; but there will be an infinite number of rays, which will form the sides of a cone of the second degree, having its vertex at the centre of the surface of wave slowness, and reciprocal to the tangent cone at the node. Again, there may be only one refracted ray, and an infinite number of waves; for if we consider that there are four tangent planes of the surface Σ, which touch the surface along an ellipse, it is evident that there might be a cone of refracted wave normals of the second degree, whose base is one of the ellipses of contact, while there would be only one refracted ray, whose direction is the perpendicular to the plane of ellipse: to find, in this case, the cone of incident waves which will be refracted into a single ray, we must project (by perpendiculars to the plane of separation) the ellipse of contact of the surface Σ upon S; then the cone, whose base is the projection and vertex the centre of surface, will be the cone of incident wave normals; while the cone whose base is the ellipse will be the cone of refracted wave normals; and the perpendicular to the plane of the ellipse will be the unique direction of the refracted ray.

In the integration of the differential equations of motion of elastic solids, I have been obliged to use a particular integral which will only represent the case of plane waves; but the differential equations themselves are general, and if a complete integral could be found for them, we should have all the knowledge we could desire upon the subject. It may be observed that I have confined my attention in this paper exclusively to the *laws of propagation*, and such knowledge of *reflection and refraction* as these laws afford: the full investigation of the latter subject is to be sought in the double integrals of which the quantity Δ, in equation (10), is composed. The surface of wave slowness and the laws of wave propagation can only give the *directions* of the reflected and refracted rays and waves, while the laws which regulate the *intensities* of the molecular vibrations, in passing from one solid to another, are to be sought in the conditions at the limits, which are all contained in the quantity Δ. For an attempt to investigate these conditions, and for a fuller account of the laws of propagation, I may refer to a paper read before the Royal Irish Academy, in May 1846, and which will shortly appear in the Transactions of that body.

January 4, 1847.

ON CERTAIN DEFINITE INTEGRALS SUGGESTED BY PROBLEMS IN THE THEORY OF ELECTRICITY.

By William Thomson.

It follows from the solution of the problem of the distribution of electricity on an infinite plane,* subject to the influence of an electrical point, that the value of the double integral,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{z\,d\xi\,d\eta}{\{(\xi-x)^2+(\eta-y)^2+z^2\}^{\frac{3}{2}}\,\{(\xi-x')^2+(\eta-y')^2+z'^2\}^{\frac{1}{2}}},$$

is

$$\frac{2\pi}{\{(x-x')^2+(y-y')^2+(z+z')^2\}^{\frac{1}{2}}}.$$

A direct analytical verification of this result is therefore interesting in connexion with the physical problem. In the following paper the multiple integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{u\,d\xi_1\,d\xi_2\ldots d\xi_n}{\{(\xi_1-x_1)^2+(\xi_2-x_2)^2+..+u^2\}^{\frac{1}{2}(n+1)}\{(\xi_1-x_1')^2+(\xi_2-x_2')^2+..+u'^2\}^{\frac{1}{2}(n-1)}}$$

is considered, and its value is shewn to be

$$\frac{\pi^{\frac{1}{2}(n+1)}}{\Gamma\frac{1}{2}(n+1)}\,\frac{1}{\{(x_1-x_1')^2+(x_2-x_2')^2+\ldots+(u+u')^2\}^{\frac{1}{2}(n-1)}},$$

a result of which the one mentioned above is a particular case. Several distinct demonstrations of this theorem are given, and some other formulæ, which have occurred to me in connexion with it, are added.

The first part of the following paper, which is a translation, with slight alterations, of a memoir in *Liouville's Journal*,† contains a demonstration suggested to me by a method followed by Green in proving the remarkable theorem in Art. (5) of his Essay on Electricity. In the second part some formulæ are given which, in the case of two variables, are such as would occur in the analysis of problems in heat and electricity, with reference to a body bounded in one direction by an infinite plane, if the methods indicated by Fourier were followed; and from them the value of the multiple integral mentioned above is deduced. In §. III. the evaluation is effected by a direct process of reduction, suggested by geometrical considerations.‡

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* See below, end of §. II.
† Vol. x. p. 137, "Démonstration d'un Théorème d'Analyse." April 1845.
‡ See "Extrait d'une lettre à M. Liouville, &c." Vol. x. p. 364.

I.

...of the multiple integral, which, if we use a very ...ment notation analogous to that of factorials, may be ... thus,

$$\left[\int_{-\infty}^{\infty}\right]^n \frac{[d\xi]^n}{\{\Sigma(\xi - x)^2 + u^2\}^{\frac{1}{2}(n+1)}\{\Sigma(\xi - x')^2 + u'^2\}^{\frac{1}{2}(n-1)}},$$

denoted by U.

Let $u + u' = a$, it being understood that u and u' are taken ositive. Then, if we assume

$$= \frac{1}{\{\Sigma(\xi - x)^2 + v^2\}^{\frac{1}{2}(s-1)}} - \frac{1}{\{\Sigma(\xi - x)^2 + (2u - v)^2\}^{\frac{1}{2}(s-1)}} \quad \text{... (1)},$$

$$= \frac{1}{\{\Sigma(\xi - x')^2 + (a - v)^2\}^{\frac{1}{2}(s-1)}} \quad \text{.................... (2)},$$

have $-2(s-1)\,uU = \left[\int_{-\infty}^{\infty}\right] R' \dfrac{dR}{dv}\,[d\xi]^s$, when $v = u$.

is easily seen that the second member of this equation vanishes when $v = \pm\infty$, and that it does not become infinite, even when one of the values 0, $2u$, or a is assigned to u. Hence the preceding equation may be written

$$-2(n-1)\,uU = \int_{-\infty}^{u}\left[\int_{-\infty}^{\infty}\right]^s\left(\frac{dR'}{dv}\,\frac{dR}{dv} + R'\,\frac{d^2R}{dv^2}\right)[d\xi]^s\,dv.$$

But we have

$$\int\left[\int_{-\infty}^{\infty}\right]^s \frac{dR'}{dv}\,\frac{dR}{dv}\,[d\xi]^s\,dv = \left[\int_{-\infty}^{\infty}\right]\int \frac{dR'}{dv}\,\frac{dR}{dv}\,dv\,[d\xi]^s$$

$$= \left[\int_{-\infty}^{\infty}\right] R\,\frac{dR'}{dv}\,[d\xi]^s - \int\left[\int_{-\infty}^{\infty}\right] R\,\frac{d^2R'}{dv}\,[d\xi]^s\,dv.$$

When we take the integral with respect to v between the limits $-\infty$ and u, the first term vanishes, since at each limit $R = 0$. Thus the preceding equation is reduced to

$$-2(s-1)\,uU = \int_{-\infty}^{u}\left[\int_{-\infty}^{\infty}\right]\left(R'\,\frac{d^2R}{dv^2} - R\,\frac{d^2R'}{dv^2}\right)[d\xi]^s\,dv.$$

Now we have $\qquad \dfrac{d^2R'}{dv^2} + \Sigma\,\dfrac{d^2R'}{d\xi^2} = 0,$

for all values of $\xi_1, \xi_2 \ldots$, provided v be not equal to a. Hence this equation is satisfied for all the values of the

variables between the limits of the integration in the preceding expression, and we may therefore employ it to eliminate $\frac{d^2R'}{dv^2}$: we thus obtain

$$-2(s-1)uU = \int_{-\infty}^{u}\left[\int_{-\infty}^{\infty}\right]^s\left(R'\frac{d^2R}{dv^2} + R\Sigma\frac{d^2R'}{d\xi^2}\right)[d\xi]^s\,dv.$$

Taking one of the terms of the second member, and integrating by parts, we have

$$\int_{-\infty}^{u}\left[\int_{-\infty}^{\infty}\right]^s R\frac{d^2R'}{d\xi_{,}^2}[d\xi]^s\,dv$$

$$=\int_{-\infty}^{u}\left[\int_{-\infty}^{\infty}\right]^{s-1}\left(\int_{-}R\frac{d^2R'}{d\xi_{,}^2}d\xi_{,}\right)[d\xi]^{s-1}\,dv$$

$$=-\int_{-\infty}^{u}\left[\int_{-\infty}^{\infty}\right]^{s-1}\left(\int_{-\infty}^{\infty}\frac{dR'}{d\xi_{,}}\frac{dR}{d\xi_{,}}d\xi_{,}\right)[d\xi]^{s-1}\,dv$$

$$=\int_{-\infty}^{u}\left[\int_{-\infty}^{\infty}\right]^s R'\frac{d^2R}{d\xi_{,}^2}[d\xi]^s\,dv,$$

since the integrated parts vanish at each limit. By applying a similar process to each term under the sign Σ, we find

$$-2(s-1)uU = \int_{-\infty}^{u}\left[\int_{-\infty}^{\infty}\right]^s R'\left(\frac{d^2R}{dv^2} + \Sigma\frac{d^2R}{d\xi^2}\right)[d\xi]^s\,dv.$$

But, if we denote by Q and Q' the two parts of R, in equation (1), so that $R = Q - Q'$, we have

$$\frac{d^2Q'}{dv^2} + \Sigma\frac{d^2Q'}{d\xi^2} = 0$$

for all values of the variables v, $\xi_{,}$, &c. within the limits of integration; hence there remains

$$-2(n-1)uU = \int_{-\infty}^{u}\left[\int_{-\infty}^{\infty}\right]^s R'\left(\frac{d^2Q}{dv^2} + \Sigma\frac{d^2Q}{d\xi^2}\right)[d\xi]^s\,dv.$$

To determine the value of this expression it may be remarked that the quantity under the integral signs vanishes for all values of the variables which differ sensibly from those expressed by

$$v = 0, \quad \xi_1 = x_1, \quad \xi_2 = x_2, \quad \&c.,$$

and moreover, that if we consider separately the terms of the second member, each is found to be a converging integral: it follows that, if we denote by P the value which R' receives

riables have these values assigned, we have

$$(s-1)\,uU = P\iiint..\left(\frac{d^2Q}{d^2v} + \Sigma\,\frac{d^2Q}{d\xi^2}\right) dv\, d\xi_1\, d\xi_2 ..\, d\xi_s ..(3),$$

ere the limits of integration must be such as to include values 0, x_1, x_2, &c., but are otherwise entirely arbitrary. considering separately the different terms of this expres-., and integrating each with respect to the variable to ich it is related, without yet assigning the limits of the egration, we find

$$2(s-1)\,uU = P\left(\iint..\frac{dQ}{dv}\, d\xi_1\, d\xi_2 .. + \iint\frac{dQ}{d\xi_1}\, dv\, d\xi_2 + \&c.\right)..(4).$$

Let us now assume

$$\xi_1 = v_1 + x_1,\quad \xi_2 = v_2 + x_2,\quad \&c.,$$

nd

$$v^2 + v_1^2 + .. + v_s^2 = r^2,$$

om which we have

$$Q = \frac{1}{r^{s-1}},\quad \frac{dQ}{dv} = -\frac{s-1}{r^{s+1}}\, v,\quad \frac{dQ}{d\xi_1} = -\frac{s-1}{r^{s+1}}\, v_1,\quad \&c.$$

The integrations in equation (3) may be extended to all the values of the variables which satisfy the condition

$$v^2 + v_1^2 + v_2^2 + .. + v_s^2 \leqq a^2;$$

and the limits in (4) will then be such as to include all the values which satisfy the equation

$$v^2 + v_1^2 + v_2^2 + .. + v_s^2 = a^2,\quad \text{or } r^2 = a^2,\ \&c.$$

Hence we have in the successive terms the second member of (4),

$$\frac{dQ}{dv} = -\frac{s-1}{a^{s+1}}\, v,\quad \frac{dQ}{dv_1} = -\frac{s-1}{a^{s+1}}\, v_1,\quad \&c.$$

If in the integrations we only take the positive values of the variables v, v_1, v_2, &c. which satisfy the limiting condition, we must multiply each integral by 2^{s+1}. Thus we have

$$uU = \frac{2^s P}{a^{s+1}}\left(\iint..v\, dv_1\, dv_2 ..\, dv_s + \iint..v_1\, dv\, dv_2 ..\, dv_s + \&c.\right)$$

$$= \frac{2^s(s+1)\,P}{a^{s+1}}\iint..(a^2 - v_1^2 - v_2^2 - .. - v_s^2)^{\frac{1}{2}}\, dv_1\, dv_2 ..\, dv_s$$

$$= (s+1)\,P\iint..(1 - l_1 - l_2 .. - l_s)^{\frac{1}{2}}\, l_1^{-\frac{1}{2}}\, l_2^{-\frac{1}{2}} ..\, l_s^{-\frac{1}{2}}\, dl_1\, dl_2 ..\, dl_s;$$

in which last expression the limits include all positive values satisfying the condition

$$l_1 + l_2 + .. + l_s \leqq 1,$$

Hence, by Liouville's theorem,

$$uU = (s+1)\,P\,\frac{\Gamma(\frac{1}{2})^s}{\Gamma\frac{1}{2}(s+1)}\cdot\tfrac{1}{2}s\int_0^1 (1-h)^{\frac{1}{2}}\,h^{\frac{1}{2}(s-1)}dh = \frac{\pi^{\frac{1}{2}(s+1)}}{\Gamma\frac{1}{2}(s+1)}P,$$

which gives the required value of the integral U.

If we denote by U' any integral corresponding to U, in which the system of variables $u, x_1, x_2 \ldots$ and $u', x_1', x_2' \ldots$ are inverted, we shall have $uU = u'U'$, since P is a function symmetrical with respect to the two systems; and we therefore deduce from the preceding result,

$$\left.\begin{aligned}
&u\left[\int_{-\infty}^{\infty}\right]^s \frac{[d\xi]^s}{\{\Sigma(\xi-x)^2+u^2\}^{\frac{1}{2}(s+1)}\{\Sigma(\xi-x')^2+u'^2\}^{\frac{1}{2}(s-1)}}\\
=\,&u'\left[\int_{-\infty}^{\infty}\right]^s \frac{[d\xi]^s}{\{\Sigma(\xi-x')^2+u'^2\}^{\frac{1}{2}(s+1)}\{\Sigma(\xi-x)^2+u^2\}^{\frac{1}{2}(s-1)}}\\
=\,&\frac{\pi^{\frac{1}{2}(s+1)}}{\Gamma\frac{1}{2}(s+1)}\;\frac{1}{\{\Sigma(x-x')^2+(u+u')^2\}^{\frac{1}{2}(s-1)}}
\end{aligned}\right\}\ldots\ldots(5).$$

I shall add another demonstration of this theorem, as an application of some remarkable analysis given by Mr. Green in his memoir "On the determination of the exterior and interior attractions of ellipsoids of variable densities."*

$$\text{Let } V = \left[\int_{-\infty}^{\infty}\right]^s \frac{u\,[d\xi]^s}{\{\Sigma(\xi-x)^2+u^2\}^{\frac{1}{2}(s+1)}\{\Sigma(\xi-x')^2+u'^2\}^{\frac{1}{2}(s-1)}}\,.\,.(6),$$

an integral which may also be expressed thus:

$$-\frac{1}{n-1}\,\frac{d}{du}\left\{\left[\int_{-\infty}^{\infty}\right]^s \frac{[d\xi]^s}{\{\Sigma(\xi-x)^2+u^2\}^{\frac{1}{2}(s-1)}\{\Sigma(\xi-x')^2+u'^2\}^{\frac{1}{2}(s-1)}}\right\}.$$

From this latter form, we see that the equation

$$\frac{d^2V}{du^2}+\Sigma\,\frac{d^2V}{dx^2}=0\ldots\ldots\ldots\ldots\ldots(7)$$

is satisfied, provided u does not vanish. Hence V is a function which satisfies this equation for all values of $x_1, x_2 \ldots$ and for all the values of u between 0 and ∞. At these limits the value of V may be easily determined, and the general value inferred in the following manner.

When $u = 0$ the quantity under the signs of integration in the expression for V vanishes for all the values of $\xi_1, \xi_2 \ldots$ which are not equal to $x_1, x_2 \ldots$ respectively. Hence it follows that, when $u = 0$,

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* Read at the Cambridge Philosophical Society, May 6, 1846. See *Trans.*, vol. v.

$$V=\frac{1}{\{\Sigma(x-x')^2+u'^2\}^{\frac{1}{2}(s-1)}}\left[\int_{-\infty}^{\infty}\right]\frac{u[d\xi]^s}{\{\Sigma(\xi-x)^2+u^2\}^{\frac{1}{2}(s+1)}}$$

$$=\frac{1}{\{\Sigma(x-x')^2+u'^2\}^{\frac{1}{2}(s-1)}}\left[\int_{-\infty}^{\infty}\right]\frac{dz_1\,dz_2\ldots dz_s}{(1+z_1^2+z_2^2+\ldots+z_s^2)^{\frac{1}{2}(s+1)}}$$

$$=\frac{1}{\{\Sigma(x-x')^2+u'^2\}^{\frac{1}{2}(s-1)}}\cdot\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdot\cdot\frac{l_1^{-\frac{1}{2}}l_2^{-\frac{1}{2}}\ldots dl_1\,dl_2\ldots}{(1+l_1+l_2+\ldots+l_s)^{\frac{1}{2}(s+1)}}$$

$$=\frac{1}{\{\Sigma(x-x')^2+u'^2\}^{\frac{1}{2}(s-1)}}\,\frac{\pi^{\frac{1}{2}s}}{\Gamma(1+\frac{1}{2}s)}\int_0^{\infty}\frac{\frac{1}{2}s\,h^{\frac{1}{2}(s-2)}}{(1+h)^{\frac{1}{2}(s+1)}}$$

$$=\frac{\pi^{\frac{1}{2}(s+1)}}{\Gamma\frac{1}{2}(s+1)}\,\frac{1}{\{\Sigma(x-x')^2+u'^2\}^{\frac{1}{2}(s-1)}}.$$

Also, when $u=\infty$, the value of V is nothing.

Thus we see that V has the same value as the expression

$$\frac{\pi^{\frac{1}{2}(s+1)}}{\Gamma\frac{1}{2}(s+1)}\cdot\frac{1}{\{\Sigma(x-x')^2+(u+u')^2\}^{\frac{1}{2}(s-1)}},$$

when $u=0$, and when $u=\infty$; which enables us to infer that

$$V=\frac{\pi^{\frac{1}{2}(s+1)}}{\Gamma\frac{1}{2}(s+1)}\cdot\frac{1}{\{\Sigma(x-x')^2+(u+u')^2\}^{\frac{1}{2}(s-1)}},$$

for all positive values of u, provided u' be taken as positive: for the second member of this equation satisfies equation (7) for all positive values of u, and for any values of the other variables, and at the limits $u=0$, and $u=\infty$ has the same value as V, and therefore, by a theorem of Green's, in the memoir referred to, must be equal to V for all positive values of u.

From what has been proved above we may deduce the solution of the following problem:

Having given for all values of ξ_1, ξ_2 ...., the value of the multiple integral

$$\mathbf{S}\frac{\rho'\,dx_1'\,dx_2'\ldots dx_s'}{\{(x_1'-\xi_1)^2+(x_2'-\xi_2)^2+\ldots+(x_s'-\xi_s)^2+u'^2\}^{\frac{1}{2}(s-1)}}\ldots.(a),$$

where u' and ρ' are any functions of x_1', x_2'....x_s', let it be required to find the value of

$$\mathbf{S}\frac{\rho'\,dx_1'\,dx_2'\ldots dx_s'}{\{(x_1'-x_1)^2+(x_2'-x_2)^2+\ldots+(x_s'-x_s)^2+(u'+u)^2\}^{\frac{1}{2}(s-1)}}\ldots(b),$$

where x_1, x_2.., x_s are any given quantities, and u a given positive quantity.

Denoting the expression (a) by Φ, and the expression (b) by ϕ, we have, from the theorem established above,

$$\phi = \frac{u\Gamma \frac{1}{2}(s+1)}{\pi^{\frac{1}{2}(s+1)}} \mathbf{S}\,\rho'\, dx_1'\, dx_2' \ldots dx_s'.$$

$$\left[\int_{-\infty}^{\infty}\right]^s \frac{[d\xi]^s}{\{\Sigma(\xi - x)^2 + u^2\}^{\frac{1}{2}(s+1)}\{\Sigma(\xi - x')^2 + u'^2\}^{\frac{1}{2}(s-1)}}$$

$$= \frac{u\Gamma \frac{1}{2}(s+1)}{\pi^{\frac{1}{2}(s+1)}} \left[\int_{-\infty}^{\infty}\right]^s \frac{[d\xi]^s}{\{\Sigma(\xi-x)^2 + u^2\}^{\frac{1}{2}(s+1)}} \cdot \mathbf{S} \frac{\rho'\, dx_1'\, dx_2' \ldots dx_s}{\{\Sigma(\xi - x')^2 + u'^2\}^{\frac{1}{2}(s-1)}},$$

or $$\phi = \frac{u\Gamma \frac{1}{2}(s+1)}{\pi^{\frac{1}{2}(s+1)}} \left[\int_{-\infty}^{\infty}\right]^s \frac{\Phi\,[d\xi]^s}{\{\Sigma(\xi - x)^2 + u^2\}^{\frac{1}{2}(s+1)}} \ldots\ldots (c).$$

But, by hypothesis, ϕ' is given for all values of $\xi_1, \xi_2 \ldots \xi_s$; and therefore this equation expresses the solution of the problem. We may also deduce from the theorem (5) the expression

$$\phi = -\frac{\Gamma(\frac{1}{2}s + 1)}{(s-1)\,\pi^{\frac{1}{2}(s+1)}} \left[\int_{-\infty}^{\infty}\right]^s \frac{\Psi\,[d\xi]^s}{\{\Sigma(\xi - x)^2 + u^2\}^{\frac{1}{2}(s-1)}} \ldots (d),$$

by means of which ϕ may be determined when the value, Ψ, of $\frac{d\phi}{du}$ corresponding to $u = 0$ is given.

For the particular case of $u' = 0$, the theorem (d) is included in a theorem given by Green, in which the number n in the exponent of the denominator may differ from the number s of variables, the sole condition being that $n - s + 1$ must be positive; but it is only in the case of $n = s$ that a general theorem such as (d), by means of which the general value of ϕ is obtained from the value $\frac{d\phi}{du}$ when $u = 0$, is obtained, can be established.

Let us now apply these formulæ to the case of $x = 2$: we may in this case conveniently replace x_1, x_2, u by x, y, z, and ξ_1, ξ_2, by ξ, η. Equations (c) and (d) become

$$\phi = \frac{z}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\Phi\, d\xi\, d\eta}{\{(\xi - x)^2 + (\eta - \gamma)^2 + z^2\}^{\frac{3}{2}}} \ldots\ldots (e),$$

and $$\phi = -\frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\Psi\, d\xi\, d\eta}{\{(\xi - x)^2 + (\eta - \gamma)^2 + z^2\}^{\frac{1}{2}}} \ldots (f),$$

where Ψ denotes the value of $\frac{d\phi}{dz}$ when $x = \xi$, $y = \eta$, $z = 0$.

The first of these theorems may be deduced from a very general theorem given by Green in his essay on Electricity and Magnetism. The second may be demonstrated in the following manner.

Let x', y', z' be considered as the coordinates of a point P', where there is situated a quantity of matter $\rho' dx' dy' dz'$, in the volume $dx' dy' dz'$. Then ϕ will be the potential on a point $P(x, y, z)$, above the plane of x, y which we may regard as horizontal, due to a quantity of matter,

$$M, (= \int\int\int \rho' dx' dy' dz')$$

situated below this plane. Now it follows from a theorem, first, so far as I am aware, given by Gauss, for a surface of any form, that there is a determinate distribution of matter upon the plane (xy) which will produce this same potential on points above the plane. Let k be the density of this distribution at a point $\Pi(\xi, \eta)$ of the plane, so that

$$\phi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{k d\xi\, d\eta}{\{(\xi - x)^2 + (\eta - y)^2 + z^2\}^{\frac{1}{2}}},$$

which gives

$$\frac{d\phi}{dz} = -z \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{k\, d\xi\, d\eta}{\{(\xi - x)^2 + (\eta - y)^2 + z^2\}^{\frac{3}{2}}}.$$

Let $z = 0$; then, denoting by k and $\left(\frac{d\phi}{dz}\right)_0$ the values of k and $\frac{d\phi}{dz}$ at the point $(x, y, 0)$, we find

$$\left(\frac{d\phi}{dz}\right)_0 = -\mathrm{k} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{z\, d\xi\, d\eta}{\{(\xi - x)^2 + (\eta - y)^2 + z^2\}^{\frac{3}{2}}},$$
$$= -\mathrm{k}.2\pi,$$

since the value of the integral in the second member is 2π, whatever be the value of z. Hence we conclude that $k = -\frac{1}{2\pi}.\Phi$, and equation (f) is established.

It should be remarked that the total quantity of matter distributed over the plane xy must be equal to the mass M, which it represents: this is readily verified from the preceding formulæ.

The same formulæ admit of an interesting application in the theory of heat. Thus let ϕ be the permanent temperature of a point P in an infinite homogeneous solid, heated by constant sources distributed below the plane (xy), (the case in which some of the sources are in this plane being of course included). If the temperature Φ at any point Π in the plane (xy) be given, the formula (e) enables us to find the temperature at any point above the plane.

As an example, let us suppose that the sources of heat are such that the temperature of a portion A of the plane (xy),

between two lines parallel to OY and at equal distances, a, on its two sides, has a constant value c, and the temperature of the remainder of the plane zero. In this case the formula (e) will give, for the temperature at a point (x, y, z) above the plane,

$$\begin{aligned}
\phi &= \frac{zc}{2\pi}\int_{-\infty}^{\infty}\int_{-a}^{a}\frac{d\xi\, d\eta}{\{(\xi - x)^2 + (\eta - y)^2 + z^2\}^{\frac{3}{2}}} \\
&= \frac{c}{\pi}\left(\tan^{-1}\frac{x+a}{z} - \tan^{-1}\frac{x-a}{z}\right) \\
&= \frac{c}{\pi}\tan^{-1}\frac{2az}{x^2+z^2-a^2}.
\end{aligned}$$

From this we conclude that the isothermal surfaces which correspond to this case are circular cylinders, which intersect the plane (xy) in the two parallel lines bounding A.

The application to this example, and all others in which the isothermal surfaces are cylindrical, may be made directly by putting $s = 1$ in the general formulæ.

II.

I now proceed to find the values, which will be denoted by V and W, of the integrals

$$\left[\int_{-\infty}^{\infty}\right]^s \frac{[d\xi]^s\,[\cos m\xi]^s}{(\Sigma\xi^2 + u^2)^{\frac{1}{2}(s-1)}}$$

and $$\left[\int_{-\infty}^{\infty}\right]^s [dm]^s\,[\cos mx]^s\,\frac{\epsilon^{-(\Sigma m^2)^{\frac{1}{2}}u}}{(\Sigma m^2)^{\frac{1}{2}}},$$

where the symbols $[\cos m\xi]^s$, $[\cos mx]^s$ denote the products

$$\cos m_1\xi_1 \,.\, \cos m_2\xi_2 \,..\, \cos m_s\xi_s,$$
$$\cos m_1x_1 \,.\, \cos m_2x_2 \,..\, \cos m_sx_s;$$

and the notation is in other respects the same as before.

By means of the formula

$$[\cos m\xi + \sin m\xi \,.\, \sqrt{(-1)}]^s = \cos(\Sigma m\xi) + \sin(\Sigma m\xi) \,.\, \sqrt{(-1)},$$

it is easily shewn that

$$V = \left[\int_{-\infty}^{\infty}\right]^s \frac{[d\xi]^s \cos\Sigma(m\xi)}{(\Sigma\xi^2 + u^2)^{\frac{1}{2}(s-1)}} \quad \ldots\ldots\ldots\ldots \quad (a).$$

Hence, by a suitable linear transformation, in which one of the assumptions is $\Sigma m\xi = \eta(\Sigma m^2)^{\frac{1}{2}}$, we have

$$V = \int_{-\infty}^{\infty}\cos\mu\eta \,.\, d\eta \,.\left[\int_{-\infty}^{\infty}\right]^{s-1}\frac{[d\zeta]^{s-1}}{(u^2 + \eta^2 + \Sigma\zeta^2)^{\frac{1}{2}(s-1)}} \quad \ldots . (b).$$

Now, by means of Liouville's theorem,* we find

$$\left[\int_{-\infty}^{\infty}\right]^{s-1} \frac{[d\zeta]^{s-1}}{(u^2+\eta^2+\Sigma\zeta^2)^{\frac{1}{2}(s-1)}} = \frac{\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\int_0^\infty \frac{2\xi\, d\xi\,.\,\xi^{s-3}}{(\xi^2+\eta^2+u^2)^{\frac{1}{2}(s-1)}}.$$

Hence $$V = \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\int_0^\infty\int_0^\infty \frac{\xi^{s-2}\cos\mu\eta\,.\,d\xi\, d\eta}{(\xi^2+\eta^2+u^2)^{\frac{1}{2}(s-1)}} \;\ldots\ldots (c).$$

Differentiating with respect to u, by which the farther reduction of the integral will be facilitated, we have

$$-\frac{dV}{du} = (s-1)\frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\int_0^\infty\int_0^\infty \frac{u\,.\,\xi^{s-2}\cos\mu\eta\,.\,d\xi\, d\eta}{(\xi^2+\eta^2+u^2)^{\frac{1}{2}(s+1)}} \;..(d).$$

Now

$$\int_0^\infty \frac{\xi^{s-2}\, d\xi}{(\xi^2+\eta^2+u^2)^{\frac{1}{2}(s+1)}} = \int_0^\infty \frac{\xi^{-3}\, d\xi}{\left(1+\frac{\eta^2+u^2}{\xi^2}\right)^{\frac{1}{2}(s+1)}} = \frac{1}{2}\int_0^\infty \frac{dt}{\{1+(\eta^2+u^2)t\}^{\frac{1}{2}(s+1)}}$$

$$= \frac{1}{s-1}\cdot\frac{1}{\eta^2+u^2}.$$

Hence $$-\frac{dV}{du} = \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\int_0^\infty \frac{u\cos\mu\eta\, d\eta}{\eta^2+u^2} \;\ldots\ldots\ldots (e),$$

$$= \frac{2\pi^{\frac{1}{2}(s+1)}}{\Gamma\frac{1}{2}(s-1)}\,.\,\epsilon^{-\mu u}.$$

From this, by integration with respect to u, we deduce the value of V: thus we have the result

$$\left[\int_{-\infty}^{\infty}\right]^{s} \frac{[d\xi]^s\,[\cos m\xi]^s}{(\Sigma\xi^2+u^2)^{\frac{1}{2}(s-1)}} = \frac{2\pi^{\frac{1}{2}(s+1)}}{\Gamma\frac{1}{2}(s-1)}\,\frac{\epsilon^{-(\Sigma m^2)^{\frac{1}{2}}u}}{(\Sigma m^2)^{\frac{1}{2}}} \;\ldots .(V).$$

To evaluate the integral W we may in the first place reduce it to a double integral by a process similar to that indicated above, for obtaining the expression (c); and we thus find

$$W = \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\int_0^\infty\int_0^\infty \frac{dm\, dn\,.\,m^{s-2}\cos(nr)\,.\,\epsilon^{-(m^2+n^2)^{\frac{1}{2}}u}}{(m^2+n^2)^{\frac{1}{2}}} \;..(a),$$

where r denotes $(\Sigma x^2)^{\frac{1}{2}}$. If we take $m=\rho\cos\vartheta$, $n=\rho\sin\vartheta$, this becomes

$$W = \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\int_0^\infty\int_0^{\frac{1}{2}\pi} d\theta\, d\rho\, \rho^{s-2}\cos^{s-2}\vartheta\cos(r\rho\sin\vartheta)\,\epsilon^{-\rho u} \;..(b).$$

Now we have

$$\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)\cos(r\rho\sin\vartheta)\,\epsilon^{-\rho u} = \rho^{2}\cos^{2}\vartheta\,.\cos(r\rho\cos\vartheta)\epsilon^{-\rho u} \;..(c).$$

* See vol. II. p. 221, First Series.

Considering first the case where s is even, let $f = \frac{1}{2}s - 1$; we thus find

$$\rho^{s-2}\cos^{s-2}\vartheta\cos(r\rho\cos\vartheta)\,\epsilon^{-\rho u} = \left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}s-1}\cos(r\rho\sin\vartheta)\,\epsilon^{-\rho u},$$

and, by substitution in (b), we have

$$W = \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\int_0^\infty\int_0^{\frac{1}{2}\pi} d\vartheta\,d\rho\,.\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}s-1}\cos(r\rho\sin\vartheta)\,\epsilon^{-\rho u}$$

$$= \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}s-1}\int_0^\infty\int_0^{\frac{1}{2}\pi} d\vartheta\,d\rho\cos(r\rho\sin\vartheta)\,\epsilon^{-\rho u}$$

$$= \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}s-1}\int_0^{\frac{1}{2}\pi}\frac{u\,d\vartheta}{u^2+r^2\sin^2\vartheta}$$

$$= \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}s-1}\frac{\frac{1}{2}\pi}{(u^2+r^2)^{\frac{1}{2}}}$$

$$= \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\,\frac{1.3^2\ldots(s-1)^2}{(u^2+r^2)^{\frac{1}{2}(s-1)}} = 2^{s-1}\pi^{\frac{1}{2}(s-1)}\Gamma\tfrac{1}{2}(s-1)\frac{1}{(u^2+r^2)^{\frac{1}{2}(s-1)}}$$

$$\ldots\ldots(d).$$

In the second case, when s is odd, let $f = \frac{1}{2}(s-1)$ in (c); then, making use of the result in (b), we have

$$W = \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}(s-1)}$$

$$\int_0^\infty\int_0^{\frac{1}{2}\pi} d\vartheta\,d\rho\,\rho\cos\vartheta\,.\cos(r\rho\sin\vartheta)\,\epsilon^{-\rho u}$$

$$= \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}(s-1)}\int_0^\infty d\rho\,.\,\frac{\sin(r\rho)}{r}\,\epsilon^{-\rho u}$$

$$= \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}(s-1)}\frac{1}{r^2+\rho^2}$$

$$= 2^{s-1}\pi^{\frac{1}{2}(s-1)}\Gamma\tfrac{1}{2}(s-1)\frac{1}{(u^2+r^2)^{\frac{1}{2}(s-1)}}\,.$$

Hence, whether s be odd or even, we conclude that

$$\left[\int_{-\infty}^{\infty}\right]^s [dm]^s\,[\cos mx]^s\,\frac{\epsilon^{-(\Sigma m^2)^{\frac{1}{2}}u}}{(\Sigma m^2)^{\frac{1}{2}}}$$

$$= 2^{s-1}\pi^{\frac{1}{2}(s-1)}\Gamma\tfrac{1}{2}(s-1)\frac{1}{(u^2+\Sigma x^2)^{\frac{1}{2}(s-1)}}\ldots.(W).$$

The investigation which we have just gone through, of the integrals (V), (W) constitutes the verification of "Fourier's theorem" in a particular case. For, by this theorem, we

have, if $F(x_1, x_2 \ldots)$ be a function which remains the same when the signs of any of the variables are changed,

$$2^s\pi^s F(x_1, x_2 \ldots)$$

$$= \left[\int_{-\infty}^{\infty}\right]^s [dm]^s [\cos mx]^s \int_{-\infty}^{\infty} [d\xi]^s [\cos m\xi]^s \, F(\xi_1, \xi_2 \ldots) \ldots (e);$$

and if we take

$$F(\xi_1, \xi_2 \ldots) = \frac{1}{(\Sigma\xi^2 + u^2)^{\frac{1}{2}(s-1)}},$$

the result of the integrations with respect to $\xi_1, \xi_2 \ldots$, is given by (V), and the second member thus becomes a multiple integral with respect to $m_1, m_2 \ldots$, which is shewn by (W) to be equal to the first member. Conversely, if we assume Fourier's theorem, we may deduce the value W, by means of it, from that of V. The integrals V and W are also connected by means of another case of Fourier's theorem, found by taking, in (e),

$$F(\xi_1, \xi_2 \ldots) = \frac{\varepsilon^{-(\Sigma\xi^2)^{\frac{1}{2}}u}}{(\Sigma\xi)^{\frac{1}{2}}}.$$

In this way, after the value of W has been found, that of V may be deduced.

The formulæ (V) and (W) may be applied to evaluate the multiple integral u, and we shall thus obtain the result of the investigation in §. I. in a different manner.

By means of the equation obtained by differentiating (W) with respect to u, we find

$$\frac{u}{\{\Sigma(\xi - x)^2 + u^2\}^{\frac{1}{2}(s+1)}} = \frac{1}{2^{s-1}(s-1)\,\pi^{\frac{1}{2}(s-1)}\,\Gamma\frac{1}{2}(s-1)}$$

$$\left[\int_{-\infty}^{\infty}\right]^s [dm]^s [\cos m(\xi - x)]^s \, \varepsilon^{-(\Sigma m^2)^{\frac{1}{2}}u};$$

Making this substitution, for one of the factors of the expression under the integral signs in U, we have

$$Uu = \frac{1}{2^{s-1}(s-1)\,\pi^{\frac{1}{2}(s-1)}\,\Gamma\frac{1}{2}(s-1)} \left[\int_{-\infty}^{\infty}\right]^s \frac{[d\xi]^s}{\{\Sigma(\xi - x')^2 + u'^2\}^{\frac{1}{2}(s-1)}}$$

$$\left[\int_{-\infty}^{\infty}\right] [dm]^s [\cos m(\xi - x)]^s \, \varepsilon^{-(\Sigma m^2)^{\frac{1}{2}}u}$$

$$= \frac{1}{2^{s-1}(s-1)\,\pi^{\frac{1}{2}(s-1)}\,\Gamma(\frac{1}{2}s - 1)}$$

$$\left[\int_{-\infty}^{\infty}\right]^s [dm]^s [\cos m(x - x')]^s \, \varepsilon^{-(\Sigma m^2)^{\frac{1}{2}}u} \left[\int_{-\infty}^{\infty}\right]^s \frac{[d\xi]^s [\cos m(\xi - x')]^s}{\{\Sigma(\xi - x')^2 + u'^2\}^{\frac{1}{2}(s-1)}}$$

$$= \frac{\pi}{2^{s-2}(s-1)\left\{\Gamma \tfrac{1}{2}(s-1)\right\}^2}$$

$$\left[\int_{-\infty}^{\infty}\right]^s [dm]^s \left[\cos m(x-x')\right]^s \epsilon^{-(\Sigma m^2)^{\frac{1}{2}} u} \frac{\epsilon^{-(\Sigma m^2)^{\frac{1}{2}} u'}}{(\Sigma m^2)^{\frac{1}{2}}}, \text{ by } (V),$$

$$= \frac{2\pi^{\frac{1}{2}(s+1)}}{(s-1)\,\Gamma\tfrac{1}{2}(s-1)} \frac{1}{\left\{\Sigma(x-x')^2+(u+u')^2\right\}^{\frac{1}{2}(s-1)}}, \text{ by } (W),$$

which agrees with the value obtained above.

III.

The value of the integral U may also be obtained by a direct process of reduction, as follows.

By a suitable linear transformation, in which assumptions such as

$$\xi_1 - x_1 = \Sigma a \zeta$$

are made, we find

$$U = \left[\int_{-\infty}^{\infty}\right] \frac{[d\zeta]^s}{(\Sigma\zeta^2+u^2)^{\frac{1}{2}(s+1)}(\Sigma\zeta^2 - 2f\zeta_1 + f^2 + u'^2)^{\frac{1}{2}(s-1)}} \ldots (a),$$

where

$$f^2 = \Sigma(x-x')^2.$$

Let us now assume

$$\zeta_1 = \rho\cos\phi, \quad \zeta_2 = \rho\sin\phi\cos\theta_1, \quad \zeta_3 = \rho\sin\phi\sin\theta_1\cos\theta_2 \ldots,$$

$$\zeta_{s-1} = \rho\sin\phi\sin\theta_1\sin\theta_2\ldots\cos\theta_{s-2},$$

$$\zeta_s = \rho\sin\phi\sin\theta_1\sin\theta_2\ldots\sin\theta_{s-2},$$

from which we deduce*

$$[d\xi]^s = \rho^{s-1}\sin^{s-2}\phi\,\sin^{s-3}\theta_1\sin^{s-4}\theta_2\ldots\sin\theta_{s-3}[d\theta]^{s-2}\,d\phi\,d\vartheta;$$

a transformation given first by Green. Equation (a) is thus reduced to

$$U = H_{s-2}\int_0^{\infty}\int_0^{2\pi} \frac{\rho^{s-1}\sin^{s-2}\phi\,d\phi\,d\rho}{(\rho^2+u^2)^{\frac{1}{2}(s+1)}(\rho^2 - 2\rho f\cos\phi + f^2 + u'^2)^{\frac{1}{2}(s-1)}} \ldots (b),$$

where H_{s-2} denotes the product

$$\int_0^{\pi}\sin^{s-3}\theta\,d\theta \,.\int_0^{\pi}\sin^{s-4}\theta\,d\theta \ldots \int_0^{\pi} d\theta.$$

Let $\rho = u\tan\frac{1}{2}\vartheta$; we thus get

$$Uu = \tfrac{1}{2}H_{s-2}\int_0^{\pi}\int_0^{2\pi}$$

$$\frac{\sin^{s-1}\vartheta\,\sin^{s-2}\phi\,d\phi\,d\vartheta}{\left\{2(f^2+u'^2+u^2)+2(f^2+u'^2-u^2)\cos\vartheta - 4uf\sin\vartheta\cos\phi\right\}^{\frac{1}{2}(s-1)}},$$

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* See vol. IV. p. 24., First Series.

and we may now conveniently assume

$$2\,(f^2 + u'^2 + u^2) = h^2 + k^2,$$

$$2\,(f^2 + u'^2 - u^2)\cos\vartheta - 4uf\sin\vartheta\cos\phi$$
$$= 2\,\{(f^2 + u^2 - u'^2)^2 + 4u^2f^2\}^{\frac{1}{2}}\cos\theta = 2hk\cos\theta,$$

and $$\sin\phi\sin\vartheta = \sin\phi\sin\theta,$$

from which we deduce

$$h^2 = (u' + u)^2 + f^2,\quad k^2 = (u' - u)^2 + f^2,$$
$$\sin\vartheta\, d\phi\, d\vartheta = \sin\theta\, d\varphi\, d\theta\,;$$

the expression for U becomes

$$Uu = \tfrac{1}{2}H_{s-2}\int_0^{\pi}\!\!\int_0^{\pi}\frac{\sin^{s-1}\theta\sin^{s-2}\varphi\, d\varphi\, d\theta}{(h^2 - 2hk\cos\theta - k^2)^{\frac{1}{2}(s-1)}}$$
$$= H_{s-1}\int_0^{\pi}\frac{\sin^{s-1}\theta\, d\theta}{(h^2 - 2hk\cos\theta + k^2)^{\frac{1}{2}(s-1)}}.$$

Let $$h\sin(\psi - \theta) = k\sin\psi\,;$$

by means of this transformation, observing that $h > k$, we readily find

$$Uu = \frac{H_s}{h^{s-1}}$$

or $$Uu = \frac{\pi^{\frac{1}{2}(s+1)}}{\Gamma\frac{1}{2}(s+1)}\,\frac{1}{\{\Sigma(x - x')^2 + (u + u')^2\}^{\frac{1}{2}(s-1)}},$$

which is the same as the result previously obtained.

St. Peter's College, Oct. 3, 1846.

ON CERTAIN FORMULÆ FOR DIFFERENTIATION, WITH APPLICATIONS TO THE EVALUATION OF DEFINITE INTEGRALS.

By Arthur Cayley.

In attempting to investigate a formula in the theory of multiple definite integrals (which will be noticed in the sequel), I was led to the question of determining the $(i + 1)^{\text{th}}$ differential coefficient of the $2i^{\text{th}}$ power of $\sqrt{(x + \lambda)} - \sqrt{(x + \mu)}$; the only way that occurred for effecting this was to find the successive differential coefficients of this quantity, which may be effected as follows. Assume

$$U_{k,i} = \{(x + \lambda)(x + \mu)\}^{\frac{1}{2}k}\,\{\sqrt{(x + \lambda)} - \sqrt{(x + \mu)})^{2i},$$

then

$$\frac{1}{U_{k,i}}\frac{d}{dx}U_{k,i} = \tfrac{1}{2}k\,\frac{2x+\lambda+\mu}{(x+\lambda)(x+\mu)} - \frac{i}{\sqrt{\{(x+\lambda)(x+\mu)\}}}$$

$$= \tfrac{1}{2}k\,\frac{\{\sqrt{(x+\lambda)}+\sqrt{(x+\mu)}\}^2 - 2\sqrt{\{(x+\lambda)(x+\mu)\}}}{(x+\lambda)(x+\mu)} - \frac{i}{\sqrt{\{(x+\lambda)(x+\mu)\}}}$$

$$= \tfrac{1}{2}k\,\frac{(\lambda-\mu)^2}{\{\sqrt{(x+\lambda)}-\sqrt{(x+\mu)}\}^2(x+\lambda)(x+\mu)} - \frac{k+i}{\sqrt{\{(x+\lambda)(x+\mu)\}}}\,.$$

Or, attending to the signification of $U_{k,i}$,

$$\frac{d}{dx}U_{k,i} = \tfrac{1}{2}k(\lambda-\mu)^2\,U_{k-2,i-1} - (k+i)\,U_{k-1,i}.$$

Hence $$-\frac{1}{i}\frac{d}{dx}U_{0,i} = U_{-1,i}$$

$$\frac{1}{i}\frac{d^2}{dx^2}U_{0,i} = \tfrac{1}{2}(\lambda-\mu)^2\,U_{-3,i-1} + (i-1)\,U_{-2,i},$$

&c.

from which the law is easily seen to be of the form

$$\frac{(-)^r}{i}\left(\frac{d}{dx}\right)^r U_{0,i} = S_\theta\,K_{r,\theta}(\lambda-\mu)^{2r-2-2\theta}\,U_{-2r+1+\theta,\,i-r+1+\theta}$$

(where the extreme values of θ are 0 and $(r-1)$ respectively) and $K_{r,\theta}$ is determined by

$$K_{r+1,\theta+1} = (r-1-\tfrac{1}{2}\theta)\,K_{r,\theta+1} + (i-3r+2+2\theta)\,K_{r,\theta}.$$

This equation is satisfied by

$$K_{r,\theta} = \frac{\Gamma(r-\frac{1}{2}-\theta)\,\Gamma(2r-1-\theta)\,\Gamma(i-r+\theta+1)}{\Gamma\frac{1}{2}\,\Gamma(\theta+1)\,\Gamma(2r-1-2\theta)\,\Gamma(i-r+1)}\,.$$

For in the first place this gives

$$(r-1-\tfrac{1}{2}\theta)\,K_{r,\theta+1}$$

$$= \frac{(r-1-\frac{1}{2}\theta)}{\Gamma\frac{1}{2}\,\Gamma(\theta+2)}\,\frac{\Gamma(r-\frac{3}{2}-\theta)\,\Gamma(2r-2-\theta)\,\Gamma(i-r+\theta+2)}{\Gamma(2r-3-2\theta)\,\Gamma(i-r+1)}$$

$$= \frac{\Gamma(r-\frac{1}{2}-\theta)\,\Gamma(2r-1-\theta)\,\Gamma(i-r+\theta+2)}{\Gamma(\frac{1}{2})\,\Gamma(\theta+2)\,\Gamma(2r-2-2\theta)\,\Gamma(i-r+1)}\,.$$

And hence the second side of the equation reduces itself to

$$\frac{\Gamma(r-\frac{1}{2}-\theta)\,\Gamma(2r-1-\theta)\,\Gamma(i-r+\theta+1)}{\Gamma(\frac{1}{2})\,\Gamma(\theta+2)\,\Gamma(2r-1-2\theta)\,\Gamma(i-r+1)}$$

$$\{2(r-1-\theta)(i-r+\theta+1) + (\theta+1)(i-3r+2-2\theta)\},$$

where the quantity within brackets reduces itself to $(i-r)$ $(2r-1-\theta)$, so that the above value reduces itself to $K_{r+1, \theta+1}$, which verifies the equation in question. Also by comparing the first few terms, it is immediately seen that the above is the correct value of $K_{r,\theta}$, so that

$$\frac{(-)^r}{i}\left(\frac{d}{dx}\right)^r U_{0,i}$$

$$= S_\theta \frac{\Gamma(r-\frac{1}{2}-\theta)\,\Gamma(2r-1-\theta)\,\Gamma(i-r+\theta+1)}{\Gamma(\frac{1}{2})\,\Gamma(\theta+1)\,\Gamma(2r-1-\theta)\,\Gamma(i-r+1)}(\lambda-\mu)^{2r-2-\theta}\,U_{-2r+1-\theta,\,i-r+\theta+1} \quad \ldots\ldots(1),$$

θ extending as before from 0 to $(r-1)$. In particular if i be integer and $r=i+1$,

$$\frac{(-)^{i+1}}{i}\left(\frac{d}{dx}\right)^{i+1}\{\sqrt{(x+\lambda)}-\sqrt{(x+\mu)}\}^{2i}$$

$$= \frac{\Gamma(i+\frac{1}{2})}{\Gamma(\frac{1}{2})}(\lambda-\mu)^{2i}\frac{1}{\{(x+\lambda)(x+\mu)\}^{i+\frac{1}{2}}} \quad \ldots.(2),$$

(since the factor $\Gamma(i-r+\theta+1) \div \Gamma(i-r+1)$ vanishes except for $\theta=0$ on account of $\Gamma(i-r+1)=\infty$). Thus also, if r be greater than $(i+1)$, $=i+1+s$ suppose, then

$$(-)^s\left(\frac{d}{dx}\right)^s\frac{1}{\{(x+\lambda)(x+\mu)\}^{i+\frac{1}{2}}}$$

$$= S_\theta \frac{\Gamma(i+s+\frac{1}{2}-\theta)\,\Gamma(2i+2s+1-\theta)\,\Gamma(\theta-s)}{\Gamma(i+\frac{1}{2})\,\Gamma(\theta+1)\,\Gamma(2i+2s+1-2\theta)\,\Gamma(-s)}$$

$$(\lambda-\mu)^{2s-2\theta}\,U_{-2i-2s-1-\theta,\,-s+\theta} \quad \ldots(3),$$

where θ extends only from $\theta=0$ to $\theta=s$, on account of the factor $\Gamma(\theta-s)\div\Gamma(-s)$, which vanishes for greater values of θ: a rather better form is obtained by replacing this factor by $(-)^\theta \dfrac{\Gamma(1+s)}{\Gamma(1+s-\theta)}$.

The above formulæ have been deduced on the supposition of i being an integer; assuming that they hold generally, the equation (2) gives, by writing $(i-\frac{1}{2})$ for i,

$$\frac{(-)^{i+\frac{1}{2}}}{i-\frac{1}{2}}\left(\frac{d}{dx}\right)^{i+\frac{1}{2}}\{\sqrt{(x+\lambda)}-\sqrt{(x+\mu)}\}^{2i-1}$$

$$= \frac{\Gamma i}{\Gamma(\frac{1}{2})}(\lambda-\mu)^{2i-1}\frac{1}{\{(x+\lambda)(x+\mu)\}^i}.$$

Or integrating $(i+\frac{1}{2})$ times by means of the formula

$$\int_0^\infty x^{i-\frac{1}{2}} fx\,dx = \frac{\Gamma(i+\frac{1}{2})}{(-)^{i+\frac{1}{2}}}\left(\int_\infty da\right)^{i+\frac{1}{2}} fa, \quad a=0\,;$$

this gives

$$\int_0^\infty \frac{x^{i-\frac{1}{2}}dx}{\{(x+\lambda)(x+\mu)\}^i} = \frac{\Gamma\frac{1}{2}\,\Gamma(i-\frac{1}{2})}{\Gamma i}\,\frac{1}{(\sqrt{\lambda}+\sqrt{\mu})^{2i-1}} \;\ldots (4),^*$$

whence also

$$\int_0^\infty \frac{x^{i-\frac{1}{2}}dx}{(x+\lambda)^{i+1}(x+\mu)^i} = \frac{\Gamma\frac{1}{2}\,\Gamma(i+\frac{1}{2})}{\Gamma i}\,\frac{1}{(\sqrt{\lambda}+\sqrt{\mu})^{2i}\sqrt{\lambda}} \;\ldots (5).$$

And from these, by simple transformations,

$$\int_\beta^\alpha \frac{(a-x)^{i-\frac{1}{2}}(x-\beta)^{i-\frac{1}{2}}dx}{\{(a-x)+m(x-\beta)\}^i} = \frac{\Gamma\frac{1}{2}\,\Gamma(i+\frac{1}{2})}{\Gamma(i+1)}\,\frac{(a-\beta)^i}{(\sqrt{m}+1)^{2i}} \;\ldots (6),$$

$$\int_\beta^\alpha \frac{(a-x)^{i-\frac{1}{2}}(x-\beta)^{i-\frac{3}{2}}dx}{\{(a-x)+m(x-\beta)\}^i} = \frac{\Gamma\frac{1}{2}\,\Gamma(i-\frac{1}{2})}{\Gamma i}\,\frac{(a-\beta)^{i-1}}{(\sqrt{m}+1)^{2i-1}} \ldots (7).$$

These last two formulæ are connected also by the following general property:

"If $\qquad (a, b, i) = \displaystyle\int_\beta^\alpha \frac{(a-x)^{a-1}(x-\beta)^{b-1}dx}{\{(a-x)+m(x-\beta)\}^i},$

then $(a, b, i) = \dfrac{\Gamma a \Gamma b}{\Gamma(a+b-i)\,\Gamma i}\,(a-\beta)^{b-i}\,(a+b-i, i, b)$"...(8),

which I have proved by means of a double integral. From (6) we may obtain for $\gamma < 1$,

$$\int_{-1}^1 \frac{(1-x^2)^{i-\frac{1}{2}}dx}{(1-2\gamma x+\gamma^2)^i} = \frac{\Gamma\frac{1}{2}\,\Gamma(i+\frac{1}{2})}{\Gamma(i+1)} \;\ldots\ldots\ldots\ldots (9),$$

which however is only a particular case of

$$\int_{-1}^1 dx\,(1-x^2)^{i-\frac{1}{2}}(1-2\gamma x+\gamma^2)^{-i}\,\frac{d}{d\beta}\left[\beta^i\left(1-2\frac{\beta}{\gamma}x+\frac{\beta^2}{\gamma^2}\right)^{-i}\right]$$

$$= \frac{\Gamma\frac{1}{2}\,\Gamma(i+\frac{1}{2})}{\Gamma(i+1)}\,\beta^{i-1}(1-\beta)^{-2i} \ldots\ldots (10),$$

which supposes γ and $\dfrac{\beta}{\gamma}$ each less than unity. This formula was obtained in the case of $(i+\frac{1}{2})$ an integer, from a theorem, *Leg. Cal. Int.*, tom. II. p. 258, but there is no doubt that it is generally true.

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* This is immediately transformed into

$$\int_0^\infty \frac{x^{i-\frac{1}{2}}dx}{(ax^2+bx+c)^i} = \frac{\Gamma\frac{1}{2}\,\Gamma(i-\frac{1}{2})}{\Gamma i}\,\frac{1}{\{b+2\sqrt{(ac)}\}^{i-\frac{1}{2}}},$$

which is a particular case of a formula which will be demonstrated in a subsequent paper.

where σ extends from 0 to λ. Hence substituting, and prefixing the summatory sign

$$W = \pi^{i+\frac{1}{2}} S \frac{(-)^\lambda \Delta^\lambda}{2^{2\lambda}\,\Gamma(\lambda+1)\,\Gamma(i+\lambda+\frac{1}{2})} \left(\frac{d}{du}\right)^{2\lambda} \int_0 \xi^{i-\frac{1}{2}} (\xi + u^2)\, d\xi,$$

where λ extends from 0 to ∞, the formula required.

ON THE CAUSTIC BY REFLECTION AT A CIRCLE.

By Arthur Cayley.

The following solution of the problem is that given by M. de St. Laurent (*Annales de Gergonne*, tom. XVII. p. 128); the process of elimination is somewhat different.

The centre of the circle being taken for the origin, let k be its radius; a, b the coordinates of the luminous point; ξ, η those of the point at which the reflection takes place; x, y of any point in the reflected ray: we have in the first place

$$\xi^2 + \eta^2 = k^2 \ldots\ldots\ldots\ldots\ldots(1).$$

There is no difficulty in finding the equation of the reflected ray*

$$(b\xi - a\eta)(\xi x + \eta y - k^2) + (y\xi - \xi\eta)(a\xi + b\eta - k^2) = 0.$$

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* To do this in the simplest way, write

$$\rho^2 = (\xi - x)^2 + (\eta - y)^2, \quad \sigma^2 = (\xi - a)^2 + (\eta - b)^2.$$

Then, by the condition of reflection,

$$\rho + \sigma = \text{min.},$$

ρ, σ being considered as functions of the variables ξ, η, which are connected by the equation (1). Hence

$$\frac{\xi - x}{\rho} + \frac{\xi - a}{\sigma} + \lambda\xi = 0,$$

$$\frac{\eta - y}{\rho} + \frac{\eta - b}{\sigma} + \lambda\eta = 0.$$

Or eliminating λ,

$$\frac{\eta x - \xi y}{\rho} + \frac{\eta a - \xi b}{\sigma} = 0,$$

whence

$$(\eta x - \xi y)^2 [(\xi - a)^2 + (\eta - b)^2] = (\eta a - \xi b)^2 [(\xi - x)^2 + (\eta - y)^2].$$

Or

$$\{(\eta x - \xi y)(\xi - a) - (\eta a - \xi b)(\xi - x)\}[(\eta x - \xi y)(\xi - a) + (\eta a - \xi b)(\xi - x)]$$
$$+ \{(\eta x - \xi y)(\eta - b) - (\eta a - \xi b)(\eta - y)\}[(\eta x - \xi y)(\eta - b) + (\eta a - \xi b)(\eta - y)] = 0.$$

The factors in { } reduce themselves respectively to ξP and ηP, where $P = \xi(b - y) - \eta(a - x) + ay - bx$, omitting the factor P, (which equated to zero, is the equation of the line through (a, b) and (ξ, η).) And replacing $\xi(\xi - a) + \eta(\eta - b)$ and $\xi(\xi - x) + \eta(\eta - y)$ by $k^2 - a\xi - b\eta$ and $k^2 - \xi x - \eta y$, respectively, we have the equation given above.

Hence, after a slight reduction,

$$V = \frac{\pi^{i+1}}{v\Gamma(i+1)} S \frac{(-)^\lambda \Gamma(i+\lambda+1)}{\Gamma(i+1)\Gamma(\lambda+1)} \frac{A^\lambda}{\{(u+v)^2\}^\lambda};$$

or finally $$V = \frac{\pi^{i+1}}{\Gamma(i+1)} \frac{1}{v\{(u+v)^2+A\}^i} \quad \ldots\ldots\ldots(16),$$

a remarkable formula, the discovery of which is due to Mr. Thomson. It only remains to prove the formula for W. Out of the variety of ways in which this may be accomplished, the following is a tolerably simple one. In the first place, by a linear transformation corresponding to that between two sets of rectangular axes, we have

$$W = \int \frac{dx\,dy\,..}{\{(x-\sqrt{A})^2 + y^2 .. + u^2\}^i};$$

or expanding in powers of A, and putting for shortness $R = x^2 + y^2 \ldots + u^2$, the general term of W is

$$(-)^\sigma A^\lambda \frac{\Gamma(i+\lambda+\sigma)}{\Gamma i\,\Gamma(\lambda-\sigma+1)\,\Gamma(2\sigma+1)} 2^{2\sigma}\int x^{2\sigma} R^{-i-\lambda-\sigma} dx\,dy\,..$$

the limits being as before $x^2 + y^2 + \ldots = \xi$. To effect the integrations, write $\sqrt{\xi}\,\sqrt{x}$, $\sqrt{\xi}\,\sqrt{y}$, &c. for $x, y \ldots$ So that the equation of the limits becomes $x + y + \ldots = 1$. Also restricting the integral to positive values, we shall multiply it by 2^{2i+1}. The integral thus becomes

$$\xi^{\sigma+i+\frac{1}{2}} \int x^{\sigma-\frac{1}{2}} y^{-\frac{1}{2}} .. \{\xi(x+y..) + u^2)^{-i-\lambda-\sigma} dx\,dy.$$

Equivalent to

$$\xi^{\sigma+i+\frac{1}{2}} \frac{\Gamma(\sigma+\frac{1}{2})\,\pi^i}{\Gamma(i+\sigma+\frac{1}{2})} \int_0^1 \theta^{i+\sigma-\frac{1}{2}} (\xi\theta + u^2)^{-i-\lambda-\sigma} d\theta;$$

i.e. to $$\frac{\Gamma(\sigma+\frac{1}{2})\,\pi^i}{\Gamma(i+\sigma+\frac{1}{2})} \int_0 \xi^{i+\sigma-\frac{1}{2}} (\xi + u^2)^{-i-\lambda-\sigma} d\xi.$$

Or after a slight reduction, the general term of W is

$$\frac{\pi^{i+\frac{1}{2}}}{\Gamma i} (-)^{\lambda+\sigma} A^\lambda \frac{\Gamma(i+\lambda+\sigma)}{\Gamma(\sigma+1)\Gamma(\lambda-\sigma+1)\Gamma(i+\sigma+\frac{1}{2})} \int_0 \xi^{i+\sigma-\frac{1}{2}} (\xi+u^2)^{-i-\lambda-\sigma} d\xi,$$

where σ may be considered as extending from 0 to λ inclusively, and then λ from 0 to ∞. But by a formula easily proved

$$\left(\frac{d}{du}\right)^{2\lambda} (\xi + u^2)^{-i} = \frac{2^{2\lambda}\,\Gamma(\lambda+1)\,\Gamma(i+\lambda+\frac{1}{2})}{\Gamma i}$$

$$S(-)^\sigma \frac{\Gamma(i+\lambda+\sigma)}{\Gamma(\sigma+1)\,\Gamma(\lambda-\sigma+1)\,\Gamma(i+\sigma+\frac{1}{2})} \xi^\sigma (\xi + u^2)^{-i-\lambda-\sigma},$$

which singularly enough is the derived equation of (7′) with respect to λ: so that the equation of the curve is obtained by expressing that two of the roots of the equation (7′) are equal: Multiplying (12) by λ and reducing by (7′),

$$-\lambda Q + 3R = 0.$$

Or combining this with (12),

$$27R^2 - Q^3 = 0.$$

Or replacing R, Q by their values,

$$27k^4.(bx - ay)^2.(x^2 + y^2 - a^2 - b^2)^2$$
$$- \{4(a^2 + b^2)(x^2 + y^2) - k^2.[(a + x)^2 + (b + y)^2]\}^3 = 0,$$

the equation of M. de St. Laurent.

ON SYMBOLICAL GEOMETRY.

By Sir William Hamilton.

(*Continued from* p. 52.)

Symbolical Expressions for a Cyclic Cone; Relations of such a Cone, and of its Cyclic Planes, to a Product of two Geometrical Fractions.

20. It is evidently a determinate* problem to construct a *cyclic cone*, that is, a cone with circular base (called usually a cone of the second degree), when three of the *sides* (or generating straight lines) of the cone are given in position, and when the plane of the base is parallel to a given *cyclic plane*, which passes through the vertex. To treat this problem, which may be regarded as a fundamental one in the theory of such cones, by a method derived from the principles of the foregoing articles, let the three given sides be denoted by the letters a, b, c; and let the two known lines, in which the given cyclic plane is cut by the planes of the two pairs,

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* The evident and known determinateness of this problem, corresponding to that of the elementary problem of circumscribing a circle about a given plane triangle, was tacitly assumed, but might with advantage have been expressly referred to, in the outline of a demonstration which was given in the note to Art. (18). The reasoning, towards the end of that note, would then stand thus:—If D be any fourth point on the determined spherical conic, which passes through the three points A, B, C, and has the arc A′B′ for a cyclic arc, it is also a fourth point on the determined spherical conic which passes through the same three points and has the arc B″C″ for a cyclic arc; therefore the two conics, determined by these two sets of conditions, coincide one with the other: or, in other words, the arc B″C″ is a *second* cyclic arc of the *same* spherical conic, of which the arc A′B′ is a *first* cyclic arc.

ab and bc, be denoted by a′ and b′; also let d denote any fourth side of the sought cyclic cone, and c′, d′ the lines of intersection of the given cyclic plane with the variable planes of cd and da; then, if suitable lengths be assigned to these straight lines, of which the relative *directions* in space are the chief object of the present investigation, the following equality between two products of certain geometrical fractions will exist, and may be regarded as a form of the *equation of the cone:*

$$\frac{\mathrm{c}}{\mathrm{b}'}\,\frac{\mathrm{a}'}{\mathrm{a}} = \frac{\mathrm{c}}{\mathrm{c}'}\,\frac{\mathrm{d}'}{\mathrm{a}} \ldots\ldots\ldots\ldots\ldots (147).$$

That is to say, when this equation is satisfied, the two lines which are the respective intersections of the planes of the fractional factors of these two equal products, namely the intersection b of the planes aa′ and b′c, and the intersection d of the planes ad′ and c′c, are two sides of a cyclic cone, which has for two other sides the lines a and c, and which has for one cyclic plane the common plane of the four lines a′, b′, c′, and d′; these eight lines, a, b, c, d, a′, b′, c′, d′, being here supposed to diverge from one common origin, namely the vertex (or centre) of the cone. This may easily be shown to be a consequence of what has been already established, respecting the connexion of the cyclic arcs of a spherical conic with the symbolic sums of certain other arcs. Or, without introducing any sphere, we may observe that, by (121) and its converse, the equation (147) may be abridged to the following:

$$\frac{\mathrm{a}'}{\mathrm{b}'} = \frac{\mathrm{d}'}{\mathrm{c}'}; \quad \text{or,} \quad \frac{\mathrm{a}'}{\mathrm{b}'}\,\frac{\mathrm{c}'}{\mathrm{d}'} = 1 \ldots\ldots\ldots (148);$$

which shows, in virtue of the notation here employed, that besides a certain proportionality of lengths, not necessary now to be considered, there exists an equality between the angles of rotation, in one common plane, which would transport the lines b′ and c′, respectively, into the directions of a′ and d′. But the four lines a′, b′, c′, d′ are respectively parallel to the four symbolic differences, b − a, c − b, d − c, a − d, or to the four straight lines BA, CB, DC, AD, that is to the successive sides of the plane quadrilateral ABCD, if we now suppose the lines a, b, c, d to terminate, in the points A, B, C, D, on a transversal plane parallel to the plane of a′ b′ c′ d′. We may therefore present the relation (148) under either of the two forms:

$$\frac{\mathrm{b}-\mathrm{a}}{\mathrm{c}-\mathrm{b}}\,\frac{\mathrm{d}-\mathrm{c}}{\mathrm{a}-\mathrm{d}} = x; \quad \text{or} \quad \frac{\mathrm{BA}}{\mathrm{CB}}\,\frac{\mathrm{DC}}{\mathrm{AD}} = x \ldots (149);$$

in which x is a positive or negative scalar; or, using the characteristic V of the operation of taking the vector part, we may write:

$$\mathrm{V}.\frac{\mathrm{b}-\mathrm{a}}{\mathrm{c}-\mathrm{b}}\frac{\mathrm{d}-\mathrm{c}}{\mathrm{a}-\mathrm{d}}=0;\quad \text{or } \mathrm{V}.\frac{\mathrm{BA}}{\mathrm{CB}}\frac{\mathrm{DC}}{\mathrm{AD}}=0. .(150).$$

When the scalar x is positive, then, by considering the two rotations above mentioned, we easily perceive that the two points B and D are at one common side of the straight line AC, and that this line subtends equal angles at those two points; being in one common plane with them, as indeed the second equation (149) sufficiently expresses, since it gives

$$\mathrm{V}\frac{\mathrm{BA}}{\mathrm{CB}}=x\mathrm{V}\frac{\mathrm{DA}}{\mathrm{CD}} \ldots\ldots\ldots\ldots (151);$$

so that the two triangles ABC, ADC, on the common base AC, have one common perpendicular to their planes, which must therefore coincide with each other. In the contrary case, namely when x is negative, the equation (151) still shows that the four points are (as above) coplanar with each other; and while the points B and D are now at opposite sides of the line AC, the angles which this line subtends at those two points are now not equal but supplementary. In each case, therefore, the four points ABCD are on the circumference of one common circle; the four lines a, b, c, d are consequently sides of a cyclic cone; and the plane of the four other lines a′, b′, c′, d′ is a cyclic plane of that cone.

21. In the foregoing article, the coplanarity of each of the four sets of three lines, a′ab, b′bc, c′cd, d′da, allows us to suppose that four other lines b″, c″, d″, a″, in the same four planes respectively, and all, like the eight former lines, diverging from the vertex of the cone, are determined so as to satisfy the four equations:

$$\frac{\mathrm{b}''}{\mathrm{b}}=\frac{\mathrm{a}}{\mathrm{a}'};\quad \frac{\mathrm{c}''}{\mathrm{c}}=\frac{\mathrm{b}}{\mathrm{b}'};\quad \frac{\mathrm{d}''}{\mathrm{d}}=\frac{\mathrm{c}}{\mathrm{c}'};\quad \frac{\mathrm{a}''}{\mathrm{a}}=\frac{\mathrm{d}}{\mathrm{d}'} \ldots (152);$$

and then, since these equations, combined with (148), give, by the associative property of the multiplication of geometrical fractions, this other equation,

$$\frac{\mathrm{b}''}{\mathrm{c}''}=\frac{\mathrm{a}''}{\mathrm{d}''} \ldots\ldots\ldots\ldots\ldots (153),$$

it follows that these four new lines are in one common plane; and also that the rotations in that plane, from b″ and c″ to

a'' and d'', respectively, are equal. And this new plane is evidently a *second* cyclic plane of the same cone;* for we may now write, instead of (147), the analogous equation:

$$\frac{c}{c''}\,\frac{b''}{a} = \frac{c}{d''}\,\frac{a''}{a} \;\ldots\ldots\ldots\ldots (154);$$

the two members being here equal respectively to the reciprocals of the two members of the first equation (148): nor is it necessary to retain the restriction that the lines a, b, c, d should terminate in one common plane. In like manner, the two members of the equation (147) are respectively equal to the reciprocals of the two members of the equation (153); a geometrical (like an arithmetical) fraction being said to be changed to its *reciprocal*, when the numerator and denominator are interchanged. We have therefore this theorem:— *A cyclic cone is the locus of the intersection of the planes of two geometrical fractions, of which the product is a constant fraction, while the numerator of the multiplier and the denominator of the multiplicand are constant lines. These two lines are two fixed sides of the cone; the plane of the two other and variable lines*, which enter as denominator and numerator into the expressions of the same two fractional factors, *is one cyclic plane of that cone;* and *the plane of the constant product is the other cyclic plane.* The investigation in the last article shows also that the condition for four points ABCD being *concircular* or *homocyclic*, that is, for their being corners of a quadrilateral inscribed in a circle, is expressed by the second equation (150); which may therefore be called the *equation of homocyclicism.* The same investigation shows that if we only know that ABCD are four points on one common plane, we may still write an equation of the form (151); which may for that reason be said to be a *formula of coplanarity.*

[*To be continued.*]

ADDITIONAL CORRECTIONS FOR THE PRECEDING PORTION OF THIS PAPER.

In Note to Art. (8), p. 138 (vol. I.), *for* CB *read* CA.
In Art. (17), p. 263, the second spherical hexagon should be
$A''A'''\,B''B'''\,C''C'''$.
In Art. (18), p. 48, line 17 (vol. II.), *for* alteration *read* alternation.

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* See the remarks made in the note to the foregoing article.

ON CERTAIN SYMBOLICAL REPRESENTATIONS OF FUNCTIONS.

By the Rev. Brice Bronwin.

A symbolical representation of Taylor's theorem has long been in use, and has been employed in integration. Sir John Herschel's and Sir William Hamilton's theorems may be thus represented. And there are some others of the kind, which I suppose to be new, and which it is the object of this paper to exhibit.

Unless it be otherwise stated, D always stands for $\dfrac{d}{do}$.

Taylor's theorem is $\phi(x) = \epsilon^{xD}\phi(o)$. Change $\phi(x)$ into $\phi(\epsilon^x)$, and it becomes $\phi(\epsilon^x) = \epsilon^{xD}\phi(\epsilon^o)$. Now change ϵ^x into x, and we have

$$\phi(x) = x^D\phi(\epsilon^o) \ldots\ldots\ldots\ldots\ldots (a).$$

This will serve to expand by the powers of $\log x$; thus

$$\phi(x) = \phi(\epsilon^o) + \frac{\log x}{1} D\phi(\epsilon^o) + \frac{(\log x)^2}{1.2} D^2\phi(\epsilon^o) + \&c.$$

We may multiply (a) by a function of x, and integrate. For

$$\int\phi(x)\, x^{n-1}dx = \frac{x^{n+D}}{n+D}\phi(\epsilon^o) = \Sigma\left\{\frac{(lx)^m}{1.2\ldots m}(n+D)^{m-1}\phi(\epsilon^o)\right\}$$

$$= \Sigma\left[\frac{(lx)^m}{1.2\ldots m}\epsilon^{-no}D^{m-1}\{\epsilon^{no}\phi(\epsilon^o)\}\right] = \Sigma\left[\frac{(lx)^m}{1.2\ldots m}D^{m-1}\{\epsilon^{no}\phi(\epsilon^o)\}\right].$$

But, by the theorem itself,

$$\int\phi(x)\, x^{n-1}dx = x^D\int\phi(\epsilon^o)\,\epsilon^{no}do$$

$$= \Sigma\left\{\frac{(lx)^m}{1.2\ldots m}D^m\int\phi(\epsilon^o)\,\epsilon^{no}do\right\} = \Sigma\left[\frac{(lx)^m}{1.2\ldots m}D^{m-1}\{\epsilon^{no}\phi(\epsilon^o)\}\right],$$

the same as before. To abridge, lx has been put for $\log x$. Now, giving to n any values whatever, and an infinity of different ones, multiplying the results by any constants, and taking the sum of all the products, we have

$$\int\phi(x)f(x)\,dx = \{\int f(x)\,x^D dx\}\,\phi(\epsilon^o).$$

It may be proved in the same manner, that Taylor's theorem may be multiplied by $f(x)$, and integrated; but the above includes the proof of it.

In (a) change $\phi(x)$ into $\phi\left(\dfrac{1}{x}\right)$, and we have $\phi\left(\dfrac{1}{x}\right) = x^D\phi(\epsilon^{-o})$.

Change in this last x into $\dfrac{1}{x}$, and there results

$$\phi(x) = x^{-D}\phi(\epsilon^{-o}) \ldots\ldots\ldots\ldots\ldots (b).$$

In (a) change x into $a-x$, and in (b) into $a+x$; these formulæ become

$$\phi(a-x) = (a-x)^{D}\phi(\epsilon^{0}), \quad \text{and} \quad \phi(a+x) = (a+x)^{-D}\phi(\epsilon^{-0});$$

whence we derive

$$\int_0^a \phi(a-x)\, x^{n-1}dx = \left\{\int_0^a (a-x)^{D} x^{n-1}dx\right\}\phi(\epsilon^{0})$$

$$= \frac{\Gamma(n)\,\Gamma(D+1)}{\Gamma(D+n+1)}\, a^{D+n}\phi(\epsilon^{0}) = \frac{\Gamma(n)\, a^{D+n}}{(D+1)(D+2)\ldots(D+n)}\,\phi(\epsilon^{0}).$$

But
$$\int\phi(a)\,da = \frac{a^{D+1}}{D+1}\,\phi(\epsilon^{0}),$$

$$\int^2\phi(a)\,da^2 = \frac{a^{D+2}}{(D+1)(D+2)}\,\phi(\epsilon^{0}),$$

$$\int^n\phi(a)\,da^n = \frac{a^{D+n}}{(D+1)(D+2)\ldots(D+n)}\,\phi(\epsilon^{0}).$$

Therefore $\displaystyle\int_0^a \phi(a-x)\, x^{n-1}dx = \Gamma(n)\int^n\phi(a)\,da^n,$

the second member to be integrated from $a=o$ to $a=a$, as the first vanishes when $a=o$.

$$\int_0^\infty \phi(a+x)\, x^{n-1}dx = \left\{\int_0^\infty (a+x)^{-D} x^{n-1}dx\right\}\phi(\epsilon^{-0})$$

$$= \frac{\Gamma(n)\,\Gamma(D-n)}{\Gamma(D)}\, a^{-D+n}\phi(\epsilon^{-0}) = \frac{\Gamma(n)\, a^{-D+n}}{(D-1)(D-2)\ldots(D-n)}\,\phi(\epsilon^{-0}).$$

$$\int\phi(a)\,da = -\frac{a^{-D+1}}{D-1}\,\phi(\epsilon^{-0}),$$

$$\int^2\phi(a)\,da^2 = (-1)^2\frac{a^{-D+2}}{(D-1)(D-2)}\,\phi(\epsilon^{-0}),$$

$$\int^n\phi(a)\,da^n = (-1)^n\frac{a^{-D+n}}{(D-1)(D-2)\ldots(D-n)}\,\phi(\epsilon^{-0}).$$

Therefore $\displaystyle\int_0^\infty \phi(a+x)\, x^{n-1}dx = (-1)^n\,\Gamma(n)\int^n\phi(a)\,da^n,$

the second member to be integrated from $a=A$ to $a=a$, A being the value of a which makes the first member vanish.

I cannot stop to comment on the several steps of these two examples, which are known,* and are only given here by

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* See vol. I. p. 114, First Series.

way of illustration. I now subjoin the integration of a differential equation, which may be more conveniently effected otherwise; but it may be well to shew that (*a*) may in some cases be thus applied.

Let
$$x^2\frac{d^2y}{dx^2} + mx\frac{dy}{dx} + ny = 0.$$

Make $y = \phi(x) = x^D\phi(\epsilon^\theta)$. With this value of y, the proposed equation becomes
$$x^D\left\{D^2 + (m-1)D + n\right\}\phi(\epsilon^\theta) = 0.$$
Or, as $\phi(\epsilon^\theta)$ is independent of x,
$$\left\{D^2 + (m-1)D + n\right\}\phi(\epsilon^\theta) = o,$$
the integral of which is $\phi(\epsilon^\theta) = A\epsilon^{b_1\theta} + B\epsilon^{b_2\theta}$, b_1, b_2 being the roots of $b^2 + (m-1)b + n = o$. Therefore
$$y = \phi(x) = Ax^{b_1} + Bx^{b_2}.$$

In (*a*) change $\phi(x)$ into $\phi(\epsilon^x)$; then $\phi(\epsilon^x) = x^D\phi(\epsilon^{\epsilon^0})$. Change in this ϵ^x into x, or x into $\log x = lx$; we have
$$\phi(x) = (lx)^D\phi(\epsilon^{\epsilon^0})\ \ldots\ldots\ldots\ldots\ (c).$$
This serves to develop $\phi(x)$ by the powers of $\log\log x = llx = l^2x$. We might apply it to integration. By a continued repetition of the same steps, we may find a formula to develop by the powers of l^nx.

We may treat $\phi(x) = (1+\Delta)^x\phi(o)$ in like manner; first changing $\phi(x)$ into $\phi(\epsilon^x)$, then x into lx. Thus we should find
$$\left.\begin{array}{l}\phi(x) = (1+\Delta)^{lx}\phi(\epsilon^0)\\ \phi(x) = (1+\Delta)^{llx}\phi(\epsilon^{\epsilon^0})\end{array}\right\}\ldots\ldots\ldots\ldots(d).$$
The following is proved by developing the second member.
$$\frac{d^n\phi(x)}{dx^n} = \phi(x+D)\,o^n\ldots\ldots\ldots\ldots(e).$$

This gives
$$\frac{d^n\phi(o)}{do^n} = \phi(D)\,o^n.$$

Therefore
$$\phi(x) = \phi(o) + \frac{x}{1}\frac{d\phi(o)}{do} + \frac{x^2}{1.2}\frac{d^2\phi(o)}{do^2} + \&c.$$
$$= \phi(D)\left\{1 + \frac{ox}{1} + \frac{o^2x^2}{1.2} + \&c.\right\}.$$

Or
$$\phi(x) = \phi(D)\,\epsilon^{ox}\ \ldots\ldots\ldots\ldots\ (f).$$

This might be thus investigated:
$$D\epsilon^{ox} = x\epsilon^{ox} = x,\quad D^2\epsilon^{ox} = x^2,\quad D^n\epsilon^{ox} = x^n.$$

Give to n an infinity of different values, which we may suppose to be either integer or fractional. Multiply each of the results by a constant, and take the sums of both members, which will be $\phi(D)\,\epsilon^{ox}$ and $\phi(x)$, and which will thus have a more general form. But if the simple operation D be performed without adding any correction, we have $D^{-n}\epsilon^{ox}=x^{-n}$; and thus negative exponents may be included, so that $\phi(x)$ may include all the forms with which we are acquainted, even log x.

As D is here separated from x, and operates only upon o; we may, according to long established principles, multiply (f) by a function of x, and integrate. This formula may supply the place of Abel's definite integral $\int \epsilon^{vx} f(v)\,dv$.

In (f) change x into $-x$, then $\phi(x)$ into $\phi(-x)$; we thus obtain

$$\phi(x)=\phi(-D)\,\epsilon^{-ox} \ldots\ldots\ldots\ldots (g).$$

By $f(r)=(1+\Delta)^r f(o)=E^r f(o)$, the last may be thus exhibited,

$$\phi(x)=E^r\phi(-D)\,\epsilon^{-rx} \ldots\ldots\ldots\ldots (h),$$

where D now denotes $\dfrac{d}{dr}$, and $E=1+\Delta$.

We will now give some examples of integration. In those which we shall select, we prefer employing the formula (h).

$$\int_0^\infty dv \cos uv\,\phi(v)=E^r\phi(-D)\int_0^\infty \epsilon^{-rv}dv\cos uv=E^r\phi(-D)\,\frac{r}{r^2+u^2}\,.$$

$$\int_0^\infty\!\!\int_0^\infty du\,dv\cos au\cos uv\,\phi(v)=E^r\phi(-D)\int_0^\infty \frac{r\,du\cos au}{r^2+u^2}$$

$$=\tfrac{1}{2}\pi\,E^r\phi(-D)\,\epsilon^{-ra}=\tfrac{1}{2}\pi\,\phi(a).$$

$$\int_0^\infty dv\sin uv\,\phi(v)=E^r\phi(-D)\int_0^\infty \epsilon^{-rv}dv\sin uv=E^r\phi(-D)\,\frac{u}{r^2+u^2}\,;$$

$$\int_0^\infty\!\!\int_0^\infty du\,dv\sin au\sin uv\,\phi(v)=E^r\phi(-D)\int_0^\infty \frac{u\,du\sin au}{r^2+u^2}$$

$$=\tfrac{1}{2}\pi\,E^r\phi(-D)\,\epsilon^{-ra}=\tfrac{1}{2}\pi\,\phi(a).$$

These two double integrals, which are known, give Fourier's theorem.

Reverting now to (f), change $\phi(x)$ into $\phi(\epsilon^x)$; and it becomes $\phi(\epsilon^x)=\phi(\epsilon^D)\,\epsilon^{ox}$. But $\epsilon^D=1+\Delta=E$. Therefore

$$\phi(\epsilon^x)=\phi(E)\,\epsilon^{ox} \ldots\ldots\ldots\ldots\ldots (i),$$

which is Herschel's theorem. Change x into $x\sqrt{(-1)}$, and into $-x\sqrt{(-1)}$, and add and subtract the results; we thus obtain

$$\tfrac{1}{2}\{\phi(\epsilon^{o\sqrt{(-1)}})+\phi(\epsilon^{-o\sqrt{(-1)}})\} = \phi(E)\cos ox,$$

$$\frac{1}{2\sqrt{(-1)}}\{\phi(\epsilon^{o\sqrt{(-1)}})-\phi(\epsilon^{-o\sqrt{(-1)}})\} = \phi(E)\sin ox,$$

$$\frac{1}{2}\int_0^\infty \frac{dx}{1+x^2}\{\phi(\epsilon^{x\sqrt{(-1)}})+\phi(\epsilon^{-x\sqrt{(-1)}})\} = \phi(E)\int_0^\infty \frac{dx\cos ox}{1+x^2}$$
$$= \tfrac{1}{2}\pi\,\phi(E)\,\epsilon^{-o} = \tfrac{1}{2}\pi\,\phi(\epsilon^{-1}),$$

$$\frac{1}{2\sqrt{(-1)}}\int_0^\infty \frac{x\,dx}{1+x^2}\{\phi(\epsilon^{x\sqrt{(-1)}})-\phi(\epsilon^{-x\sqrt{(-1)}})\} = \phi(E)\int_0^\infty \frac{x\,dx\sin ox}{1+x^2}$$
$$= \tfrac{1}{2}\pi\,\phi(E)\,\epsilon^{-o} = \tfrac{1}{2}\pi\,\phi(\epsilon^{-1})\,\ldots\ldots\ldots\ldots(k).$$

Make $\phi(\epsilon^{o\sqrt{(-1)}})+\phi(\epsilon^{-o\sqrt{(-1)}}) = \dfrac{\epsilon^{ao\sqrt{(-1)}}+\epsilon^{-ao\sqrt{(-1)}}}{\epsilon^{bo\sqrt{(-1)}}+\epsilon^{-bo\sqrt{(-1)}}} = \dfrac{\cos ax}{\cos bx}$

$$\phi(\epsilon^{o\sqrt{(-1)}})-\phi(\epsilon^{-o\sqrt{(-1)}}) = \frac{\epsilon^{ao\sqrt{(-1)}}-\epsilon^{-ao\sqrt{(-1)}}}{\epsilon^{bo\sqrt{(-1)}}+\epsilon^{-bo\sqrt{(-1)}}} = \sqrt{(-1)}\,\frac{\sin ax}{\cos bx}.$$

Here a must be less than b, or the second members would reduce to a different form. These functional equations solved give

$$\phi(\epsilon^{o\sqrt{(-1)}}) = \tfrac{1}{2}\,\frac{\epsilon^{ao\sqrt{(-1)}}+\epsilon^{-ao\sqrt{(-1)}}}{\epsilon^{bo\sqrt{(-1)}}+\epsilon^{-bo\sqrt{(-1)}}}$$

for the first, and

$$\phi(\epsilon^{o\sqrt{(-1)}}) = \tfrac{1}{2}\,\frac{\epsilon^{ao\sqrt{(-1)}}-\epsilon^{-ao\sqrt{(-1)}}}{\epsilon^{bo\sqrt{(-1)}}+\epsilon^{-bo\sqrt{(-1)}}}$$

for the second. With these values of $\phi(\epsilon^{o\sqrt{(-1)}})$, (k) give immediately

$$\int_0^\infty \frac{dx}{1+x^2}\,\frac{\cos ax}{\cos bx} = \tfrac{1}{2}\pi\,\frac{\epsilon^a+\epsilon^{-a}}{\epsilon^b+\epsilon^{-b}}\qquad \int_0^\infty \frac{x\,dx}{1+x^2}\,\frac{\sin ax}{\cos bx} = -\tfrac{1}{2}\pi\,\frac{\epsilon^a-\epsilon^{-a}}{\epsilon^b+\epsilon^{-b}}.$$

In (i) change ϵ^o into x, and it gives

$$\phi(x) = \phi(E)\,x^o\,\ldots\ldots\ldots\ldots\ldots(l),$$

which is Hamilton's theorem.

Change in (l) $\phi(x)$ into $\phi(\epsilon^x)$, and then x into $\log x$; there results
$$\phi(x) = \phi(\epsilon^E)\,(lx)^o\,\ldots\ldots\ldots\ldots(m).$$

By a repetition of the same steps, we might add more theorems; and let it be remembered that we can always replace o by r, if it appear desirable to do so.

Let $\phi(a,x) = \Sigma(A_n x^n)$; then $\left(x\dfrac{d}{dx}\right)^r\phi(a,x) = \Sigma(n^r A_n x^n)$,

and consequently $\psi\left(x\dfrac{d}{dx}\right)\phi(a,x) = \Sigma\{\psi(n)\,A_n x^n\}$. Make

$\psi(n) = \left(\frac{d}{da}\right)^n$, then $\left(\frac{d}{da}\right)^{x\frac{d}{dx}} \phi(a, x) = \Sigma\left\{\left(\frac{d}{da}\right)^n A_n x^n\right\}$. Suppose, now, $\phi(a, x) = \phi(a)\, \epsilon^x$, and therefore $A_n = \frac{\phi(a)}{1.2\ldots n}$; the last becomes

$$\left(\frac{d}{da}\right)^{x\frac{d}{dx}} \{\phi(a)\, \epsilon^x\} = \Sigma\left(\frac{d^n\phi(a)}{da^n}\ \frac{x^n}{1.2\ldots n}\right) = \phi(a + x).$$

Or $$\phi(a + x) = \left(\frac{d}{da}\right)^{x\frac{d}{dx}} \{\phi(a)\, \epsilon^x\} \ldots\ldots\ldots (n).$$

This appears a mere curiosity; but we do not know what may prove useful. The preceding theorems will of course apply, if there be more variables than one. Thus we shall have

$$\phi(x, y) = x^D y^{D'} \phi(\epsilon^o, \epsilon^{o'}),\quad \phi(x, y) = \phi(D, D')\, \epsilon^{ox + o'y},$$

$\phi(x, y) = \phi(E, E')\, x^o y^{o'}$, &c.; and similarly for more variables. It must be observed, that D and E operate upon o, D' and E' upon o'.

Perhaps the following, derived from (f), may not be utterly unworthy of notice.

$$\left.\begin{aligned} &\frac{d^n\phi(x)}{dx^n} = \phi(D)\, o^n \epsilon^{ox},\ \int^n \phi(x)\, dx^n = \phi(D)\, o^{-n}\, \epsilon^{ox}, \\ &\Delta^n\phi(x) = \phi(D)\, (\epsilon^o - 1)^n\, \epsilon^{ox},\ \Sigma^n\phi(x) = \phi(D)(\epsilon^o - 1)^{-n}\, \epsilon^{ox}, \end{aligned}\right\} \ldots (o),$$

where $\Delta x = 1$. We must not be startled at such quantities as 0^{-n}; for by means of the arbitraries of integration, all terms containing such quantities may be made to disappear.

In the development of such theorems as those which have been investigated, formulæ to facilitate reduction, similar to those which Sir John Herschel has given in his Examples, might be of use. But it would take up too much room to enter upon the subject here. We will only observe, that the theorems themselves will supply such formulæ, by giving particular values to x, particular forms to the function ϕ, and by comparing the coefficients of the same term, given by different developments of the function. And we may also change the function ϕ and the variable x, as we have done in this paper; and thus may multiply formulæ.

For example, by Taylor's theorem,

$$\epsilon^{xD}\, o^n = x^n,\quad \epsilon^{2xD}\, o^n = 2^n x^n,\quad \&c.$$

Also $\qquad \epsilon^{D} o^{n} = 1^{n}, \quad \epsilon^{2D} o^{n} = 2^{n}, \quad$ &c.

Therefore $\qquad \epsilon^{xD} o^{n} = x^{n} \epsilon^{D} o^{n}, \quad \epsilon^{2xD} o^{n} = x^{n} \epsilon^{2D} o^{n}, \quad$ &c.

And hence also $\qquad f(\epsilon^{xD})\, o^{n} = x^{n} f(\epsilon^{D})\, o^{n}.$

Or $\qquad f\{(1+\Delta)^{x}\}\, o^{n} = x^{n} f(1+\Delta)\, o^{n}.$

Or $\qquad f(xD)\, o^{n} = x^{n} f(D)\, o^{n}.$

If we expand (a) by the powers of lx, and if we change x in (l) into ϵ^{lx}; and expand that in like manner, and then compare like terms; we shall find

$$D^{n}\, \phi(\epsilon^{t}) = \phi(1+\Delta)\, o^{n}.$$

We have treated only of the general form $\phi\,(x)$; but it is in certain particular functions, that the symbols of operation D and Δ give those simple expressions of them, which afford such easy and elegant means of integration. And here too we can sometimes employ other and more complex symbols than D and Δ with great effect.

Gunthwaite Hall, near Barnsley, Dec. 11, 1846.

ON PRINCIPAL AXES OF A BODY, THEIR MOMENTS OF INERTIA, AND DISTRIBUTION IN SPACE.

By Richard Townsend.

(*Continued from* p. 42.)

27. Any or all of the above constructions for principal axes (17, 18, 26) verify the anticipations of (2), shewing immediately, that an axis taken at random in a body may not be a principal axis at all; those axes alone being principal which are normals to surfaces of the second order confocal with the ellipsoid of gyration; and their principal points, or the points for which they are principal, being the points on these surfaces at which they are normals, and their corresponding principal planes being of course the tangent planes at those points to the same surfaces.

In distinguishing between axes in a given body, we have therefore to determine respecting every given axis, 1st, whether it be principal or not; and, 2nd, if it be, where will be its principal point, or points, if it have more than one.

Towards this we have the well-known theorem:

The normal and tangent plane at every point of any surface of the second order will meet each of its three principal planes in a point and line, which will always be pole and polar to each other with respect to the focal conic in that plane.

Hence, to find whether an assumed axis is principal or not, draw any plane perpendicular to that axis, and produce both axis and plane to meet a principal plane of the ellipsoid of gyration. If then the line of meeting of the plane be parallel to the polar of the point of meeting of the axis with respect to the focal conic in that principal plane, the axis will be principal, but otherwise it will not.

To find the principal point on the axis when the necessary condition is fulfilled, we have but to draw through that polar a plane perpendicular to the axis, that plane will meet it at its principal point, and for that point will be itself a principal plane.

(As the point on a principal axis for which it is principal has been called the principal point of that axis, so may the plane which at the principal point of a principal axis intersects it at right angles, as containing the other two accompanying principal axes, be called the principal plane of that axis.)

From the same theorem, it appears that in every plane drawn at will in a body, there exists a point for which one principal axis is perpendicular to that plane; that point in every plane may be called its principal point, since for it the plane, as containing two of its principal axes, is a principal plane.

To find that point in a given plane, from the pole of the right line in which it intersects a principal plane of the ellipsoid of gyration with respect to the focal conic in that plane, let a perpendicular be dropped on the given plane; that perpendicular will meet it at its principal point, and for that point will be itself a principal axis.

(As the plane perpendicular to a principal axis at its principal point may be called the principal plane of that axis, so may the axis perpendicular to a principal plane at its principal point be called the principal axis of that plane.)

Using for convenience these definitions, it appears at once from the above that, *every* plane has a corresponding principal axis, and that, every *principal* axis has a corresponding principal plane.

28. The above general test for principal axes, and general construction for their principal points, may be applied, and of course hold, in all particular cases: but there are some very important particular cases of principal axes which, being of frequent occurrence, should be familiarly known

without requiring the application of either; nor indeed is such application in their case necessary, as they appear more readily from the following simpler considerations.

If from any vertex situated on one of its axes a cone envelope any surface of the second order, then will one axis of that cone always coincide with that axis, and its remaining axes will be always parallel to the remaining axes of the surface.

Hence, in a body, a central principal axis is principal at every point along its whole length, and at all its points the other principal axes remain always parallel to each other and to the other central principal axes.

The three infinitely distant right lines in which the three principal planes of any cone of the second order, or of any hyperboloid of one or of two sheets, intersect the plane infinitely, will always be normals, both to that surface itself and also to its whole system of confocal surfaces, and will each be met by every different surface of that system obviously at a different point.

Hence, in a body, the three infinitely distant right lines in which the three central principal planes intersect the plane infinity, are also three other principal axes which are always principal at every point along their whole length.

The three central principal axes and these three particular infinitely distant axes are in this respect unique; they alone possess the property of being principal at *every* point, while all other principal axes are principal for but a single point.

Every line passing through the centre of a sphere is always a normal to that surface, and every system of surfaces of the second order confocal with an ellipsoid contains always a concentric sphere of infinite radius. Hence,

All axes passing through the centre of gravity of a body are principal axes, and the points for which they are principal are situated all at an infinite distance.

That system of principal axes alone possesses this property; all other principal axes which are not themselves infinitely distant having their principal points at a finite distance.

Moreover, since they all pierce the sphere of infinite radius perpendicularly at their principal points, it follows that the plane infinity is principal *at every point.*

Every line perpendicular to a principal plane of a surface of the second order is always a normal to the infinitely flat confocal surface, which is bounded by the focal conic in that plane. Hence, in a body,

At every point of a central principal plane one principal axis is always perpendicular to that plane, so that all axes parallel to a central principal axis are principal, and also, a central principal plane is principal *at every point.*

The three central principal planes and the plane infinity are in this respect unique; they alone possessing the property of being principal at every point, and every other plane having but a single point for which it is principal.

The poles of all diameters to a conic are always infinitely distant from its centre. Hence, in a body,

The principal points of all planes passing through the centre of gravity are situated all at an infinite distance.

That system of planes alone possesses this property; all other planes which are not themselves infinitely distant having their principal points at a finite distance.

Every line lying in the plane of a conic is a normal either to it or to some confocal conic; and to find the point for which it is a normal, drop upon it a perpendicular from its pole with respect to the given conic, the line and perpendicular will then be the normals to the two confocal conics which pass through their points of intersection. Hence, in a body,

All axes which lie in a central principal plane are principal axes; and to find the points for which they are principal, from the pole of each individual axis with respect to the focal conic in that plane of the ellipsoid of gyration, let a perpendicular be dropped on that axis; the point of meeting will then be the principal point, and the perpendicular will be the accompanying principal axis.

The same is true of the plane infinity, all axes which lie in that plane being also principal; for, the asymptotic cones to the system of hyperboloids confocal with the ellipsoid of gyration intersect at right angles the plane infinity in a system of confocal conics; and to some one of these, and therefore to the particular surface on which it lies, every line taken at will in that plane is a normal at some point or other, and therefore a principal axis of the body.

The common foci of this system of conics are the points where the assymptotes of the focal hyperbola to the ellipsoid of gyration pierce the plane infinity. Hence at once a construction for finding the principal point of an infinitely distant axis, the same exactly as determines it for an axis lying in a central principal plane; and hence also the construction established on other principles in Art (26) for finding the principal axes at an infinitely distant point.

Join that point with the centre of gravity, the joining line will be one principal axis; and to find the other two, draw two planes passing through that line and the asymptotes of the focal hyperbola, and bisect the two supplemental angles, acute and obtuse, between the lines in which they meet the plane infinity, the bisecting lines will be then the two principal axes sought.

At every point in a central principal plane the three principal axes may also, and for the same reason (26), be found by a construction quite elementary; at the point erect a perpendicular to the plane, it will be one principal axis, and to find the other two, connect the point with the two foci of the focal conic in that plane of the ellipsoid of gyration, and bisect the supplemental angles, acute and obtuse, between the connecting lines, the two bisectors will be then the remaining principal axes sought.

The three central principal planes and the plane infinity alone possess the above property; for no other plane are *all* axes which lie therein principal, though (as shall presently appear) there exists in every plane *an infinite number* of principal axes which are always distributed therein according to a simple and very elegant law.

The principal results contained in this article may be briefly summed up, by saying, that in every body the three central principal planes and the plane infinity possess always the remarkable and unique properties—that every axis perpendicular to, and also every axis lying in, any one of these four planes is always principal; that they are always themselves principal at *every* point; that all axes passing through any one of their four points of intersection are always principal; and that their six lines of intersection, all principal axes, possess always the important and exclusive property of being principal for *every point* along their whole length.

29. The general test (27) for ascertaining whether an axis be ordinary or principal leads at once to the following, which often serves the same purpose more easily, and also leads itself to some very general geometrical properties of principal axes.

The right line in which any two planes taken at random in a body intersect will be a principal axis if the principal axes of those planes intersect, but otherwise it will not; and when they do intersect, their plane will be the principal plane corresponding to that principal axis.

For producing the planes with their principal axes to meet one of the principal planes of the ellipsoid of gyration, the points of meeting of the axes will be the poles with respect to the focal conic in that plane of the traces of the planes, and therefore the intersecting point of these traces will be the pole of the line joining the feet of the axes: if then the axes intersect, a plane passing through this line and containing them both, will be perpendicular to the intersection of the planes, which intersection will therefore (27) be a principal axis at the point of meeting; but if the axes do not intersect, no plane passing through that line can be parallel to them both, that is, perpendicular to the intersection of the planes; in that case, therefore, that intersection (27) will not be a principal axis.

Conversely, (for the same reason) if any two planes be drawn at will through a principal axis, the corresponding principal axes will intersect, and their plane will be the principal plane of the axis; but of no two planes drawn through an axis which is not principal, will the principal axes ever intersect.

Now, through an axis, of which we do not know whether it is principal or not, we may often be able to draw two planes so conveniently as to be able to determine immediately their principal axes: if we can do this, the nature of the axis will by means of the above be immediately determined. Instances will appear as we proceed.

Hence, also, we immediately deduce the following consequences:—

If a system of planes pass all through a *principal* axis, their corresponding system of principal axes will lie *all* in the same plane, viz. the principal plane of the axis.

But if a system of planes pass all through an axis which is not principal, then will no two of their whole corresponding system of principal axes lie in the same plane.

We shall presently see that in the latter and obviously more general case, the system of axes will always generate an hyperbolic paraboloid, and that in the former particular case they will always envelope a parabola; in fact, the *gauche* surface, which alone in the general case could be generated, degenerates in the particular case into a developable, and the limit to an hyperbolic paraboloid, when it flattens down into a developable surface, is obviously a portion of a plane bounded by a parabola, the generatrices of that limiting surface being the tangents to that curve.

of the latter is the polar reciprocal with respect to that conic of the curve locus of the former.

This property, which is but a particular case of one more general, will be often found very useful, for in several remarkable cases one of these curves in each principal plane is easily seen to be a conic, and in all such cases we see from this that the other also must be of the second order.

Suppose, for instance, that a system of planes generate by their successive intersections a developable surface circumscribing the ellipsoid of gyration, or any one of its confocal system, along its curve of intersection with any concentric and coaxal surface of the second order, or more generally along any curve whose orthographic projections on the three central principal planes are conics, then will the surface generated by their corresponding system of principal axes intersect the same three planes also in conics; for in that case, the curves of intersection of such a developable being polars reciprocal with respect to the principal sections of the surface enveloped to the orthographic projections of the curve of contact on the same respectively, are of the second order, and therefore so are also *their* polars reciprocal with respect to the focal conics in the same planes.

Hence, again, if (as often happens) a system of planes determined by some law envelope a cone of the second order, as, for instance, if such a cone be taken arbitrarily in the body, and that the system of planes be its system of tangents, then will the surface generated by the corresponding system of principal axes intersect the three central principal planes in conics; these three conics will be all parabolas, if the cone pass through the centre of gravity; and if its vertex lie in either of the three central principal planes, the conic in that plane will dwindle into a finite portion of a right line, for the particular conic in which the cone intersects that plane will be in this case two right lines real or imaginary, and the polar reciprocal with respect to any curve of the second order of such a conic is always infinitely flat, the portion viz. of a right line bounded by the poles of the two lines; and if moreover the cone touch that plane, then will the corresponding conic dwindle into a point, the pole with respect to the focal conic of the side of contact.

And, conversely, if a system of principal axes generate
e of the second order, then will the developable
velope of the corresponding system of principal
ect each central principal plane in a conic;

In the present case also the principal point of every tangent plane to either of the two developables is on its edge of contact, hence (29) the two surfaces intersect in a line of curvature common to both; which is also evident *à priori*, since, if any surface whatsoever envelope another along a line of curvature, the curve of contact will be obviously a line of curvature also of the enveloping surface.

Hence, also, (29) if that curve be made the arête de rebroussement of a third developable surface, all its edges, that is all the tangents to the curve, will be principal axes, principal at their points of contact, which in the present case is also evident *à priori*, for the tangent at every point on the curve of intersection of any two of the surfaces confocal with the ellipsoid of gyration is the normal to the third confocal surface which passes through that point; the whole system of tangents all round that curve is therefore a system of principal axes, principal at their points of contact.

The developable surface generated by the system of tangents all round a line of curvature of any one of the surfaces confocal with the ellipsoid of gyration, we shall have occasion to notice again, for all its edges are not only principal but also equimomental axes, and the whole system of such, subdivisible in various ways into an infinite number of smaller systems according to different arbitrary laws of division, possesses many curious and interesting properties.

The fourth developable (29) remains still to be noticed: if, therefore, we take any line of curvature on any one of the whole system of surfaces confocal with the ellipsoid of gyration, the system of normal planes to that curve will generate by their successive intersections a developable surface, all whose edges will be principal axes.

31. *Every* ruled surface, whether gauche or developable, which is generated by a system of principal axes, is connected with the corresponding developable surface envelope of the corresponding system of principal planes by the following relation; from which if either be given, the other may be readily determined.

Their curves of intersection with each principal plane of the ellipsoid of gyration are always polars reciprocal to each other with respect to the focal conic in that plane.

For, the points in which the different principal axes pierce that plane are the poles with respect to the focal conic therein of the lines in which the corresponding principal planes intersect the same; the curve, therefore, envelope

of the latter is the polar reciprocal with respect to that conic of the curve locus of the former.

This property, which is but a particular case of one more general, will be often found very useful, for in several remarkable cases one of these curves in each principal plane is easily seen to be a conic, and in all such cases we see from this that the other also must be of the second order.

Suppose, for instance, that a system of planes generate by their successive intersections a developable surface circumscribing the ellipsoid of gyration, or any one of its confocal system, along its curve of intersection with any concentric and coaxal surface of the second order, or more generally along any curve whose orthographic projections on the three central principal planes are conics, then will the surface generated by their corresponding system of principal axes intersect the same three planes also in conics; for in that case, the curves of intersection of such a developable being polars reciprocal with respect to the principal sections of the surface enveloped to the orthographic projections of the curve of contact on the same respectively, are of the second order, and therefore so are also *their* polars reciprocal with respect to the focal conics in the same planes.

Hence, again, if (as often happens) a system of planes determined by some law envelope a cone of the second order, as, for instance, if such a cone be taken arbitrarily in the body, and that the system of planes be its system of tangents, then will the surface generated by the corresponding system of principal axes intersect the three central principal planes in conics; these three conics will be all parabolas, if the cone pass through the centre of gravity; and if its vertex lie in either of the three central principal planes, the conic in that plane will dwindle into a finite portion of a right line, for the particular conic in which the cone intersects that plane will be in this case two right lines real or imaginary, and the polar reciprocal with respect to any curve of the second order of such a conic is always infinitely flat, the portion viz. of a right line bounded by the poles of the two lines; and if moreover the cone touch that plane, then will the corresponding conic dwindle into a point, the pole with respect to the focal conic of the side of contact.

And, conversely, if a system of principal axes generate a surface of the second order, then will the developable surface envelope of the corresponding system of principal planes intersect each central principal plane in a conic;

these three conics, as above, will be all parabolas if the generated surface pass through the centre of gravity. And if (as not unfrequently happens in consequence of every axis which lies in either of the three planes being (28) principal) it touch any of the central principal planes, that is if it intersect it in two right lines, then apparently will the conic in which the corresponding developable intersects that plane be infinitely flat, the right line viz. which passes through the poles of the two lines with respect to the focal conic: in this case, however, since one of the two lines must be one of the generating axes, the whole right line is due to *its* principal plane, and the conic is therefore properly but a point, the pole of the other line.

If the surface of the second order generated by the system of principal axes touch the three central principal planes, then will the three conics in which the developable envelope of the corresponding system of principal planes intersects those planes, all dwindle into points; that developable therefore in this case will be an infinitely slender cylinder. Hence we see that if a surface of the second order generated by a system of principal axes touch any two of the central principal planes, it must also touch the third, and that if a system of planes pass all through the same right line, that is, if they all touch an infinitely slender cylinder of the second order, then will the surface generated by the corresponding system of principal axes touch the three central principal planes; that surface will in fact be a paraboloid of the second order. But of this more hereafter.

In general, if the developable surface envelope of any system of planes be an infinitely flat cone or cylinder of any order, and therefore intersect the central principal planes in finite portions of a right line bounded by points, then will the surface generated by the corresponding system of principal axes touch those three planes, for it intersects them in right lines, the poles of the above points, and every plane which passes through a right line on a gauche surface is a tangent plane to that surface at some point or other.

32. As an example illustrative of the preceding article, Let a system of planes touch the ellipsoid of gyration or any one of its confocal surfaces along any plane section; their developable envelope will be of course a cone of the second order, intersecting the three central principal planes in conics, and the corresponding system of principal axes, that is the system of normals to the surface along the plane curve,

will generate a gauche surface of the fourth order, which (31) will therefore intersect the same planes also in conics, the remaining portion of each curve of the fourth order consisting of two right lines real or imaginary, the particular pair of generating principal axes normals to the surface of the second order at the two points, real or imaginary, where the plane of the curve of contact intersects each principal section of that surface.

Let the plane of the section pass through the centre of gravity; then will the cone become a cylinder, and its intersection with each central principal plane a conic, concentric with the principal section on that plane. Hence we see that every system of principal axes normals to any one of the surfaces confocal with the ellipsoid of gyration along any central section of that surface, will intersect each central principal plane in a conic whose centre will be the centre of gravity,* the remaining portion of each curve of the fourth order, in which the surface generated by that system of axes intersects these planes, being two real and parallel right lines, the pairs of normals viz. to the surface of the second order at the diametrically opposite pairs of points, in which the plane of the central section intersects each central principal section.

Let now the particular surface be one of the confocal hyperboloids of one sheet, and let the plane of the central section be one of the system of tangent planes to its asymptotic cone. In this case the section will be two right lines parallel to each other and to the side of contact; and since all planes passing through either of these are tangent planes to the surface, the cylinder enveloping it along that section

* The same principle enables us to find immediately the curve in which the surface locus of its centres of curvature intersects each principal plane of any surface of the second order: that curve consists of two parts, corresponding to the two sheets of the surface of centres; of these, one in each plane is obviously the evolute of the principal section in that plane, and to find the other we have but to find the locus of the ultimate intersections with that plane of a system of normals drawn infinitely near to it, which is immediately done by means of the principle in question; for the envelope of the intersections with the same plane of the corresponding system of tangent planes to the surface is ultimately the principal section therein, and of this envelope the locus required is the polar reciprocal with respect to the focal conic in that central plane. Hence in every surface of the second order, the curves of intersection with each principal plane of its surface of centres are the evolute of the principal section in that plane, and, a conic concentric and coaxal with, and having its semiaxes, thirds proportional to those of the focal and principal conics in the same principal plane.

will be infinitely flat, being either portion indifferently of the central plane divided into two regions bounded by the two parallel lines. Such being always the limiting and transition state between an elliptic and an hyperbolic cylinder, and the system of tangent planes to its asymptotic cone determining on every hyperboloid of one sheet the bounding system of central sections, on one side of which the enveloping cylinders to that surface are all elliptic, and on the other side of which they are all hyperbolic, the particular class of parabolic enveloping cylinders being confined to the two paraboloids.

Since, here, the system of planes touching the hyperboloid pass all through two parallel right lines, equidistant from and in a plane passing through the centre of gravity, it follows that the developable surface, their envelope, intersects each central principal plane in two points equidistant and in opposite directions from that centre, and therefore the gauche surface generated by the corresponding system of principal axes will (31) touch these three planes, and besides intersecting them each in the two normals to the hyperboloid at the two diametrically opposite points in that principal plane, will also intersect them in two parallel right lines, the polars of these points with respect to the focal conic.

The complete curve of the fourth order in which the surface generated by the normals intersects each central principal plane, consists therefore in this case of four real right lines, in pairs parallel to each other and similarly situated in opposite directions from the centre of gravity, one pair of these parallel lines being generating axes, and the other two forming the polar reciprocal with respect to the focal conic of the particular curve, in which the developable envelope of the corresponding system of principal planes intersects that central plane; a result confirmatory of the concluding remarks in (31).

That such should be the nature of the intersecting curves we might easily have seen *à priori*, for in the particular case in question the gauche surface of the fourth order, generated by the system of principal axes, breaks up into two paraboloids of the second order, equal, similar, and similarly placed, but in opposite directions from the centre of gravity, one corresponding to one of the two parallel generatrices of the hyperboloid of one sheet, and the other to the second; they both touch the three central principal planes, and therefore intersect them each in two right lines, the analogous lines for each being of course parallel to each other.

For, taking arbitrarily any rectilinear generatrix of any one of the confocal hyperboloids of one sheet, the system of principal axes normals along it to the surface will generate an hyperbolic paraboloid, for (27) they pass all through the three polars with respect to the focal conics of the points, whence the assumed generatrix pierces the three central principal planes, and therefore generate a surface of the second order. Again, they are all perpendicular to that generatrix, that is, all parallel to the same plane, and therefore the generated surface is a paraboloid.

This paraboloid intersecting each central principal plane in a right line, of course intersects it in another, and therefore touches it; that other is the particular generating principal axis which lies in the principal plane, that is, the normal to the principal section of the hyperboloid in that plane at the point where the generatrix meets it; the intersection therefore of that normal with the polar of the same point with respect to the focal conic, is the point of contact of the paraboloid. Also, since one of its lines of intersection with each plane is one of the generating principal axes of that surface, the corresponding system of principal planes will intersect that plane in a system of lines whose envelope ought (31) to be an evanescent conic, the pole of the other line of intersection with respect to the focal conic; and such, since they all pass through a line, it is in the present instance. But we forbear at present to consider this surface any further, as it will be fully and more generally discussed in a subsequent article.

33. Having spoken of curves plane or of double curvature enveloped by principal axes, and of rule surfaces gauche or developable generated by systems of principal axes, it may, before we proceed, be satisfactory to examine briefly the possibility of either taking place, and the conditions which are necessary to be fulfilled in such cases.

Now a principal axis being a normal to one of a system of surfaces which (VIII.) contain in their equation a variable parameter δ, its equations contain in their parameters four variable quantities, viz. xyz, and δ where xyz are the coordinates of the point on the surface δ at which it is a normal; and these four quantities are connected by but a single equation, that of the surface, so that of the four parameters in the equations of a principal axis three are absolutely independent.

In order therefore that a system of principal axes should generate a ruled surface of either species, it is necessary that

they be restricted by two conditions; and for the same reason, conversely, if a system of principal axes be restricted to fulfil any two independent conditions, the axes of that system will generate a surface gauche or developable as the case may be.

In the latter case they will obviously be all tangents to a curve, the arête de rebroussement of the generated surface; in order therefore that a system of principal axes should envelope a curve, it is necessary that the axes of the system be restricted by two conditions: but, conversely, it does not always follow that a system so restricted will envelope a curve, unless that the given conditions be such as to involve in their very nature the additional circumstance that the surface generated by the system of axes conformable to them must be developable. But in general this is a result purely accidental, and arising in particular cases from some coincidence between the conditions of a nature altogether casual; since, when the principal axes of a system are restrained by two arbitrary and independent conditions, the surface they generate is of course completely fixed, and may or may not be developable as the case may be.

Suppose, for instance, that a system of principal axes be restrained to lie all in a given plane; here being restricted by two conditions, they will envelope a curve in that plane, the arête de rebroussement of the developable surface, which from the very nature of the conditions is in this case necessarily described. Or suppose that they be all required to pass through a given point, here also being restrained by two conditions, they will generate a surface, which from the nature of the conditions is in this case also developable. But let the conditions be, that they all pass through two given curves, plane or of double curvature, or that they all touch two given surfaces, or that they all touch a given surface along a given curve, then is it plain that the fact of the resulting surface being developable would be purely accidental, and arise from a coincidence peculiar to the particular case proposed.

If one of the two conditions be that every consecutive pair of the axes should lie in the same plane, and therefore intersect, then certainly would the resulting surface generated by the system of axes be necessarily developable, whatever might be the other condition: but then the question would be indeterminate, and a given single condition combined with the first could only determine the nature of the developable, but never in any case the particular surface itself. For every system of principal axes which are restricted to

fulfil but a single condition may always be divided into a multitude of smaller systems, each forming a developable surface, and consequently the problem, to find the developable surface generated by a system of principal axes which fulfil one given condition, is essentially indeterminate.

Suppose, for instance, it were required to find the developable of principal axes which should pass through a given curve, the problem would be indeterminate: for, from each point of the curve there diverges a cone of principal axes, and we might select at random a side of one of these cones as an edge of our developable; then take the infinitely near side of the consecutive cone which intersects that as the second edge, and so on, and thus get an indefinite number of developable surfaces, three being an indefinite number of sides to a cone. The cone, which from the centre of gravity as vertex passes through the given curve, fulfils the conditions (since all axes passing through the centre of gravity are (28) principal), and is therefore one of these developables; and so are also the three cylinders which orthographically project the curve on the central principal planes, since all axes perpendicular to any of those planes are principal (28). The same exactly might be said if it were required to circumscribe a given surface with a developable of principal axes. Or suppose that it were required to find a developable of axes in a body which should be all both principal and equimomental, then (30) should we have a multitude of developables fulfilling the conditions, of which the arêtes de rebroussement would be lines of curvature on the sufaces of the system confocal with the ellipsoid of gyration. In this last instance the developables present themselves to our consideration immediately and naturally, and their arêtes de rebroussement generate, by the continuity of their change of position from surface to surface, a very remarkable and familiar surface, which will form the subject of a subsequent article. (See Art. 15, p. 25).

34. Let us now see what takes place when we have a system of principal axes restricted by but a single condition, or, algebraically speaking, whose parameters are connected by but a single equation.

In such a case, though they will envelope, they will not of course generate a surface, but the whole system comprising them may be divided in a multitude of different ways into an infinite number of smaller systems, each of which will form a continuous surface. For, we may arbi-

trarily *introduce* an additional and independent condition, or, which is the same thing, we may connect the parameters by a second and perfectly arbitrary equation, and thus detach from the system a group which fulfilling two conditions will generate a surface. We may then, in the introduced equation, cause an arbitrary constant to vary, and thus obtain a different group forming a new surface of the same species with the former; and finally, we may give to that constant all possible values in continued succession, and thus have the whole original system of axes divided into a multitude of groups forming a system of surfaces all of the same species. Again, this division may obviously be performed in an infinite number of different ways, for the number of equations containing each an arbitrary constant, which might be introduced between the parameters, is of course infinite.

Examples of this division of systems subject to but a single restriction will appear as we proceed, and also instances illustrative of the advantages which may be derived from the power we have of, 1st, Selecting in most cases whatever law of division we may find convenient; 2nd, Of sometimes changing that law when occasion may seem to require it; and, 3rd, Of considering the same given system as made up of two or more different and distinct groups of smaller systems according to different and arbitrary laws of division.

In every case of the division of a system of principal axes restricted by but one condition, the surfaces formed by the smaller systems, as containing each but a single parameter variable from one to the other, will admit of an envelope, this will obviously be the surface generated by their successive characteristics or the curves in which they intersect, two and two consecutively; and to find the equation of that surface, we have but to proceed in the usual manner, setting out from the equation containing but one parameter which expresses the system of surfaces enveloped.

Now, though there exists an infinite number of ways in which the division of a system of surfaces may be performed, and therefore an infinite number of groups of surfaces enveloped, still for all of them the envelope will be the same, but the circumstances of its determination will be considerably different in the different cases; these will readily appear from the following considerations.

Whenever we have a system of right lines which are restrained by any two independent conditions, or, which is the same thing, when their parameters are connected by two independent equations, that system will of course envelope

a surface, which surface is fixed and implicitly determined when the conditions are given. Hence, a system of principal axes which are restricted by but a single condition will always envelope a surface, which will, for the same reason, be fixed and implicitly determined when the condition is given.

This envelope, moreover, is obviously the same as that of any system of surfaces whatsoever into which we may divide the system of axes; hence, by whatever arbitrary law we may divide a given system of principal axes restricted by but one condition into an infinite number of smaller systems, each forming a surface, the resulting system of surfaces will have invariably the same envelope, the surface viz. which touches the whole given system of axes.

To find that surface when the system of axes is given, we have therefore but to introduce a condition, and having thus divided the system (as stated above) into an infinite number of surfaces, proceed from the equation, which containing one variable parameter expresses that system of surfaces, to determine their envelope in the usual manner.

But for every different introduced condition we have a different equation, and it is obvious that on the form of that equation depends the facility, and perhaps the possibility, of determining the equation of the envelope; we must therefore endeavour to find among all the different systems of surfaces into which the original system of axes is divisible, that particular system which has the simplest equation.

This would no doubt be often difficult, and no rule can perhaps be given which will hold in all cases. But, if we may conjecture any thing from the known general laws of envelopes, the most manageable form of the equation expressing the system of surfaces enveloped will in most instances correspond to the case where one of these surfaces, and therefore their whole system, is developable.

To find, therefore, the surface enveloped by a given system of principal axes, restrained by but a single condition, we must first divide the whole system into an infinite number of groups, each forming a developable surface, and then, having thrown into its simplest form the equation expressing that system of developables, proceed in the usual way to determine their envelope.

This particular way of performing the division of the system of axes possesses moreover the important advantage of enabling us (as we shall just now see) to form a tolerably

clear conception of the nature of the envelope itself, and at the same time it also leads indirectly to the result, that whenever there exists one way of forming from the axes a system of developable surfaces, then will there always exist a second and entirely different way of performing a similar division; and that, hence, generally there always exist two, and not more than two, different and distinct systems of developable surfaces, into either of which a given system of principal axes restricted by a single condition may be always divided.

Now, when the nature of the condition is such that the axes admit at all of a real envelope, the division of the system into at least one system of developable surfaces is always possible. For in all such cases, introducing at random any other condition whatever, we shall then, by the variation of the constant in the equation expressing the introduced condition, have the whole system divided into a series of rule surfaces of some sort or other, generally not developable; these surfaces, like every other system of consecutive surfaces which admit of a real envelope, will intersect two and two consecutively in a system of real curves, and through every point on the curve, in which any one of them taken arbitrarily from the whole system is intersected by the consecutive surface, there will pass a generatrix of each of these two surfaces, that is, a consecutive pair of the original system of axes will there intersect each other. Again, of these two generatrices at each point of this curve, one will always meet the consecutive curve which lies on its own surface, and through the point of meeting there will pass also a generatrix of the third consecutive surface, that is, a third consecutive axis of the given system will there meet the second. This will again meet the third consecutive curve, and through the point of meeting there will pass a fourth consecutive axis of the given system; and so on, at curve after curve, the same thing will take place successively, until the whole series of curves will be all exhausted. Hence, passing through a point on every curve of the whole system, we shall have a developable surface formed of a system of the original axes, and hence therefore, in the transition from point to point of any one individual curve, we shall have that whole system of axes completely exhausted, and divided into a series of developable surfaces.

The suggested division of every such system of principal axes into a system of developable surfaces, as a preparatory step towards endeavouring to find the surface envelope

of that system of axes, being therefore possible whenever that for which we seek has a real existence, we may suppose that division as having actually taken place in every individual instance, and we thus learn respecting the nature of the envelope in general. That, like the surface envelope of the whole system of normals to every algebraic surface, it consists always of two different and distinct sheets, which, like that same class of surfaces, may in some cases divide into two different and distinct surfaces, and which in others may dwindle either wholly or in part into a curve or evanescent surface, but which rarely, if ever, run into each other like the two sheets in Fresnel's "biaxal wave surface of double refracting media," or like the three sheets in Mr. Haughton's "surface of wave slowness of crystalline elastic solids," being for the most part separated from each other, and to the eye appearing to be two distinct surfaces, even in the great majority of cases, where, algebraically speaking, they are but parts of a single surface, and are contained, one and both, in the same unresolvable equation. To see this, let us take some one of the developable surfaces into which we may consider any particular system of axes divided, and let us follow it in its variation until the whole system be exhausted; we shall then perceive the envelope to consist always of a sheet generated by the system of consecutive curves, in which the different consecutive pairs of developables ultimately intersect, a sheet to which these developables will obviously be all circumscribed along their respective curves of ultimate intersection, and also of another and wholly distinct sheet generated by the system of arêtes de rebroussement of all these developable surfaces; a sheet to which, as well as to the other, the axes will be all tangents, but which with respect to *the same* system of developables will not be circumscribed by that system of surfaces, but will be merely the locus of their arêtes de rebroussement.

With respect to the enveloping system of axes, these two sheets possess however exactly the same properties. For, taking that second sheet with its whole system of lines of regression, that is, with its system of generating curves, considered in the above method, and conceiving as traced out upon it what may be called the conjugate system of curves, those viz. which all intersect every one of the former, so that the tangents to the two intersecting elements shall at every point of the surface be there a pair of conjugate tangents, let a system of developable surfaces be

circumscribed to the sheet along this new system of curves. Then, since every edge of a developable circumscribed to any algebraic surface forms always with the corresponding tangent to the curve of contact a pair of conjugate tangents to the surface at its point of contact, will that system of circumscribing developables possess the property that their edges will be all tangents to the original system of curves, and therefore also all tangents to the other sheet of the envelope and consequently all principal axes of the original system. This new system of developable surfaces, different altogether and distinct from the system which we have considered as producing the envelope, is therefore formed also out of the original system of principal axes, and equally with the former exhausts that whole system. Again, the system of curves arêtes de rebroussement of the new developable system must lie all upon the first sheet of the envelope; for, the edges of each surface of that system being all tangents to that sheet, if they did not, the developables themselves would be all circumscribed to the first sheet also, and therefore to the two sheets together: this certainly might take place for one particular surface of the system, which surface in certain cases might even break up into two, three, or more different surfaces; but it would be impossible that the whole system infinite in number should be all circumscribed to the two sheets simultaneously. Hence always the arêtes de rebroussement of the new system lie all on the first sheet, upon which, for the same reason as above, they are, conversely, the system of curves conjugate to the lines of contact of the former system.

Hence we see that, for every system of principal axes restricted by a single condition, there exist always two and but two different and distinct systems of developable surfaces, into either of which that whole system may be always divided; and also that the surface envelope of every such system of axes consists generally of two different and distinct sheets, separated from, and rarely if ever running into each other, of which sheets each will be enveloped by one of the two component systems of developable surfaces into which that system of axes may be divided, and will be the locus of the arêtes de rebroussement of the other system, and upon both of which the two opposite systems of generating curves, the lines of contact and the lines of regression, will be always conjugate to each other. These properties bear an obvious and close analogy to those of the whole system of normals to every algebraic surface, for

every such system of right lines in space, there being always two different and distinct systems of developable surfaces, into either of which they may be always resolved, and the surface their envelope being always of the same nature as that just considered in the present case.

As in the latter class of surfaces, it is obvious that if, in the class of envelopes we are now considering, either of the two systems of lines of regression be all plane curves, then will the sheet generated by the other system be always a developable surface; for that sheet, being the envelope of the system of developable surfaces of which the plane curves are the arêtes de rebroussement, will in that case be the envelope of a system of planes whose common equation contains but a single variable parameter. It will be presently proved (but let that be assumed, if necessary, throughout the present article) that every *plane* curve, whose tangents are all principal axes, will be always a parabola of the second order.

Such is the general type of the surface envelope of a system of principal axes restricted by a single condition: like the class of surfaces to which we have compared it, it is generally (algebraically speaking) a single surface consisting of two distinct sheets, which for the most part are separated from and rarely if ever run into each other, though as in their case the two sheets may sometimes be two separate surfaces, or even one of them or perhaps both may dwindle into a curve; but respecting the individual sheets themselves, whatever be their nature, there is nothing whatever which similarly restricts their character or limits them to any particular class, description, or form of surfaces, they may each or both, as the cases may be, have themselves one, two, or any number of sheets, they may either or even both be developable surfaces, they may be both closed or both open, or they may either or both be limited in one direction and extend to infinity in the other, or either or both may return back into themselves or extend to infinity in any or in every direction; their curve or curves of intersection moreover may be of any nature whatever, they may be altogether imaginary or they may be wholly or partly real, and if real wholly or in part, they may wholly or partly be closed and return back into themselves, or they may wholly or in part be open and extend to infinity in any or in every direction.

Suppose that a system of principal axes were restricted by the single condition of passing all through a curve

plane or of double curvature assumed arbitrarily in the body; then, from every point of that curve will a cone of the axes diverge, this system of cones will be one of the two component systems of developable surfaces into which that system of axes may be divided. The other also may be easily found: take arbitrarily a side of any one of these cones as the basis of a developable of the new system, this will intersect the consecutive cone in a number of points equal to the order of that cone; of the sides of that cone passing through these points take that which is consecutive to the assumed side of the first, this will be the second side of the developable, and will intersect the cone from the third consecutive point; of this the consecutive intersecting side will be the third edge of the developable, and so on. The developable so found will be one of the second system, and the others of that system may be found in a similar manner. In this case the two sheets of the envelope will be quite distinct from each other; one, the locus of the arêtes de rebroussement of first system of developables, that is, the locus of the vertices of the system of cones will be the assumed curve, the other will generally be a surface, the locus of the arêtes de rebroussement of the other system of developables, or, which is the same thing, the envelope of the system of cones.

Hence we see that in every body there exists an infinite number of systems of principal axes restricted by a single condition, for which one of the sheets of the envelope will not be a surface, but a curve plane or of double curvature as the case may be.

Suppose again that a system of principal axes were restricted by the single condition of being tangents all to a developable surface given or arbitrarily assumed in the body. Here again as in the former example we readily obtain the two component developable systems and the two sheets of the complete envelope; for in every tangent plane to the assumed surface there lies an infinite number of principal axes which envelope a parabola in that plane; hence the system of tangent planes themselves to that surface, or rather the system of portions of each tangent plane bounded by their respective parabolas, will be one of the two component developable systems: the other, as in the example above, may also be easily found, take arbitrarily a tangent to the parabola in any one of the tangent planes, as the basis of a developable of the new system, this will intersect the consecutive tangent plane in a point, of the

two tangents to the parabola in this new plane which pass through the point of meeting take that which is consecutive to the first assumed tangent, this will be the second edge of the developable and will meet the third consecutive tangent plane in a point, the consecutive tangent through which to the parabola in that third plane will be the third edge of the developable, and so on to the end. The developable so found will be one, the second system and the others of that system may be found in a similar manner; in this case also the two sheets of the envelope will be quite distinct from each other, one, the envelope of the first system of developables, that is of the system of tangent planes to the assumed developable surface, will be that developable surface itself, the other, the locus of their system of arêtes de rebroussement, will be the surface generated by the system of parabolas envelopes of the systems of principal axes in the system of tangent planes to that same surface.

Hence we see that in every body there exists an infinite number of systems of principal axes restricted by a single condition, for which one of the sheets of the envelope will be a developable surface; also, that in all such cases, one of the two component developable systems will be always a system of planes, the system of tangent planes to the developable sheet itself; and that moreover, the system of lines of contact with that sheet will be always a system of right lines, the system of edges of the sheet itself, while the system of lines of regression on the other sheet will be always a system of plane curves and all parabolas of the second order. As for the second sheet itself, the system of lines of contact on that sheet, and the system of lines of regression on the developable sheet, they obviously all vary in their nature and properties with the system of axes themselves, and remain to be determined when that system is given in every particular case.

Again, more generally, suppose that a system of principal axes were restricted by the single condition of being tangents all to a surface of any nature whatever, given, or arbitrarily described in the body, and that it were required to find the two systems of component developables, and the surface envelope of the system of axes. Here, as indeed also in the preceding case, we might at first sight suppose that the latter were already known and that the surface itself were the envelope, but this would not be the case exactly; the surface itself, unless indeed (which of course would scarcely

ever happen) it chanced to be given or assumed so fortunately as that every tangent which was a principal axis would have double contact with it, would not be the whole envelope, it would be only one of its two sheets, and the other, which therefore in the vast multitude of cases of this nature will be always a distinct surface, would still remain to be determined.

Hence we see that in every body cases without number exist, for which the two sheets of the surface envelope of a system of principal axes restricted by a single condition are two distinct surfaces; and, moreover, that these surfaces are not confined to any particular class or species, but though of course in every case inseparably connected with each other, that either may be of any nature whatever; for in that extensive class of cases where a system of principal axes are restricted to touch a given surface, that surface itself which is absolutely arbitrary is always one of the sheets of the envelope, and the other fixed of course and implicitly determined when the first is given, may, according to the varieties of that first, be also of any nature whatever; the curve or curves, moreover, in which the two surfaces intersect each other, may, as depending on these surfaces, be also of any kind whatever, it may be altogether imaginary or it may be real, and if real, it may either be one continuous curve closed or extending to infinity, or it may consist of two or more detached curves separated from each other by intervals of both surfaces. If the two surfaces happen to touch each other at one or more points, then obviously will every point of contact be a double point, nodal, cuspal, or conjugate, on the curve of their mutual intersection.

It is not to be supposed however that every point of these two surfaces, or more generally of the surface whatever be its nature envelope of a system of principal axes restricted by a single condition, is in all cases actually touched by an axis of the enveloping system; on the contrary it more frequently happens that only a portion of the whole envelope is actually available, and that upon that surface, or to speak more generally, upon every surface whatever given or arbitrarily assumed in a body there exists two distinct regions, continuous or discontinuous, separated from each other by a very remarkable curve or curves, such that for all the points of one region some among the whole system of tangents to the surface at each point are principal axes, while at every point of the other not one of the whole system of tangents possesses that property.

This is easily conceivable in the case of the complete envelope itself, whatever be its nature; for since every principal axis which touches that surface must touch it at two points, in order to find the direction at any point on either of its sheets of the particular tangent, or the directions of all the particular tangents if there be more than one, which will possess the property of being principal axes, we have only to make that point the point of contact of a tangent plane to its own sheet of the envelope, and the vertex of a cone enveloping the other sheet, the cone obviously will always intersect the tangent plane in all the lines passing through the point which will have double contact with the surface, and therefore, *a fortiori*, in all the principal axes sought; and it is easy to see that if the particular determinate number of the intersecting sides which are principal axes be in general even, these particular sides may as often be all imaginary as real.

Now that number is always even, and moreover it can never exceed two; for, from every point of the envelope, or more generally from every point of any surface whatever given or arbitrarily assumed in the body, there diverges a cone of principal axes, which cone also will of course always intersect the tangent plane at each point of the surface in all the tangents at that point which are principal axes; but every cone of principal axes in a body wherever be its vertex (let this be assumed for the present) is always of the second order, and can therefore intersect any plane passing through its vertex in never more than two right lines, and these may as often be both imaginary as real.

Hence at no point on either sheet of the surface envelope of a system of principal axes restricted by a single condition can ever more than two axes of the system touch the surface, and hence also both sheets may possess regions for which no axis of the system will touch at any point whatever, in which case the curve or curves separating the available from the untouched regions will on each sheet possess of course the property that at all its points the two tangent principal axes will coincide with each other; the same, moreover, may exactly be said of any surface whatever given or arbitrarily assumed in the body, it possesses generally (though not universally) two distinct regions, for all the points on one of which two principal axes containing between them an angle of finite magnitude will touch the surface, and for all the points on the other of which no tangent whatever will be a principal axis,

the curve or curves separating these two regions of real and imaginary contact being the locus or loci of that system or systems of points on the surface for which the two tangent principal axes coincide with each other; the first of these regions on every surface obviously contains all those points for which the diverging cone of the second order of principal axes intersects the tangent plane in two real and different right lines; the second for the same reason contains all those for which the two intersecting sides are both imaginary, and the separating curve or curves is the locus of that particular system or systems of points for which the tangent plane to the surface is also a tangent plane to the cone of principal axes.

These two different regions always exist on every closed surface of any form which is of small dimensions in comparison with its distance from the center of gravity, for, at the different points of such a surface the diverging cone of principal axes varies but little in magnitude, position, and figure, while between the same limits the tangent plane passes through every possible variety of position.

Moreover, since for every point on such a surface there generally exists a second on the opposite side for which the tangent plane is parallel to that at the first point, the bounding curve separating the regions of real and imaginary contact, consists generally of two distinct closed curves, returning each into themselves, and dividing the surface into three distinct portions; hence on such a surface; the regions of the two species of contact consist generally, one of the two opposite and unconnected caps separated from each other by the intermediate interval, and the other of the continuous zone between them.

On the contrary, the region of real contact for the most part monopolises the whole of every closed surface which contains within it the centre of gravity and which is such as every direction to present its concavity towards that point; for at the different points of every such surface, the diverging cone of principal axis experiences considerable variations both in position and figure; while the corresponding tangent plane also goes simultaneously through every possible variety of position. Also on every such surface the separating curves are of course altogether imaginary, and moreover the two conjugate systems of curves so intimately connected with the distribution of the enveloping system of principal axes, viz. the lines of contact and the lines of regression of the two component

systems of developoble surfaces experience in their case considerable modifications from the general type, and become peculiarly simple in their nature and distribution.

But, in general, different surfaces assumed arbitrarily in a body present every imaginable variety both with respect to the nature and magnitude of the two different regions, and with respect to the nature and form of the two bounding curves; in some cases the region of real contact will take up the whole surface, in others the region of imaginary contact may occupy the whole of it; the two regions, either or both finite or extending to infinity in any or in every direction according to the nature of the assumed surface in each particular case, will in some cases be both continuous, while in others they will be discontinuous and consist each of two or more separate and detached portions of the surface, or they will be one continuous and the other consisting of several unconnected portions; and also, the curves of separation bounding the different regions will in some cases be altogether imaginary, while in others they will be one or both real, in some cases the real curve or curves will either or both consist of a single continuous curve finite or extending to infinity as the case may be, and in others, either or both will consist of two or more detached curves separated from each other by intervals of the surface and all closed or all open, or some closed and finite and others extending to infinity in either or in both directions; all of which will be perfectly manifest from a property to be presently established, viz. that two different curves of this nature distinct from each other in their respective properties exist on every surface whatever assumed arbitrarily in a body, and that they, and therefore with them the two regions of real and imaginary contact, are always determined by the intersections with that surface of two other determinable surfaces distinct also from each other and differing themselves in their nature and properties.

In the case of every surface, whatever be its nature, which is the complete envelope of a system of principal axes restricted by a single condition, there exist on each sheet two distinct curves of this nature, both possessing the property that at all their points the two tangent principal axes coincide with each other, and therefore both separating an untouched from an available region of that surface, but otherwise differing from each other in their nature and properties and arising each from a different cause. Of these two curves, thus independent of each other on the same

sheet, but each intimately connected with the corresponding curve on the other sheet, the two corresponding pairs possessing always each the same properties and being always together both real or both imaginary, one on each is the same for both sheets and is a portion of their common intersecting curve, the other on each is a portion of its curve of contact with their common circumscribing developable surface.

This there is no difficulty in seeing, for, at every point of the curve of contact with either sheet of the circumscribing developable common to the two sheets, the angle vanishes between two of the particular tangents which also touch the other sheet, that is, the two tangent principal axes all along the portion of that curve corresponding to those particular lines of double contact coincide with each other, and, at every point of the intersecting curve common to the two sheets, the angle between two of the tangents to either sheet, which also touch the other, is equal in magnitude to two right angles; hence at all points of the proper portion of that curve also the two tangent principal axes to either sheet coincide with each other.

Moreover, in the former case, the two coincident principal axes all along either curve of contact coincide obviously at each point of that curve with the corresponding edge of the circumscribing developable, that is with the tangent to the surface at that point conjugate to the tangent to the curve itself at the same point; and in the latter case, all along the common intersecting curve, they obviously coincide with the tangent itself at every point of that curve; hence on each sheet of the surface envelope of a system of principal axes restricted by a single condition, the two curves which separate the available from the untouched regions are not merely the loci of the two systems of points on that sheet for which the two tangent principal axes coincide with each other, but are moreover, one the envelope of the corresponding system of coincident axes themselves, and the other the envelope, not of its corresponding system of axes themselves but of the system of tangents to the surface conjugate to that system of axes.

The two developable surfaces generated by these two particular systems of axes, both surfaces of principal axes, and of principal axes which all touching the two sheets of the envelope belong to the system of axes enveloped by that surface, are in other respects also two very remarkable surfaces with respect to that system; for, notwithstanding that every system of principal axes subject to a single con-

dition may always be divided into two and not more than two different and distinct systems of developable surfaces, of which the surfaces of one will be all circumscribed to one sheet of the envelope and will have their arêtes de rebroussement all situated on the other, while the surfaces of the other system will be all circumscribed to the latter sheet and will have their arêtes de rebroussement all placed on the former. These two particular developable surfaces, the one circumscribed equally to the two sheets of the envelope, and the other having its arête de rebroussement equally on both, belong to neither of the above two systems, which are the only developable systems into which the original system of axes can be divided, and yet they are both developable surfaces of principal axes, and of principal axes which belong to that system.

This apparent paradox may be explained in almost exactly the same way as a particular solution of an ordinary differential equation has been generally explained, a solution which satisfies the equation, but which nevertheless does not belong to the only system or systems of solutions into which by the variation of the arbitrary constant or constants the complete integral of that equation may be always resolved, and in both cases the geometrical interpretations are almost precisely similar; for, as we shall just now see, these two developable surfaces of principal axes, though not belonging to either of the two component developable systems, contain each one or more of the edges of every individual surface of both those systems; and moreover, the two curves of contact with the two sheets of the enveloping surface of their common circumscribing developable are the envelopes each of the whole system of lines of contact of the component developable system circumscribed to its own sheet of that surface, while, at the same time, the common intersecting curve arête de rebroussement of the other developable surface is also the envelope common to the two sheets of the two systems of lines of regression, the arêtes de rebroussement of the two developable systems; the lines of regression of one component system and the lines of contact of the other lying always on the same sheet of the envelope and forming these two systems of curves conjugate to each other.

These properties there is no difficulty in establishing, for, assuming arbitrarily any tangent to the curve of intersection on the portion of that curve corresponding to the principal axes, that tangent may be made the basis of two

developable surfaces, one circumscribed to one of the sheets of the envelope and having its arête de rebroussement on the other, and the other circumscribed to the latter sheet and having its arête de rebroussement on the former, and each belonging therefore to one of the two different component developable systems: and in proceeding in both directions to complete these two developables, it is obvious that the two lines of regression will both touch the intersecting curve at the point of contact of the assumed tangent, while the two lines of contact will consist each of two branches both terminating abruptly at the same point, and (since the edges of every developable circumscribed to any surface and the corresponding tangents to its curve of contact are always pairs of conjugate tangents to the surface) both having the same tangent at that point, viz. the tangent on their own sheet conjugate to the assumed tangent to the curve of intersection, but lying themselves on opposite sides of that tangent, and both turning their convexities towards it; hence the two lines of contact of the two developable surfaces have each a cusp of the ramphoid species at the point of contact of the assumed tangent, the tangents at the two cusps being the two conjugates to that tangent on the two sheets of the envelope; and proceeding thus in the same way from tangent to tangent of the curve of intersection we see that the whole system of tangents to that curve may be made the bases of two different and distinct systems of developable surfaces, one having their system of arêtes de rebroussement all on one sheet of the envelope and being all circumscribed to the second, and the other having their system of arêtes de rebroussement all on the latter sheet and being all circumscribed to the former; and again, that the two systems of lines of regression will all touch at one or more points the curve of intersection, and that the two systems of lines of contact will have each one or more cusps of the ramphoid species all ranged on that same curve, the cuspal points of each pair of the latter lines being equal in number and coinciding with the points of contact of the corresponding pair of the former, and every pair of cuspal tangents being conjugate each on its own sheet of the envelope to the corresponding tangent common to the corresponding pair of lines of regression.

Again, assuming arbitrarily any side of the common circumscribing developable surface, on the portion of that surface corresponding to the principal axes, that side may similarly be made the common basis of two developable

surfaces, one circumscribed to one of the sheets of the envelope and having its arête de rebroussement on the second, and the other circumscribed to the latter sheet and having its arête de rebroussement on the former, and therefore, as in the other case, belonging each to one of the two component developable systems; and proceeding as before in both directions to complete these two developable surfaces, it is equally obvious that the two lines of contact will in this case touch each the curve of contact with its own sheet of the common circumscribing developable, while the two lines of regression will have each a cusp of the ramphoid species at the point where the corresponding line of contact touches its enveloping curve, the tangent of each cusp being, as before, conjugate to the corresponding tangent of the curve of contact. Hence in this case, proceeding from edge to edge, the whole system of edges of the common circumscribing developable may be made the bases of two different and distinct systems of developable surfaces, one having their system of arêtes de rebroussement all on one sheet of the envelope and being all circumscribed to the second, and the other having their system of arêtes de rebroussement all on the latter sheet and being all circumscribed to the former; and of these, the two systems of lines of contact will each envelope the curve of contact with its own sheet of the common circumscribing developable, every individual line of the system touching that curve in one or more points as the case may be, while the two systems of lines of regression will in this case have each one or more cusps of the ramphoid species ranged all on the same two curves, and on every individual line of either system being equal in number and coinciding with all the points of contact of the corresponding line of contact on the same sheet, the cuspal tangent, as before, being in all cases conjugate to the tangent of the corresponding curve of the other system with respect to the sheet of the envelope on which they both lie.

[*To be continued.*]

ON LOGARITHMIC INTEGRALS OF THE SECOND ORDER.

PART II.

By FRANCIS W. NEWMAN.

(*Continued from* p. 100).

$$\left.\begin{aligned} \textit{On the Integrals } \Lambda(x, a) &= \tfrac{1}{2}\int_0 \frac{\log x\,(x-\cos a)\,dx}{x^2 - 2x\cos a + 1} \\ \lambda(x, a) &= \tfrac{1}{2}\textstyle\int_0 \log(x^2 - 2x\cos a + 1)\,\dfrac{dx}{x} \end{aligned}\right\}$$

which are related by the equation $\Lambda(x, a) + \lambda(x, a) = \frac{1}{2}\log x \log X$, *if* X *stands for* $(x^2 - 2x\cos a + 1)$.

§ I.—*Simplest cases of* Λ.

1. It was shewn in Part I. that all the integrals included in $\int F_1x \log F_2x\,dx$ are reducible to common forms in conjunction with three peculiar integrals, Lx, ꜧx, and $\Lambda(x, a)$, of which the last alone remains to be treated. We suppose a to be between 0 and π, unless the contrary is stated. We also generally suppose x to be positive. When it comes out negative in any formula, we can restore it by help of the identical equation

$$\Lambda(-x, a) = -\Lambda(x, \pi - a),$$

which subsists by virtue of the convention (already proposed) that *log x* is always to mean $\frac{1}{2}\log(x^2)$.

$$\text{Assuming } x = \frac{\sin\omega}{\sin(\omega + a)}, \text{ or } \tan\omega = \frac{x\sin a}{1 - x\cos a},$$

$$\text{we get } \sqrt{X} = \frac{\sin a}{\sin(\omega + a)};$$

$$\text{whence } \Lambda(x, a) = \int \log\frac{\sin(\omega + a)}{\sin\omega}\,.\,d\log\sin(\omega + a);$$

which we shall hereafter denote by $\chi(\omega, a)$; so that $\Lambda(x, a)$ and $\chi(\omega, a)$ are identical forms. This substitution is chiefly of use in enabling us to understand the nature of other transformations at which we shall arrive. For the present, when ω is named, it is supposed to bear this relation to x.

2. To find the *complete function* $\Lambda(1, a)$, which $= -\lambda(1, a)$.

$$\text{Since } \lambda(x, a) = \tfrac{1}{2}\int_0 \log X\,\frac{dx}{x},\quad \frac{d\lambda}{da} = \int_0 \frac{\sin a}{X}\,dx = \omega.$$

$$\text{Make } x = 1, \therefore \tan\omega = \frac{\sin a}{1 - \cos a} = \cot\tfrac{1}{2}a, \text{ or } \omega = \tfrac{1}{2}(\pi - a).$$

Integrate $\dfrac{d\lambda}{da} = \frac{1}{2}(\pi - a)$; $\therefore \lambda(1, a) = c + \frac{1}{2}\pi a - \frac{1}{4}a^2$.

To find c, make $a = 0$; $\lambda(x, 0) = \int_0 \log(1 - x)\dfrac{dx}{x} = L(1 - x)$,

$$\text{whence } \lambda(1, 0) = L0, \text{ or } c = -\tfrac{1}{6}\pi^2,$$

$$\left.\begin{aligned} &\therefore \Lambda(1, a) = -\lambda(1, a) = \tfrac{1}{6}\pi^2 - \tfrac{1}{2}\pi a + \tfrac{1}{4}a^2. \\ &\text{Hence also } \Lambda(1, \pi - a) = \tfrac{1}{4}a^2 - \tfrac{1}{12}\pi^2. \end{aligned}\right\} \ldots.(1).$$

3. To find Λ at special values of a.

At the extreme values, ($a = 0$ and $a = \pi$), X is an algebraic square, $(x \pm 1)^2$. Hence

$$\left.\begin{aligned} &\Lambda(x, 0) = Lx - L0 \\ &\Lambda(x, \pi) = L(-x) - L0 = lxl(1 + x) - L(1 + x) \end{aligned}\right\} \ldots(2).$$

In the following process, we for a moment suppose a to increase from 0 to any magnitude; and, to shew both variables, write $f(x, a)$ instead of X. Then, by a well-known formula of Trigonometry,

$$f(x^n, na) = f(x, a) . f\left(x, a + \frac{2\pi}{n}\right) . f\left(x, a + \frac{4\pi}{n}\right) \ldots f\left(x, a + \frac{2n-2}{n}\pi\right).$$

Differentiate logarithmically: multiply by $(2n)^{-1}.\log(x^n) = 2^{-1}\log x$, and integrate: then

$$\frac{1}{n}\Lambda(x^n, na) = \Lambda(x, a) + \Lambda\left(x, \ a + \frac{2\pi}{n}\right)$$

$$+ \Lambda\left(x, a + \frac{4\pi}{n}\right) + \ldots. + \Lambda\left(x, a + \frac{2n-2}{n}\pi\right) \ldots.(3).$$

In particular, if $n = 2$, $\cos(a + \pi) = \cos(\pi - a)$,

$$\left.\begin{aligned} \therefore \quad \tfrac{1}{2}\Lambda(x^2, 2a) &= \Lambda(x, a) + \Lambda(x, \pi - a) \\ &= \Lambda(x, a) + \Lambda(-x, a) \end{aligned}\right\} \ldots.(4),$$

which has a certain analogy to $Lx + L(-x) = \frac{1}{2}L(x^2) + \frac{3}{2}L0$.

When $a = \dfrac{\pi}{n}$, we have

$$\left.\begin{aligned} &\frac{1}{n}\Lambda(x^n, \pi) \text{ or } L(-x^n) - L0 \\ &= \Lambda\left(x, \frac{\pi}{n}\right) + \Lambda\left(x, \frac{3\pi}{n}\right) + \Lambda\left(x, \frac{5\pi}{n}\right) + \ldots + \Lambda\left(x, \frac{2n-1}{n}\pi\right) \end{aligned}\right\}.$$

Introvert the terms; and to make every a fall between 0 and π, observe that $\cos\dfrac{2n-r}{n}\pi = \cos\dfrac{r\pi}{n}$. Then we find that

$$\left.\begin{aligned}&\text{if } S = \Lambda\left(x, \frac{\pi}{n}\right) + \Lambda\left(x, \frac{3\pi}{n}\right) + \Lambda\left(x, \frac{5\pi}{n}\right) + \ldots\ldots \\ &\text{when } n \text{ is even, } S \text{ to } \frac{n}{2} \text{ terms} = \frac{1}{2n}\Lambda(x^n, \pi), \\ &\text{and when } n \text{ is odd,} \\ &\qquad S \text{ to } \frac{n-1}{2} \text{ terms} = \frac{1}{2n}\Lambda(x^n, \pi) - \tfrac{1}{2}\Lambda(x, \pi)\end{aligned}\right\}\ldots(5).$$

In particular,

$$\left.\begin{aligned}&\text{If } n = 2, \quad \Lambda(x, \tfrac{1}{2}\pi) = \tfrac{1}{4}\Lambda(x^2, \pi): \\ &\text{If } n = 3, \quad \Lambda(x, \tfrac{1}{3}\pi) = \tfrac{1}{6}\Lambda(x^3, \pi) - \tfrac{1}{2}\Lambda(x, \pi)\end{aligned}\right\}\ldots(6).$$

If in equation (3) we make $na = 2\pi$, and n is odd, we similarly have

$$\Lambda\left(x, \frac{2\pi}{n}\right) + \Lambda\left(x, \frac{4\pi}{n}\right) + \ldots. + \Lambda\left(x, \frac{n-1}{n}\pi\right)$$

$$= \frac{1}{2n}\Lambda(x^n, 0) - \tfrac{1}{2}\Lambda(x, 0)\ldots\ldots(7):$$

If $n = 3$ in the last,

$$\Lambda(x, \tfrac{2}{3}\pi) = \tfrac{1}{6}\Lambda(x^3, 0) - \tfrac{1}{2}\Lambda(x, 0)\ldots\ldots\ldots(8).$$

Thus we know $\Lambda(x, a)$ in finite functions of x, by means of L, when a has any of the values 0, π, $\frac{1}{2}\pi$, $\frac{1}{3}\pi$, $\frac{2}{3}\pi$. It will afterwards appear, that the assertion may be extended to the case of $a =$ any of these, *divided by* 2^n, when n is an arbitrary integer.

4. To find Λ, when x (at the upper limit) is a given function of a.

Generally, if $u = Fx$, $V = \psi(x, a)$, and $U = \int u dV$,

$$\frac{dU}{da} = \int u \frac{d^2V}{dx\,da} dx = \int u \frac{dV'}{dx} dx, \text{ if } V' = \frac{dV}{da};$$

Integrate by parts, and the last is $(uV' - \int V'du)$; and the *total* differential

$$d(U) = \frac{dU}{dx} dx + \frac{dU}{da} da = u\frac{dV}{dx} dx + \left(u\frac{dV}{da} - \int V'du\right) da$$

$$= u.d(V) - (\int V'du)\, da.$$

In the present case, let $u = \frac{1}{2}\log x$, $V = \log X$, and observe that $V' = 2xX^{-1}\sin a$, which vanishes with x; and $\int V'du = \omega$, which also vanishes with x, as $\frac{dU}{da}$ or (here) $\frac{d\Lambda}{da}$ ought to do

No constant then is needed in integrating; and we get

$$d(\Lambda) = \tfrac{1}{2}\log x.d\,(\log X) - \omega da. \ldots\ldots.(9)$$

for the total differential, when x is a function of a.

The extreme case of $x = (\cos a)^0 = 1$, reproduces equation (1), as it ought.

Assume $x = 2\cos a$, $X = 1$, $\tan\omega = -\tan 2a$, $\omega = \pi - 2a$; $\therefore\ d\Lambda = -(\pi - 2a)\,da$. Observe that Λ vanishes when $x = 0$, and consequently (here) when $a = \frac{1}{2}\pi$, and we get

$$\Lambda \text{ or } \Lambda(2\cos a, a) = (\tfrac{1}{2}\pi - a)^2 \ldots\ldots (10).$$

This is the most remarkable equation we have yet met.

It may also be denoted (if $a = \cos a$) by

$$\int_0^{2a} \frac{\log x(x-a)}{x^2 - 2ax + 1}\,.\,dx = (\sin^{-1} a)^2.$$

Once more, assume $x = \cos a = a$, $\therefore X = 1 - a^2$, $\tan\omega = \cot a$, $\omega = \frac{1}{2}\pi - a$; whence

$$d\Lambda = \tfrac{1}{2}\log a\, d\log(1 - a^2) - (\tfrac{1}{2}\pi - a)\,da,$$

$$\therefore\quad \Lambda = \tfrac{1}{4}L(a^2) + \tfrac{1}{2}(\tfrac{1}{2}\pi - a)^2 + \text{const};$$

$$\text{or } \Lambda(\cos a, a) = \tfrac{1}{4}L(\cos^2 a) + \tfrac{1}{2}(\tfrac{1}{3}\pi^2 - \pi a + a^2). \ldots.(11):$$

$$= \tfrac{1}{4}L(\cos^2 a) + \Lambda(1, a) + \tfrac{1}{4}a^2.$$

The constant is determined as before, by observing that Λ must vanish when $a = \frac{1}{2}\pi$; also $L0 = -\frac{1}{6}\pi^2$.

There is an infinity of other assumptions ($x = Fa$) which make (9) integrable in finite terms. Again, ω may be expanded in a series of cosines or sines of multiples of a, after which we may try to integrate. But the cases in which these processes succeed appear to be more easily treated by some of the methods which follow.

§ II.—*On changing* $\Lambda(x, a)$ *to* $\Lambda(y, a)$.

5. As long as x is small (say $x < \pm\frac{1}{2}$), we may develop $\log X$ by the well-known series, and obtain

$$-\lambda(x, a) = \frac{x\cos a}{1^2} + \frac{x^2\cos 2a}{2^2} + \frac{x^3\cos 3a}{3^2} + \&\text{c.}\ldots.(12),$$

from which $\lambda(x, a)$ and $\Lambda(x, a)$ are found. But when x is not small enough, we may try to reduce $\Lambda(x, a)$ to $\Lambda(y, a)$, in which y is small.

For the present, let Y stand for $y^2 - 2y\cos a + 1$.

6. Then, *first*, put $xy = 1$, $\therefore$ $Y = Xx^{-2}$, $\log y = -\log x$, $d\log Y = d\log X - 2d\log x$;

$$\Lambda x + \Lambda y = \tfrac{1}{2}\int lx\,(dlX - dlY) = \int lx\,d\,lx$$
$$= \tfrac{1}{2}\log^2 x + \text{const};$$
$$\text{or } \Lambda x + \Lambda x^{-1} = 2\Lambda 1 + \tfrac{1}{2}\log^2 x \;\ldots\ldots\ldots (13),$$

which serves to reduce Λx, when x is > 1, to Λy, where y is < 1.

Cor. 1. When $x = \infty$, Λx converges to $2\Lambda 1 + \frac{1}{2}\log^2 x$.

Cor. 2. As $Lx + Lx^{-1} = \frac{1}{2}\log^2 x$,

$$\therefore (\Lambda x - Lx) + (\Lambda x^{-1} - Lx^{-1}) = 2\Lambda 1.$$

Secondly, suppose $x + y = 2\cos a$, $X = 1 - xy = Y$;

$$\therefore \Lambda x + \Lambda y = \tfrac{1}{2}\int (lx + ly)\,dl\,(1 - xy)$$
$$= \tfrac{1}{2}\int l(xy).dl(1 - xy) = \tfrac{1}{2}L(xy) + C:$$

$$\text{whence } \left.\begin{array}{l}\Lambda x + \Lambda y - \Lambda(2\cos a) = \tfrac{1}{2}L(xy) - \tfrac{1}{2}L0,\\ \text{when } x + y = 2\cos a.\end{array}\right\} \ldots (14).$$

As we know $\Lambda(2\cos a)$ from equation (10), this enables us to find Λx by means of Λy, whenever x is near to $2\cos a$. It also gives

$$2\Lambda\cos a - \Lambda(2\cos a) = \tfrac{1}{2}L\cos^2 a - \tfrac{1}{2}L0,$$

which is verified by (10) and (11).
Make $x = 1$, and we find

$$\Lambda(2\cos a - 1) = \tfrac{1}{2}L(2\cos a - 1) + \tfrac{1}{8}\pi^2 - \tfrac{1}{2}3\pi a + \tfrac{1}{4}5a^2 \ldots (14^*).$$

We may combine the two last integrations, by supposing $x^{-1} + y = 2\cos a$, which gives

$$\Lambda x - \Lambda y + \tfrac{1}{2}a^2 = \tfrac{1}{2}\log^2 x - \tfrac{1}{2}L\frac{y}{x} \;\ldots\ldots (15).$$

Again, in this write y^{-1} for y, and eliminate Λy^{-1} by means of (13), and $L(x^{-1}y^{-1})$ by means of the known properties of L; then

$$\left\{\begin{array}{l} x^{-1} + y^{-1} = 2\cos a\,; \\ \Lambda x + \Lambda y - \pi\left(\tfrac{1}{3}\pi - a\right) = \tfrac{1}{2}L(xy) + \tfrac{1}{4}\log^2\left(\dfrac{x}{y}\right)\end{array}\right\} \ldots (16).$$

But in this, the arbitrary constant is liable to change by reason of discontinuity, if x or y passes through zero.

7. The four suppositions here made have something in common. In (13), (14), and (16), we find

$$\frac{dx}{X} = \frac{dy}{Y}\,; \text{ and in (15), } \frac{dx}{X} = -\frac{dy}{Y}.$$

Let us in all suppose $x = \dfrac{\sin \omega}{\sin(\omega + a)}$;

then if $y = \dfrac{\sin \theta}{\sin(\theta - a)}$, $xy = 1$, when $\theta = \omega + a$

......, but $x + y = 2 \cos a$, when $\theta = \omega + 2a$.

8. By equation (13) we can obtain $\Lambda(\sec a)$ and $\Lambda(\frac{1}{2} \sec a)$ from $\Lambda(\cos a)$ and $\Lambda(2 \cos a)$. Observing that

$$L \cos^2 a + L \sec^2 a = 2 \log^2 \cos a,$$

we have
$$\left.\begin{array}{l} \Lambda \sec a = \frac{1}{4} L \sec^2 a + \frac{1}{2}\pi(\frac{1}{3}\pi - a) \\ \Lambda(\frac{1}{2}\sec a) = \frac{1}{2}\log^2(\frac{1}{2}\sec a) + \frac{1}{12}\pi^2 - \frac{1}{2}a^2 \end{array}\right\} \ldots\ldots(17).$$

9. Farther, since we fulfil the relation $x^{-1} + y = 2 \cos a$, by supposing

$$x = \frac{\sin \omega}{\sin(\omega + a)}, \quad y = \frac{\sin(\omega - a)}{\sin \omega},$$

it is evident that if Λx is known, we can by the repeated use of (15) find $\Lambda \dfrac{\sin\{\omega - (n+1)a\}}{\sin(\omega - na)}$. Or conversely, if Λy is known, we can deduce $\Lambda \dfrac{\sin(\omega + na)}{\sin\{\omega + (n+1)a\}}$.

For example: *first*, let m_n stand for $\dfrac{\cos na}{\cos(n-1)a}$; then $m_n^{-1} + m_{n+1} = 2 \cos a$;

$$\therefore\ 2\Lambda m_{n+1} - 2\Lambda m_n = a^2 + L(m_n^{-1} m_{n+1}) - \log^2 m_n.$$

For n write 1, 2, 3....$(n-1)$, and add the results, taking Λm_1 from (11);

$$\left.\begin{array}{l} \therefore\ 2\Lambda \dfrac{\cos na}{\cos(n-1)a} = \frac{1}{3}\pi^2 - \pi a + na^2 \\ \quad + \frac{1}{2} L \cos^2 a + L(m_1^{-1} m_2) + L(m_2^{-1} m_3) + \ldots + L(m_{n-1}^{-1} m_n) \\ \quad - \log^2 m_1 - \log_2 m_2 - \&c. \ldots\ldots - \log^2 m_{n-1}: \end{array}\right\}$$

where for $\cos^2 a$ we may write $(m_0^{-1} m_1)$.

Similarly, if $m_n = \dfrac{\sin na}{\sin(n+1)a}$, $m_n^{-1} + m_{n-1} = 2 \cos a$; and Λm_1 is known by (17); so that

$$2\Lambda \frac{\sin na}{\sin(n+1)a} = \frac{1}{6}\pi^2 - na^2 - L\frac{m_1}{m_2} - \ldots\ldots - L\frac{m_{n-1}}{m_n}$$
$$+ \log^2 m_1 + \log^2 m_2 + \ldots\ldots + \log^2 m_n.$$

10. A more general relation between Λx and Λy is obtainable by Mr. Fox Talbot's Method of Symmetrical Integrals.

Let $X = (m - x)\,v$, $Y = (m - y)\,v$; where m is constant, and x, y functions of v. Then x and y are the two roots of

$$x^2 - (2\cos a - v)\,x + (1 - mv) = 0\,;$$

consequently $x + y = 2\cos a - v$, $xy = 1 - mv$;

and, eliminating v,

$$(m - x)(m - y) = 1 - 2m\cos a + m^2 = M.$$

Now $\Lambda x + \Lambda y = \tfrac{1}{2}\int lx\,\{dlv + dl(m - x)\} + \tfrac{1}{2}\int ly\,\{dlv + dl(m - y)\}$

$$= \tfrac{1}{2}\int l(xy)\,dlv + \tfrac{1}{2}\int lx\,dl(m - x) + \tfrac{1}{2}\int ly\,dl(m - y).$$

The first integral

$$= \int l(1 - mv)\,dlv = L(1 - mv) = L(xy)\,;$$

the second

$$= lml(m - x) + L\frac{x}{m}\,; \text{ the third} = lml(m - y) + L\frac{y}{m}\,.$$

Observe that $lml(m - x) + lml(m - y) = lmlM = \text{const.}$;

$$\therefore\ 2\Lambda x + 2\Lambda y - \text{const.} = L(xy) + L\frac{x}{m} + L\frac{y}{m}\,.$$

Let $x = e$, when $y = 0$; then $e = 2\cos a - m^{-1}$; so that Λe and Λm will be known from one another by (15). Also

$$\left(1 - \frac{x}{m}\right)\left(1 - \frac{y}{m}\right) = 1 - \frac{e}{m} = 1 - 2e\cos a + e^2 = E.$$

Finally

$$2(\Lambda x + \Lambda y - \Lambda e) = L(xy) + L\frac{x}{m} + L\frac{y}{m} - L\frac{e}{m} - 2L0 \ldots (18).$$

By slightly varying the integration, we get

$$2(\lambda x + \lambda y - \lambda e)$$
$$= L(1 - xy) + L\left(1 - \frac{x}{m}\right) + L\left(1 - \frac{y}{m}\right) - L\left(1 - \frac{e}{m}\right) \ldots (18^*),$$

which may be convenient when m, or one of the variables, is negative.

By giving special values to m, such as make Λm and Λe known functions, the equations become available to us in many ways. But in order to understand our result, it will be well to transform it by means of ω and the function χ of Art. 1.

11. First observe that as m is arbitrary, it may be so settled (x being given), as to assign any required value to y. This amounts to making x and y independent variables, and m a function of them. And in fact,

$$m = \frac{1 - xy}{2 \cos \alpha - (x + y)},$$

by which m may be eliminated, and e expressed in terms of x and y.

But, leaving m constant, put

$$x = \frac{\sin \omega}{\sin (\omega + \alpha)}, \quad y = \frac{\sin \theta}{\sin (\theta + \alpha)}, \quad e = \frac{\sin \eta}{\sin (\eta + \alpha)};$$

Now $d\omega + d\theta = \dfrac{\sin \alpha \, dx}{X} + \dfrac{\sin \alpha \, dy}{Y} = \dfrac{\sin \alpha}{v} \left(\dfrac{dx}{m - x} + \dfrac{dy}{m - y} \right)$

$$= \frac{-\sin \alpha}{v} d \log \{(m - x).(m - y)\} = \frac{-\sin \alpha}{v} d \log M = 0;$$

$\therefore \omega + \theta = \text{const.}$

When $\omega = 0$, $x = 0$; $\therefore y = e$, and $\theta = \eta$; or const. $= \eta$ and

$$\omega + \theta = \eta.$$

Again, $m^{-1} = 2 \cos \alpha - e = \dfrac{\sin (\eta + 2\alpha)}{\sin (\eta + \alpha)}$. To save room, let $\psi(\omega)$ denote the function of ω to which x is equal;

$$\left.\begin{aligned} &\therefore 2 \{\chi\omega + \chi\theta - \chi(\omega + \theta)\} = L(\psi\omega.\psi\theta) - 2L0 \\ &+ L \frac{\psi\omega}{\psi(\omega + \theta + \alpha)} + L \frac{\psi\theta}{\psi(\omega + \theta + \alpha)} - L \frac{\psi(\omega + \theta)}{\psi(\omega + \theta + \alpha)} \end{aligned}\right\} \ldots (19):$$

in which m and e or η have been eliminated, and ω, θ remain as independent variables.

Thus, if $\chi\omega$ and $\chi\theta$ are known for any particular values of ω and θ, $\chi(\omega + \theta)$ may be hence deduced. It is at once evident, that $\chi(\omega - \theta)$, $\chi(n\omega)$ and in fact $\chi\left(\dfrac{m}{n}\omega\right)$ may also be found, in finite terms, and by equations of the first degree. Yet this is rather a theoretical truth, than one of utility for calculation.

Since we already know Λx for the values $x = 1$, $x = 2 \cos \alpha$, $x^{-1} = 2 \cos \alpha$, $x = \cos \alpha$, $x^{-1} = \cos \alpha$; which correspond to $\omega = \frac{1}{2}(\pi - \alpha)$, $\omega = \pi - 2\alpha$, $\omega = \alpha$, $\omega = \frac{1}{2}\pi - \alpha$, $\omega = \pm \frac{1}{2}\pi$; we may start from any of these; and by addition or subtraction obtain a variety of results, and many more by combining division. If, however, we seek for $\chi(n\alpha)$ in this way, the result is far more complicated than in Art. 9. The cases

which may chiefly deserve to be pointed out as attainable, are the following:

$\omega = \pi - a$	$\omega = \frac{1}{2}\pi + a$	$\omega = \frac{1}{2}(\pi - 3a)$.
$\omega = \frac{1}{2}a$	$\omega = \frac{1}{4}\pi \pm \frac{1}{2}a$	$\omega = \frac{1}{4}\pi - a$.
$\omega = \pm\frac{1}{4}\pi$	$\omega = \frac{1}{4}(\pi \pm a)$	

The process of bisection, transferred to (18), consists in supposing Λe known, and making $x = y = m(1 \mp \sqrt{E})$; then $2\Lambda x$ is found from Λe.

As equation (18) is virtually an integral of the equation

$$\frac{dx}{X} + \frac{dy}{Y} = 0,$$

and contains an additional arbitrary constant m; no more general result is attainable in this direction. If we had assumed $X = v(m - x)(n - y)$, $Y = v(m - y)(n - y)$, the integral of $d(\Lambda x + \Lambda y)$, hence arising, would be a mere combination of (18) with (15), and would tell us nothing new.

§ III.—*On changing* $\Lambda(x, a)$ *to* $\Lambda(y, \beta)$.

12. The properties hitherto attained have more show of utility than they make good, in regard to the general reduction of $\Lambda(x, a)$ to another calculable function. In that respect more advantage is derived from changing a in Λ simultaneously with x.

With a view to this, put now Y for $y^2 - 2y\cos\beta + 1$.

For a first integration, *assume* $x\cos a + y\cos\beta = 1$, and $a + \beta = \frac{1}{2}\pi$;

$$\therefore\ X = \sin^2 a\,(x^2 + y^2),\ Y = \sin^2\beta\,(x^2 + y^2);$$

$$dlX = dlY = dl\left(1 + \frac{x^2}{y^2}\right) + dl.y^2.$$

$$\text{Also}\quad \frac{dx}{\sin a} + \frac{dy}{\sin\beta} = 0;\quad \frac{\sin a\,dx}{X} + \frac{\sin\beta\,dy}{Y} = 0.$$

$$\text{Now}\ \Lambda(x, a) - \Lambda(y, \beta) = \tfrac{1}{2}\int(lx - ly)\,dl(x^2 + y^2)$$

$$= \tfrac{1}{2}\int l\frac{x}{y}.dl\left(\frac{x^2}{y^2} + 1\right) + \int l\frac{x}{y}\,dly.$$

The former integral $= \frac{1}{4}L(-x^2y^{-2})$. In the latter, observe that $(xy^{-1}) = \tan a\,\{(y\cos\beta)^{-1} - 1\}$. Put $y\cos\beta = v^{-1}$, $dly = -dlv$,

$\therefore \int l(xy^{-1})\,dly = -\int\{l\tan a + l(v - 1)\}\,dlv$

$$= l\tan a.l(y\cos\beta) - L(1 - v) + \text{const.}$$

To correct, make $x = 0$, $y = \sec\beta$; then, since $v - 1 = \dfrac{x \cos\alpha}{y \cos\beta}$,

$$\left.\begin{aligned}&\Lambda(x, \alpha) - \Lambda(y, \beta) + \Lambda(\sec\beta, \beta)\\ &= \tfrac{1}{4}L\left(-\frac{x^2}{y^2}\right) - L\left(-\frac{x\cos\alpha}{y\cos\beta}\right) + l\tan\alpha ly\cos\beta + \tfrac{3}{4}L0\end{aligned}\right\} \dots (20).$$

To avoid the negatives under L, we may use the equation

$$L(-z) = lzl(1+z) - L(1+z) + L0;$$

reducing by which, we get for the right-hand member

$$\tfrac{1}{2}l(xy^{-1}).lX - \tfrac{1}{4}L(Xy^{-2}\cos^{-2}\beta) + L(y\cos\beta)^{-1}. \dots (20^*).$$

This may be named the *Complementary Equation.*† If in it we make $\beta = \frac{1}{3}\pi$, $\Lambda(y, \beta)$ is a known function of y. Hence we can by it determine $\Lambda(x, \frac{1}{6}\pi)$ as a known function of x.

If in the last, $x = \dfrac{\sin\omega}{\sin(\omega + \alpha)}$, $y = \dfrac{\sin\theta}{\sin(\theta+\beta)}$, then from $\dfrac{\sin\alpha dx}{X} + \dfrac{\sin\alpha dy}{Y} = 0$, we get $d\omega + d\theta = 0$, or $\omega + \theta = \text{const.}$ Let $y = 0$, $x\cos\alpha = 1$; $\sin\omega\cos\alpha = \sin(\omega+\alpha)$, or $\omega = \frac{1}{2}\pi$, and $\theta = 0$;

$$\therefore \omega + \theta = \tfrac{1}{2}\pi; \quad \alpha + \beta = \tfrac{1}{2}\pi; \quad \sin(\omega+\alpha) = \sin(\theta+\beta).$$

13. For a second integration, *assume*

$$x = 2y\cos\beta - 1, \quad \alpha = 2\beta;$$

$$\therefore X = 4Y\cos^2\beta, \quad dlX = dlY, \quad dx = 2dy\cos\beta;$$

$$\frac{\sin\alpha\, dx}{X} = \frac{2\sin\alpha\, dy\cos\beta}{4Y\cos^2\beta} = \frac{\sin\beta\, dy}{Y},$$

which gives $d\omega = d\theta$. When $\omega = 0$, $x = 0$, $y^{-1} = 2\cos\beta$, $\tan\theta = 2y\sin\beta$, or $\theta = \beta$; $\therefore \omega = \theta - \beta$; or $\omega + \alpha = \theta + \beta$. So much for the algebraic relations of the variables.

$$\text{Now } \Lambda(x, \alpha) - 2\Lambda(y, \beta) = \tfrac{1}{2}\textstyle\int \log(xy^{-2})\, dlY.$$

Let $y^{-1} = z$, $Z = z^2 - 2z\cos\beta + 1$; whence $Y = Zy^2$,

$$xy^{-2} = 1 - Z = (2y\cos\beta - 1)y^{-2};$$

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† In many formulas, it would appear that if our tables of Lx gave $\log x$ rather than x for the *argument*, this would be more convenient in the application. This suggests that *the same* table might give x, hyp log x and Lx.

$$\therefore \tfrac{1}{2}\int l(xy^{-2})\, dl Y = \tfrac{1}{2}\int l(1 - Z)\, dl Z + \int \log \frac{2y \cos \beta - 1}{y^2}\, dl y$$
$$= \tfrac{1}{2} L(1 - Z) + L(1 - 2y \cos \beta) - \log^2 y.$$

Hence

$$\Lambda(x, \alpha) - 2\Lambda(y, \beta) = \tfrac{1}{2} L(xy^{-2}) + L(-x) - \log^2 y + C. \ldots . (21)^*.$$

To find C, let $x = 0$, $y = \frac{1}{2}\sec\beta$; and after a few reductions, $C = \frac{1}{4}\alpha^2 + \frac{1}{12}\pi^2$. The same result is found by making $x = -1$, $y = 0$; but the infinites under L and *log* are a little more troublesome.

This may be called the *Equation of Bisection*, since $\alpha = 2\beta$, and since $\Lambda(x, \alpha) - 2\Lambda(y, \beta)$ is expressed in known functions.

It follows that if $\Lambda(x, \alpha)$ is a known function of x for some one value of α, so is $\Lambda(x, \frac{1}{2}\alpha)$. For we have only to make $x' = 2x \cos \frac{1}{2}\alpha - 1$, and determine $\Lambda(x, \frac{1}{2}\alpha)$ from $\Lambda(x', \alpha)$ by equation (21).

Hence $\Lambda\left(x, \dfrac{\pi}{2^n}\right)$ and $\Lambda\left(x, \dfrac{\pi}{3.2^n}\right)$ can be obtained in finite terms as known functions of x, since we know $\Lambda(x, \frac{1}{2}\pi)$ and $\Lambda(x, \frac{1}{3}\pi)$.

14. By a repeated use of the equation of bisection, it is evident that $\Lambda(x, \alpha)$ is reducible to $\Lambda(x_n, 2^{-n}\alpha)$, which, when $n = \infty$, is $\Lambda(x_n, 0)$ a known function. It may be worth while to enter into a few details concerning this.

Let α_n represent $2^{-n}\alpha$, and from x suppose $x_1, x_2 \ldots$ to be derived by the law

$$2x_1 \cos \alpha_1 = 1 + x;\quad 2x_2 \cos \alpha_2 = 1 + x_1;\quad \&c.\ \&c. \ldots .$$

It is easy to compute these by the intervention of ω. For we had

$$\omega + \alpha = \theta + \beta \text{ or } = \omega_1 + \alpha_1 = \omega_2 + \alpha_2 = \omega_3 + \alpha_3 = \&c.,$$
$$\text{whence } \omega_n = \omega + \alpha - \alpha_n.$$

Thus $x_n = \dfrac{\sin(\omega + \alpha - 2^{-n}\alpha)}{\sin(\omega + \alpha)}$, which, when $n = \infty$, converges to 1, and nearly $= 1 - 2^{-n}\alpha \cot(\omega + \alpha)$. (We must entirely except the case of $\omega + \alpha = 0$ or $= \pi$, which gives $x = \infty$.) Hence $2^n.\Lambda(x_n\ \alpha_n) = 2^n.\Lambda(x_n\ 0) = 2^n.\{Lx_n + \frac{1}{6}\pi^2\} = 2^n.(x_n - 1) + 2^n.\frac{1}{6}\pi^2 = -\alpha\cot(\omega + \alpha) + 2^n.\frac{1}{6}\pi^2$. Apply equation (21) n times: multiply the results by $2^0, 2^1, 2^2 \ldots 2^{n-1}$, and add all together. Substitute for $2^n.\Lambda(x_n\ \alpha_n)$ as above, and be careful to note that $2^{-2} + 2^{-3} + \ldots . + 2^{-\infty} = \frac{1}{2}$,

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* It is easy to combine equations (20), (21) with (13).

and $(2^0 + 2^1 + 2^2 + \ldots\ldots + 2^{n-1})\,\frac{1}{12}\pi^2 + 2^n . \frac{1}{6}\pi^2$

$$= \tfrac{1}{6}\pi^2 + (2^0 + 2^1 + \ldots\ldots + 2^{n-1})\,\tfrac{1}{4}\pi^2 :$$

after which, making $n = \infty$, we get

$$\left.\begin{aligned}\Lambda(x, a) = \tfrac{1}{2}a^2 - a \cot(\omega + a) + \tfrac{1}{6}\pi^2 & \\ + 2^0 \left\{\tfrac{1}{4}\pi^2 + L(-x) - \log^2 x_1 + \tfrac{1}{2} L(x\, x_1^{-2})\right\} & \\ + 2^1 \left\{\tfrac{1}{4}\pi^2 + L(-x_1) - \log^2 x_2 + \tfrac{1}{2} L(x_1\, x_2^{-2})\right\} & \\ + 2^2 \left\{\tfrac{1}{4}\pi^2 + L(-x_2) - \log^2 x_3 + \tfrac{1}{2} L(x_2\, x_3^{-2})\right\} & \\ + \&c.\ \&c.\ldots\ldots & \end{aligned}\right\} \ldots(22).$$

This always converges, yet not rapidly. When x_n is approaching its limit 1, we may approximately determine the remnant of the series, by the formulas

$$\left.\begin{aligned}& L(1-h) = -h - \tfrac{1}{4}h^2 - \tfrac{1}{9}h^3;\ \tfrac{1}{4}\pi^2 + L(-1+h) = \tfrac{1}{4}h^2 + \tfrac{1}{6}h^3; \text{ when } h \text{ is very small.} \\ & \log^2(1-h) = h^2 + h^3. \text{ Also } h_n = a_n\left\{(1 - \tfrac{1}{6}a_n^2)\cot(\omega + a) + \tfrac{1}{2}a_n\right\}; \\ & \text{and } h_{n+1} = (\tfrac{1}{2}h_n - \tfrac{1}{8}a_n^2)(1 + \tfrac{1}{8}a_n^2), \text{ very nearly.}\end{aligned}\right\}$$

But the great defect of the method is, that even if we start with x nearly $= 1$, we still do not any the more rapidly reach the limit $x_n = 1$: hence the series has no practical interest, unless indeed at once both x is very near to 1, and $\cos a$ between x and 1, a case which is the most troublesome of all in the method of Art. 16.

15. The equation of bisection would farther enable us to increase the number of functions $x = Fa$, which give $\Lambda(x, a)$ as a known function of a. For let $x = Fa$ be any one function, for which $\Lambda(x, a)$ is known; put $2x_1 \cos \frac{1}{2}a = 1 + x$, or put $x' = 2x \cos a - 1$; and $x_1\ x'$ are new functions of a, for which $\Lambda(x_1, \frac{1}{2}a)$ and $\Lambda(x', 2a)$ are known.

If we could integrate so as to obtain $\Lambda(x, a) + m\Lambda(y, \beta)$ in known functions, when $d\omega \propto d\theta$, by means of some *general* relations uniting a, β, m; it would more than anything else perfect what is wanting in this theory.

§ IV.—*To calculate* $\Lambda(x, a)$ *in any case.*

16. We have now the means of reducing $\Lambda(x, a)$ in all cases to another function $\Lambda(x', a')$ in which x' shall be less than $\frac{1}{2}$; which will enable us to apply equation (12).

Avoiding details, it will suffice here to shew the possibility of the transformation.

First, when $a > 60^\circ$; if x is > 1, we may reduce Λ to the case of $x < 1$ by equation (13). If then the new x is between $\frac{1}{2}$ and 1, put $x' = -x$, $a' = \pi - a = 2\beta$, $x' = 2y \cos\beta - 1$; $\Lambda(x, a) = \Lambda(x', a')$. Apply the equation of bisection to reduce $\Lambda(x, a)$ to $\Lambda(y, \beta)$. Now as a' is $< 120^\circ$, β is $< 60^\circ$, $2\cos\beta > 1$, $2y\cos\beta > y$; $\therefore 1 + x'$ or $1 - x > y$, or $y < \frac{1}{2}$.

Next, when a is $< 30^\circ$, put $a + \beta = \frac{1}{2}\pi$, and use the Complementary Equation. Since β is $> 60^\circ$, this case is reduced to the former.

Thirdly, when a is between 60° and 30°. Here x is by hypothesis between 1 and $\frac{1}{2}$, and $2\cos a$ between $\sqrt{3}$ and 1; so that $2x\cos a$ is between $\sqrt{3}$ and $\frac{1}{2}$.—We separate the case of $x > \cos a$; in which we can proceed exactly as when a was $> 60^\circ$. For since $2y \sin\frac{1}{2}a = 1 - x$, which is $< 1 - \cos a$ or than $2\sin^2\frac{1}{2}a$, $\therefore y$ is $< \sin\frac{1}{2}a < \frac{1}{2}$.—When x is *not* $> \cos a$, $2x\cos a$ does not exceed $2\cos^2 a$ or $\frac{3}{2}$; so that its limits are $\frac{3}{2}$ and $\frac{1}{2}$. Put $y = 2x\cos a - 1$, and y is between $+\frac{1}{2}$ and $-\frac{1}{2}$. If then $2a = \beta$, we can reduce by equation (21), only exchanging x with y, and a with β.

The simplicity of the coefficients in equation (12), which are known by common tables, would lead us to prefer that series when other things are equal. Yet if x is near to $\frac{1}{2}$, its convergence is not such as to give accuracy to many decimal places without great labour; and some of the following methods may become preferable.

§ V.—*To take advantage of a lying within certain limits.*

17. If a is extremely small, and x is $< \frac{1}{2}$; or if, x being near to 1, the product $2\sin\frac{1}{2}a \, . \left(\dfrac{x}{1-x}\right)$ is still very small.

Put $b = 2\sin\frac{1}{2}a$, $z = \dfrac{x}{1-x}$, or $x = \dfrac{z}{1+z}$; $1 - x = \dfrac{1}{1+z}$;

$$X = (1-x)^2 + b^2 x = (1-x)^2 . \{1 + b^2 z . (1+z)\}$$

$$d\log x = d\{\log z - \log(1+z)\} = \frac{dz}{z(1+z)};$$

$$\therefore \lambda(x, a) = \tfrac{1}{2}\textstyle\int_0 \log X \, d\log x = \int_0 \log(1-x)\, d\log x$$
$$+ \tfrac{1}{2}\textstyle\int_0 \log\{1 + b^2 z . (1+z)\} \dfrac{dz}{z(1+z)}$$
$$= L(1-x) + \tfrac{1}{2}P \ldots\ldots\ldots\ldots\ldots\ldots (23),$$

if $P = \int_0 \{b^2 - \frac{1}{2}b^4 z . (1+z) + \frac{1}{3}b^6 z^2 . (1+z)^2 - \&c.\ldots\} \, dz$
$$= b^2 z - \tfrac{1}{2}b^4(\tfrac{1}{2}z^2 + \tfrac{1}{3}z^3) + \tfrac{1}{3}b^6(\tfrac{1}{3}z^3 + 2\,\tfrac{1}{4}z^4 + \tfrac{1}{5}z^5) - \&c.\ldots (23^*),$$
which converges rapidly, since bz is very small.

18. If, on the contrary, a is very near to π (which is always the more favourable case, x being supposed positive), let $x = \tan^2 \frac{1}{2}\omega$; then $\lambda(x, a) = L(1 + x) - 2\Omega$,

$$\text{if } \Omega = -\tfrac{1}{2}\textstyle\int_0 \log\left(1 - \cos^2 \tfrac{1}{2}a \sin^2 \omega\right) \dfrac{d\omega}{\sin \omega}.$$

If we develop the logarithm, we readily see that Ω may take the form

$$A_0 - 2A_1 \cos \omega + 2A_3 \frac{\cos 3\omega}{3} - 2A_5 \frac{\cos 5\omega}{5} + \&c.$$

To find A_0, let $\omega = \frac{1}{2}\pi$, $\Omega = A_0$, $x = 1$, $\therefore \lambda(1, a) = L2 - 2A_0$;

whence $2A_0 = \dfrac{\pi^2}{12} + \Lambda(1, a) = \left(\dfrac{\pi - a}{2}\right)^2$.—Let $\pi - a = 4\beta$,

$$\therefore A_0 = 2\beta^2.$$

Next $\dfrac{d\Omega}{d\omega} = 2A_1 \sin \omega - 2A_3 \sin 3\omega + 2A_5 \sin 5\omega - \&c.$,

$$\text{also } \frac{d\Omega}{d\omega} = -\tfrac{1}{2}\log\left(1 - \sin^2 2\beta \,.\, \sin^2 \omega\right) \frac{1}{\sin \omega}.$$

Put $b = \tan \beta$, $\sin 2\beta = \dfrac{2b}{1 + b^2}$; and the value of $\sin \omega \,.\, \dfrac{d\Omega}{d\omega}$

$$\text{is } \log(1 + b^2) - \tfrac{1}{2}\log(1 + 2b^2 \cos 2\omega + b^4),$$

or $\log(1 + b^2) - b^2 \cos 2\omega + \frac{1}{2}b^4 \cos 4\omega - \frac{1}{3}b^6 \cos 6\omega + \&c.$, which is to be made equal to

$$2 \sin \omega \left\{A_1 \sin \omega - A_3 \sin 3\omega + A_5 \sin \omega - \&c.\right\},$$

or $A_1(1 - \cos 2\omega) - A_3(\cos 2\omega - \cos 4\omega) + A_5(\cos 4\omega - \cos 6\omega) - \&c.$
Hence we get $\quad A_1 = \log(1 + b^2)$,

$$A_1 + A_3 = \frac{b^2}{1};\ \text{and generally } A_{2n-1} + A_{2n+1} = \frac{b^{2n}}{n}.$$

In the First Part of these investigations we have used $\phi_n x$ to denote $\int_0 \tan^{n-1} x\, dx$; which yields $\phi_1 x = x$, $\phi_2 x = \frac{1}{2}\log(1 + \tan^2 x)$ or $\log \sec x$; and $\phi_n x + \phi_{n+2} x = \dfrac{\tan^n x}{n}$.

Thus $A_1 = 2\phi_2\beta$; $A_3 = 2\phi_4\beta$; $A_5 = 2\phi_6\beta$; &c. . . .

$$\left.\begin{aligned} \text{and } \lambda(x, a) = L(1 + x) - 4\beta^2 + 8\phi_2\beta \,.\, \frac{\cos \omega}{1} &- 8\phi_4\beta \,.\, \frac{\cos 3\omega}{3} \\ + 8\phi_6\beta \,.\, \frac{\cos 5\omega}{5} &- 8\phi_8\beta \,.\, \frac{\cos 7\omega}{7} \\ + \&c. \ldots \&c. \ldots & \end{aligned}\right\} \ldots (24),$$

which converges best when β is least, or a nearest to π.

19. To find Λ and λ, when a is near to $\frac{1}{2}\pi$.

Put $x = \tan(\frac{1}{4}\pi - \frac{1}{2}\omega)$;

$$\therefore \lambda(x, a) = \tfrac{1}{4}L(1+x^2) - \Omega, \text{ if } \Omega = \tfrac{1}{2}\int \log(1 - \cos a \cos\omega)\frac{d\omega}{\cos\omega}.$$

Assume $\Omega = C - C_0\omega - 2C_1 \sin\omega - 2C_2 \dfrac{\sin 2\omega}{2} - 2C_3 \dfrac{\sin 3\omega}{3} - \&c.$

To determine C, put $\omega = 0$, $x = 1$, $\Omega = C$, $\lambda(1, a) = \frac{1}{4}L2 - C$,

or $C = \frac{1}{4}L2 + \Lambda(1, a) = \frac{1}{16}3\pi^2 - \frac{1}{2}\pi a + \frac{1}{4}a^2$.

To determine C_0,

we have $-\dfrac{d\Omega}{d\omega} = C_0 + 2C_1 \cos\omega + 2C_2 \cos 2\omega + \&c. \dots$

Multiply by $d\omega$, and integrate from $\omega = 0$ to $\omega = \pi$, observing that $\int_0^\pi \cos n\omega\, d\omega = 0$, for all integer values of n; also $\int_0^\pi C_0 d\omega = \pi C_0$.

Then $\pi C_0 = \int_0^\pi -\dfrac{d\Omega}{d\omega}\, d\omega = \int_1^{-1} -\dfrac{d\Omega}{dx}\, dx = \Omega$ (from $x = -1$ to $x = 1$) $= \{\frac{1}{4}L2 - \lambda(1, a)\} - \{\frac{1}{4}L2 - \lambda(-1, a)\} = \Lambda(1, a) - \Lambda(1, \pi - a) = (\frac{1}{6}\pi^2 - \frac{1}{2}\pi a + \frac{1}{4}a^2) - (\frac{1}{4}a^2 - \frac{1}{12}\pi^2) = \frac{1}{4}\pi^2 - \frac{1}{2}\pi a.$

Whence $C_0 = \frac{1}{2}(\frac{1}{2}\pi - a)$. Call this γ. $\therefore C = \frac{1}{2}\pi\gamma + \gamma^2$.

Farther, put $c = \tan\gamma$, $\cos a = \sin 2\gamma = \dfrac{2c}{1+c^2}$;

$$\cos\omega . \frac{d\Omega}{d\omega} = \tfrac{1}{2}\log\left(1 - \frac{2c\cos\omega}{1+c^2}\right)$$

$$= -\tfrac{1}{2}\log(1+c^2) - c\cos\omega - \tfrac{1}{2}c^2\cos 2\omega - \tfrac{1}{3}c^3\cos 3\omega - \&c.\dots$$

But $-\cos\omega \dfrac{d\Omega}{d\omega} = \cos\omega\,\{C_0 + 2C_1\cos\omega + 2C_2\cos 2\omega + \&c.\dots\}$

$$= C_1 + (C_0 + C_2)\cos\omega + (C_1 + C_3)\cos 2\omega + (C_2 + C_4)\cos 3\omega + \&c.\dots$$

$\therefore\ C_1 = \frac{1}{2}\log(1+c^2) = \phi_2\gamma$; $\quad C_2 = c - C_0 = \tan\gamma - \gamma = \phi_3\gamma$;

$C_3 = \frac{1}{2}c^2 - C_1 = \phi_4\gamma$; $\quad C_4 = \frac{1}{3}c^3 - C_2 = \phi_5\gamma$; &c. . . .

$$\left.\begin{aligned} &\text{Whence } \lambda(x, a) = \tfrac{1}{4}L(1+x^2) - (\tfrac{1}{2}\pi\gamma + \gamma^2) + \gamma\omega \\ &+ 2\phi_2\gamma . \frac{\sin\omega}{1} + 2\phi_3\gamma . \frac{\sin 2\omega}{2} + 2\phi_4\gamma . \frac{\sin 3\omega}{3} + \&c. \end{aligned}\right\}\dots(25),$$

where $\gamma = \frac{1}{4}\pi - \frac{1}{2}a$, $x = \tan(\frac{1}{4}\pi - \frac{1}{2}\omega)$.

The convergence is rapid when a is very near to $\frac{1}{2}\pi$.

In equation (24), put $\omega = 0$, $\Omega = 0$;

$$\therefore \tfrac{1}{2}\beta^2 = \phi_2\beta - \tfrac{1}{3}\phi_4\beta + \tfrac{1}{5}\phi_6\beta - \&c.$$

In the value of $\dfrac{d\Omega}{d\omega}$ corresponding, make $\omega = \frac{1}{2}\pi$;

$$\therefore -\tfrac{1}{2}\log\cos 2\beta = \phi_2\beta + \phi_4\beta + \phi_6\beta + \&c.\dots$$

In equation (25), if we change γ into $-\gamma$, $\phi_{2n}\gamma$ remains unchanged, and $\phi_{2n-1}\gamma$ changes sign. By adding the two results thus obtained, we might easily reproduce equation (24).

Put $\omega = \pi$ in the value of $\dfrac{d\Omega}{d\theta}$ corresponding to (25);

$$\therefore \tfrac{1}{2}\log(1+\sin 2\gamma) = \phi_1\gamma - 2\phi_2\gamma + 2\phi_3\gamma - 2\phi_4\gamma + \&c.,$$

$$\text{so } \tfrac{1}{2}\log(1-\sin 2\gamma) = -\phi_1\gamma - 2\phi_2\gamma - 2\phi_3\gamma - 2\phi_4\gamma - \&c.;$$

which gives not only

$$-\tfrac{1}{4}\log\cos 2\gamma = \phi_2\gamma + \phi_4\gamma + \phi_6\gamma + \&c.,$$

but also $\frac{1}{2}\log\tan(\frac{1}{4}\pi+\gamma) = \phi_1\gamma + 2\phi_3\gamma + 2\phi_5\gamma + \&c.\ldots.$

These are mere properties of the functions $\phi_1, \phi_2, \phi_3 \ldots$ and can in several ways be verified.

The series (24), (25) cannot be practically used with advantage, unless we have tables of $\phi_n a$; but these might be computed with so much ease, within the limits $a = 0$, $a = 45°$, that this is apparently the best method of adding completeness to this branch of the calculus. The following section will shew that the use of ϕ_n is not confined to the particular cases contemplated in equations (24), (25).

§ VI.—*To find* Λ, *when* x *is near to* 1.

20. We shall suppose a to be $< 90°$, and deal with

$\Lambda(x, \pi - a)$ and $\Lambda(x, a)$ separately.

Put $\cos a = \dfrac{1-m^2}{1+m^2}$, or $m = \tan\frac{1}{2}a$; $X' = 1 + 2x\cos a + x^2$;

$$\therefore (1+m^2)X' = (1+x)^2 + m^2(1-x)^2. \quad \text{Let } y = \frac{1-x}{1+x}.$$

Then $\Lambda(x, \pi - a) = \frac{1}{2}\log x \log X' - L(1+x) + R$,

$$\text{if } R = \int \log\frac{1+m^2y^2}{1+m^2}\cdot\frac{dy}{1-y^2}.$$

$$\text{Assume } -\frac{dR}{dy} = M_0 - M_2y^2 + M_4y^4 - \&c.\ldots;$$

$$\therefore \log(1+m^2) - \frac{m^2y^2}{1} + \frac{m^4y^4}{2} - \frac{m^6y^6}{3} + \&c.\ldots.$$

$$= (1-y^2)\{M_0 - M_2y^2 + M_4y^4 - \&c.\ldots.\},$$

which gives

$M_0 = \log(1+m^2) = 2\phi_2\frac{1}{2}a$; $M_2 = 2\phi_4\frac{1}{2}a$; $M_4 = 2\phi_6\frac{1}{2}a$; &c....

$$\therefore\ 2lx = l(\cos^2 a - v^2) - l\left(\frac{1+y}{1-y}\right).$$

Say $\left.\begin{aligned} T &= \tfrac{1}{4}\int l(\cos^2 a - v^2)\, dl\,(v^2 + \sin^2 a); \\ U &= \tfrac{1}{4}\int_0 l\left(\frac{1+y}{1-y}\right) dl\,(y^2 + \tan^2 a); \end{aligned}\right\} \therefore \Lambda(x, a) = c + T - U.$

Let $\cos^2 a - v^2 = V$, $v^2 + \sin^2 a = 1 - V$, $T = \frac{1}{4}\int l V\, dl\,(1 - V) = \frac{1}{4}L(V)$. To find c, let $y = 0$, $\therefore \Lambda(\cos a, a) = c + \frac{1}{4}L \cos^2 a$, or $c = \frac{1}{6}\pi^2 - \frac{1}{2}\pi a + \frac{1}{2}a^2$. Observe also that $V = 1 - X$.

To find U, we have only to compare it with $\frac{1}{2}S$ of Art. 21, and write $\tan a$ for m; that is, a for $\frac{1}{2}a$. Hence if $u = y \cot a = (\cos a - x) \operatorname{cosec} a$,

$$\left.\begin{aligned} \Lambda(x, a) &= (\tfrac{1}{6}\pi^2 - \tfrac{1}{2}\pi a + \tfrac{1}{2}a^2) + \tfrac{1}{4}L(2x \cos a - x^2) \\ &\quad + a \tan^{-1} u + \frac{u}{1}\,\phi_1 a + \frac{u^3}{3}\,\phi_3 a + \frac{u^5}{5}\,\phi_5 a + \&c\ldots \end{aligned}\right\}(29).$$

§ VIII.—*Geometrical idea of the function* $y = \Lambda(x, a)$.

24. When a is $< 90°$, $\dfrac{dy}{dx}$ or $\dfrac{\log x\,.\,(x - \cos a)}{X}$ is positive from $x = 0$ to $x = \cos a$, and then negative until $x = 1$; after which it is perpetually positive. Thus Λx increases up to $\Lambda \cos a$, which is a maximum, and decreases down to $\Lambda 1$, which is (geometrically) a minimum. But it is not certainly a numerical minimum, if it has become negative.

Since $\Lambda 1 = \left(\dfrac{\pi - a}{2}\right)^2 - \dfrac{\pi^2}{12}$, this cannot be negative, unless $(\pi - a)^2 < \frac{1}{3}\pi^2$, or $a > (1 - 3^{-\frac{1}{2}})\,\pi$, which brings a near to the limit $\frac{1}{2}\pi$. If a is $< (1 - 3^{-\frac{1}{2}})\,\pi$, Λx never becomes negative; and $\Lambda 1$ is a numerical minimum.

When $x = 0$, $\dfrac{dy}{dx} = -\log x\,.\cos a = +\infty$ when a is $< 90°$, or is $-\infty$ when a is $> 90°$. When $x = \cos a$, or $x = 1$, $\dfrac{dy}{dx} = 0$. Again, the curve has an infinite branch corresponding to $x = \infty$, which gives $\Lambda x = 2\Lambda 1 + \frac{1}{2}\log^2 x$.

When a is $< 90°$, but so little less as to make $\Lambda 1$ negative, there are *two* values of x (one on each side of $x = 1$), such as to make $\Lambda x = 0$; besides the value $x = 0$.

When a is $> 90°$, $\dfrac{dy}{dx}$ is negative from $x = 0$ to $x = 1$, after which it is always positive; and, as before, Λx is positive infinity when $x = \infty$. There is then *one* value x that makes

the coefficients M_1, M_3, M_5. . . . become $1, \frac{1}{3}, \frac{1}{5}$. . . . so that the series always converges faster than

$$y + 3^{-2}y^3 + 5^{-2}y^5 + \ldots .\&c.\ldots.$$

and when x is $> \frac{1}{2}$, y^2 is $< \frac{1}{9}$; which is a far better convergence than we ordinarily get from equation (12).

22. We may increase the convergence (for the latter case only) by representing the given function as $\Lambda(x^2, 2a)$, and using the formula

$$\tfrac{1}{2}\Lambda(x^2, 2a) = \Lambda(x, \pi - a) + \Lambda(x, a),$$

taking $\Lambda(x, \pi - a)$ from (26) and $\Lambda(x, a)$ from (27). Thus if we wish to estimate $\Lambda(h, \mu)$, where h and μ are given, put $x^2 = h$, $2a = \mu$; then $y = \dfrac{1 - \sqrt{h}}{1 + \sqrt{h}}$, which is smaller than if we had made $x = h$, or $y = \dfrac{1 - h}{1 + h}$. In fact, this will enable us to restrict the use of equation (12) to the case of $x < \frac{1}{4}$; for if the variable is $> \frac{1}{4}$, call it x^2; $\therefore y^2$ is $< \frac{1}{9}$, and we find Λ by combining (26) and (27).

Supposing tables of ϕ_n to have been formed, it would perhaps be worth while, for the sake of the method just suggested, to add to them the values of f_n; where

$f_1 a = \cot a\, \phi_1 a + \phi_2 a$	$3f_3 a = \cot^3 a\, \phi_3 a - \phi_4 a$
$5f_5 a = \cot^5 a . \phi_5 a + \phi_6 a$	$7f_7 a = \cot^7 a\, \phi_7 a - \phi_8 a$
&c. . . .	&c. . . .

whence we obtain

$$\left.\begin{aligned} &\tfrac{1}{2}\Lambda(x^2, 4a) = a^2 + (\tfrac{1}{2}\pi - a)^2 \\ &+ lxl(1 + x) + \tfrac{1}{2}lxl(1 + 2x\cos 2a + x^2) - 2L(1 + x) \\ &+ 2a . \tan^{-1}(y \cot a) - 2\{yf_1 a + y^3 f_3 a + y^5 f_5 a + \ldots\} \end{aligned}\right\} \ldots (28).$$

This is more compact to the eye: yet we here lose the advantage of regularity in the decrease of the coefficients.

§ VII.—*To find Λ when x is near to $\cos a$.*

23. When x is near to $2 \cos a$, we may reduce $\Lambda(x, a)$ by means of equation (14); but no such property has occurred with reference to $\cos a$.

Let $y = 1 - \dfrac{x}{\cos a}$; $X = (y \cos a)^2 + \sin^2 a$;

$$dlX = dl(y^2 + \tan^2 a).$$

Put $y \cos a = v$, $x^2 = (\cos a - v)^2 = (\cos^2 a - v^2) \div \dfrac{\cos a + v}{\cos a - v}$;

binations, would appear to be particular cases of the more general problem, whose solution is $Q_{x,y,z}$.

The object of this paper is to assign $Q_{x,3,2}$; and to establish the following theorem:

If Q_x denote the greatest number of triads that can be formed with x symbols, so that no duad shall be twice employed, then

$$3Q_x = x\frac{x-1}{2} - V_x,$$

if for V_x we put $6k+4$, when $x = 6n-1$; $\frac{1}{2}x + 3k + 1$, when $x = 6n-2$; 0, when $x = 6n+1$ or $6n+3$; and $\frac{1}{2}x$, when $x = 6n$ or $6n+2$: where $2^m(2k+1) = n$; x, n, m, k, being all integers $\overline{\gtrless}\ 0$.

Q_x being defined as above,
let V_x be the number of duads possible with x things, that are excluded from Q_x.

Let Q_x' denote a system of triads formed with x things, in which no duad is twice employed, Q_x' being not of necessity a maximum, but such that v_x is the number of duads possible with x things, not employed in Q_x'.

The duads *bc cd d....r rs sb*, n duads in number, are a circle of n, or of n duads.

$(v_x = C_n)$ signifies, v_x can be made a circle of n; or Q_x' can be so formed, that v_x shall be left out, a circle of n duads.

$D_x = x\dfrac{x-1}{2}$, = the duads possible with x things.

It will create no inconvenience if we speak of V_x, Q_x, and D_x either as numbers, or as visible arrangements of things.

(A) Prop. If $V_x = 0$, $V_{2x+1} = 0$ and $(v_{2x-1} = C_{2x-2})$ $[x = 2n+1]$.

Q_x being formed of triads made with the x letters $ABC...$, let D_{x+1} be written out as follows, the $x+1$ letters being $a\,b\,c\,d$......Consider the line XX' as a circle continuous at XX', and divided into x parts:

$$\underline{X \qquad\qquad\qquad\qquad X'}$$

beginning at X, write out ab, ac, ad, &c., proceeding always to the right and in alphabetical order, and placing a duad under every division, as you proceed round the circle, except when it is necessary to omit a column (or division) in order to avoid repeating a letter in that column. By this means D_{x+1}, $(x+1 = 2m)$, will be written out in x columns, each containing all the $x+1$ letters a, b, c, d....and no duad will be twice written, nor will any column contain the same letter twice. This is the simplest of the various rules that may be

given for writing out D_{x+1} in x columns, so as to answer the conditions imposed. Before these x columns, in any order, place the x letters $A, B, C. \dots$ in succession as initial letters, completing with each letter (A) a column of $\frac{1}{2}(x+1)$ new triads. Add the x columns, of $\frac{1}{2}(x+1)$ triads each, to those already formed with $A, B, C. \dots$ in Q_x, and the sum will be Q_{2x+1}; for, since all the duads possible with A, B, C, &c., and all those possible with a, b, c, &c., have been exhausted, and since each of the letters A, B, C, &c. is combined with each of a, b, c, &c., $V_{2x+1} = 0$.

Collect now all the triads in which a and b are found, and erase the letters a and b. The duads that remain after that erasure will be a circle of $2x-2$ duads, and the triads remaining, containing only $2x-1$ letters, are Q'_{2x-1}. It would be absurd to multiply symbols in order to demonstrate this result; as the operation is merely mechanical, and proves itself on the first trial. An example will presently be given.

(B) Prop. If $(v_{x+1} = C_x)$, $V_{2x+1} = 0$,

and $v_{2x-1} = C_{2x-2})$ $[x = 2m]$.

Let $BC, CD, DE, E. \dots U, UB$ be the circle of x duads, and let A be the $(x+1)^{\text{th}}$ letter in Q'_{x+1}, which is supposed to be formed with the $x+1$ letters $A, B, C\dots$ Let $a, b, c\dots t$ be other x letters. Write out as before D_x $(x = 2m)$ in $x-1$ columns each containing all the x letters $a, b, c\dots$ Erase now the x duads $ab, bc, cd, d. \dots s, st, ta$, except the two last, st and ta, and write these $x-2$ duads in two additional columns, thus: placing the duads st, ta as erased duads below those two columns, (st) under ab, and (ta) under bc,*

$$\begin{array}{cc} ab & bc \\ cd & de \\ ef & fg \\ \vdots & \vdots \\ (st) & (ta) \end{array}$$

$D_x = \frac{1}{2}x + x\,\frac{1}{2}(x-2)$, is now written out in $x+1$ columns, one column containing $\frac{1}{2}x$ duads, in which all the x letters occur, and x columns containing each $(\frac{1}{2}x - 1)$ duads, and in each of which all the letters except two, viz. those in the erased duad, are found.

Using as a key, $\quad BCDE. \dots TU,$

$\qquad a\,b\,c\,d \dots s\,t,$

* Where (st), (ta) denote that st and ta are erased.

from the x triads following:

$$aBC, bCD, cDE, dEF. \ldots \ldots sTU, tUB.$$

Place now A as initial letter to the column of $\frac{1}{2}x$ duads; and A is thus combined with all the x letters $a, b, c. \ldots \ldots$ Place B as initial letter to that column of $\frac{1}{2}x - 1$ duads from which ta has been erased; C as initial to that column from which ab has been erased; D as initial to that from which bc has been erased; and so on. By this process, each of the letters $B, C, D. \ldots U$ completes a column of $\frac{1}{2}x - 1$ triads, and is thus combined with all the x letters $a, b, c. \ldots$ The new triads, in number $= x + D_x$, being added to Q'_{x+1}, will complete it into Q_{2x+1}; and $V_{2x+1} = 0$; because all the duads possible with the $2x + 1$ letters $A, B, C. \ldots a, b, c. \ldots$ have been exhausted.

The first of the new triads beginning with A will always be Aas, s being the $(x - 1)^{\text{th}}$ of the x letters $a, b, c. \ldots \ldots$ Collect now all the triads in which a and s are found; erase a and s; and the duads remaining will always be a circle of $2x - 2$ duads. This process being, like the last, merely mechanical, needs no demonstration besides inspection of an example.

E.g. Q_3 is completed into Q_7 by the addition of D_4, and Q_7 into Q_{15} by the addition of D_8, as follows.

$$\begin{array}{lll} Q_7 = ABC & & \\ \quad Aab & Bac & Cad \\ \quad Acd & Bbd & Cbc. \end{array}$$

D_8 may be written according to the rule given page 192, thus:

hi	*hk*	*hl*	*hm*	*hn*	*ho*	*hp*
—	—	*ik*	*il*	*im*	*in*	*io*
—	*ip*	—	—	*kl*	*km*	*kn*
ko	—	—	*kp*	—	—	*lm*
ln	*lo*	—	—	—	*lp*	—
—	*mn*	*mo*	—	—	—	—
mp	—	—	*no*	—	—	—
—	—	*np*	—	*op*	—	—:

or it may be thus written:

hi	*hk*	*hl*	*hm*	*hn*	*ho*	*hp*
kl	*il*	*ik*	*in*	*im*	*ip*	*io*
mn	*mo*	*mp*	*ko*	*kp*	*mk*	*nk*
op	*np*	*no*	*lp*	*lo*	*nl*	*ml,*

which is done by a rule that could easily be assigned, and is the more symmetrical of the two. But symmetry has little to do with these combinations: they are essentially unsymmetrical.

$Q_{15} = ABC$

ADE	BDF	CDG				
AFG	BEG	CEF				
Aab	Bac	Cad	Dae	Eaf	Fag	Gah
Acg	Bbh	Cbc	Dbd	Ebe	Fbf	Gbg
Adf	Bdg	Ceg	Dch	Ecd	Fce	Gcf
Aeh	Bef	Cfh	Dfg	Egh	Fdh	Gde.

Collecting the triads in which a and b occur, we have

Aab	Bac	Cad	Dae	Eaf	Fag	Gah
	Bbh	Cbc	Dbd	Ebc	Fbf	Gbg;

and, erasing a and b, we obtain

$$Bc\ \ cC\ \ Cd\ \ dD\ \ De\ \ eE\ \ Ef\ \ fF\ \ Fg\ \ gG\ \ Gh\ \ hB,$$

a circle of 12 duads.

Q'_{13} is completed into Q_{25} as follows. First, Q'_{13} is written in capitals:

$A'B'C'$

$A'D'E'$	$B'D'F'$	$C'D'G'$				
$A'F'G'$	$B'E'G'$	$C'E'F'$				
$A'CG$	$B'DG$	$C'EG$	$D'CH$	$E'CD$	$F'CE$	$G'CF$
$A'DF$	$B'EF$	$C'FH$	$D'FG$	$E'GH$	$F'DH$	$G'DE$
$A'EH$	$B'C$	$C'C$	$D'D$	$E'E$	$F'F$	$G'G$
	$B'H$	$C'D$	$D'E$	$E'F$	$F'G$	$G'H$.

Next, D_{12} is written thus:

C	F	C'	F'	D	G	D'	G'	E	A	E'	H	B'
(ab)	ac	ad	ae	af	ag	ah	ai	ak	al	am	ab	bc
cl	bm	(bc)	bd	be	bf	bg	bh	bi	bk	bl	cd	de
dk	dl	el	cm	(cd)	ce	cf	cg	ch	ci	ck	ef	fg
ei	ek	fk	fl	gl	dm	(de)	df	dg	dh	di	gh	hi
fh	fi	gi	gk	hk	hl	il	em	(ef)	eg	eh	ik	kl
gm	(gh)	hm	(hi)	im	(ik)	km	(kl)	lm	fm	(fg)	(lm)	(ma)

Using the key $B'CC'DD'EE'FF'GG'H$

$a\ b\ c\ d\ e\ f\ g\ h\ i\ k\ l\ m,$

... ...ads

$$C\ bC'C\ cC'D\ dDD'\ eD'E\ fEE'\ gE'F$$
$$hFF'\ iF'G\ kGG'\ lG'H\ mHB';$$

...n complete each of the above columns into triads by the ...dition of the letter marked over it, and Q_{25} is formed.

ABC												
ADE	BDF	CDG										
AFG	BEG	CEF										
ACG	BDG	CEG	DCH	ECD	FCE	GCF						
ADF	BEF	CFH	DFG	EGH	FDH	GDE						
AEH	BCa	CCb	DDd	EEf	FFh	GGk						
Aal	BHm	CDc	DEe	EFg	FGi	GHl						
Abk	Bbc	Cad	Dah	Eam	Fae	Gai	Hab	Eak	Daf	Cel	Fac	Gag
Aci	Bde	Cel	Dbg	Ebl	Fbd	Gbh	Hcd	Ebi	Dbe	Cdk	Fbm	Gbf
Adh	Bfg	Cfk	Dcf	Eck	Fcm	Gcg	Hef	Ech	Dgl	Cei	Fdl	Gce
Aeg	Bhi	Cgi	Dil	Edi	Ffl	Gdf	Hgh	Edg	Dhk	Cfh	Fek	Gdm
Afm	Bkl	Chm	Dkm	Eeh	Fgk	Gem	Hik	Elm	Dim	Cgm	Ffi	Ghl

And it is plain that $V_{25} = 0$.

If now the triads be collected in which the letters a and l occur, and those letters be erased, we obtain the following circle of 22 duads,

$$B'C\ Cc\ cF\ Fd\ dC'\ C'e\ eF'\ F'f\ fD\ Dg\ gG\ Gh$$
$$hD'\ D'i\ iG'\ G'H\ Hb\ bE'\ E'm\ mE\ Ek\ kB',$$

which proves that $(v_{23} = C_{22})$. In the same way this may be verified for any number.

Having established the two fundamental propositions,

$$\text{If } V_x = 0, \quad V_{2x+1} = 0, \quad \text{and } (v_{2x-1} = C_{2x-2};$$
$$\text{If } (v_{x+1} = C_x), \quad V_{2x+1} = 0, \quad \text{and } (v_{2x-1} = C_{2x-2};$$

we deduce the following, of which the first is self-evident:

$$V_3 = 0;$$

Because	$V_3 = 0,$	$V_7 = 0,$	and $(v_5 = C_4);$
,,	$(v_5 = C_4),$	$V_9 = 0,$	and $(v_7 = C_6);$
,,	$(v_7 = C_6),$	$V_{13} = 0,$	and $(v_{11} = C_{10});$
,,	$V_7 = 0,$	$V_{15} = 0,$	and $(v_{13} = C_{12});$
,,	$V_9 = 0,$	$V_{19} = 0,$	and $(v_{17} = C_{16});$
,,	$(v_{11} = C_{10}),$	$V_{21} = 0,$	and $(v_{19} = C_{18});$
,,	$(v_{13} = C_{12}),$	$V_{25} = 0,$	and $(v_{23} = C_{22});$ &c.

Generally, $V_{6n+1} = 0 = V_{6n+3}$, for all values of n.

(C) Prop. $V_{6n} = 3n$; and $V_{6n+2} = 3n + 1$).

For if x be $6n$ or $6n+2$, $V_{x+1} = 0$: also any letter a, occurring in Q_{x+1}, is found in $\frac{1}{2}x$ different triads, being combined in Q_{x+1} with x different letters. Let a be erased from Q_{x+1}: the remaining triads are Q_x, and the $\frac{1}{2}x$ duads that appear after the erasure are V_x, containing each of the x letters once. That this V_x is the least possible is plain from the following considerations.

Whether x is odd or even, any letter a that is found in Q_x m times, is therein combined with $2m$ different letters, and must appear in combination with $x-1-2m$ letters, i.e. $x-1-2m$, times, in V_x. When x is even this cannot be less than once.

Generally, any letter that is found in V_{2x}, is found in it an odd number of times; and any letter found in V_{2x+1} appears in it an even number of times.

The relation between Q_x and V_x may be expressed thus: Counting the duads, of which there are three in every triad, the following is always true, from the definitions of the terms:

$$3Q_x + V_x = D_x,$$

whether x is odd or even.

$$\text{If} \quad V_x = 0, \quad Q_x = \tfrac{1}{3}D_x = x \cdot \frac{x-1}{6}.$$

$$\text{If} \quad V_x = \tfrac{1}{2}x, \quad Q_x = \tfrac{1}{3}D_x - \frac{x}{6} = x \cdot \frac{x-2}{6};$$

$\therefore$ for the values of x, $x = 6n+1$, $x = 6n+3$, $x = 6n$, $x = 6n+2$;

$$Q_x = x \cdot \frac{x - 2^{\cos^2 x \frac{1}{2}\pi}}{6}.$$

To find Q_x when $x = 6n - 1$ or $6n - 2$, which are the only cases not included in the above formula, is a more difficult matter; and the results about to be offered, although they will perhaps be assented to as certain, will yet be found deficient in mathematical rigour.

To consider Q_{2x+1} in general: all those letters that appear in it less than x times will be found in V_{2x+1}; and no one that appears x times in Q_{2x+1} can be exhibited in V_{2x+1}. Let there be y different letters found, and $x - y$ different letters not found, in V_x: $(x = 2n + 1)$. Let the y letters be $a, b, c, \ldots$ and the $x - y$ be $A, B, C, \ldots$

The triads in Q_x will in general be of the four forms

$$(ABC),\ (ADe),\ (Afg),\ (acd).$$

Of these forms let the numbers of triads in Q_x be, in that order,

$$I_x \ \ M_x \ \ R_x \ \ T_x.$$

These numbers may vary much, as well as the number y, whilst the numbers x, Q_x, and V_x remain unchanged; so highly indeterminate is the structure of Q_x. But the following relations are true, of the duads employed.

(1) $$3I_x + M_x = D_{x-y},$$

for these duads are all exhausted in these two forms of triads;

(2) and $3T_x + R_x = D_y - V_x$.

(3) Also $3(I_x + M_x + R_x + T_x) = D_x - V_x = 3Q_x$,

(4) $\therefore\ 2(M_x + R_x) = D_x - (D_y + D_{x-y}) = y.(x-y)$.

If any relation can be assigned among these variables that shall lead to an equation determining V_x, a minimum, in terms of x; such an equation must be satisfied by $V_x = 0$, as well as by $V_x = fx$, fx being some simple function of the numbers x, 3, and 2; for V_x is not always reducible to 0: *e.g.* V_5. If now we suppose $y = V_x$, (4) − (2) gives

$$M_x - 3T_x = \tfrac{1}{2}y.(x-y) - D_y + V_x$$

$$= V_x.\frac{(x - V_x)}{2} - V_x.\frac{V_x - 1}{2} + V_x,$$

(5) or $$M_x - 3T_x = V_x.\left(\frac{x+3}{2} - V_x\right).$$

If either $M_x = 0 = T_x$ or $M_x = 3T_x$, we have an equation giving either $V_x = 0$ or $V_x = \tfrac{1}{2}(x+3)$. Since we know that the former is a minimum, we know that the conditions which lead to it are the conditions of a minimum in those cases in which $V_x = 0$ is possible; and this tempts us to conclude that we have hit upon the conditions of a minimum in general, and that the value of V_x is always $\tfrac{1}{2}(x+3)$, when it is not 0, (x being $2n+1$). Since this supposition is verified by trial in all cases that have been investigated, it is hoped that, until it is disproved by trial, the following theorem may be considered to have a good foundation,

$$(V_x =)\ V_{6n-1} = \frac{6n+2}{2} = 3n+1.$$

It is however to be remarked, that the equation $V_x = \tfrac{1}{2}(x+3)$ is true only of x, such that V_x is excluded in the exhaustion of D_x by the formation of its duads into the triads of Q_x. If Q_x is of the form $Q_x = Q_{x'} + D_{x-x'}$, when $D_{x-x'}$ is added entire to $Q_{x'}$, as in the process above described, in which $Q_{15} = Q_7 + D_8$, D_8 being added entire; we have no right to expect a formula exhibiting V_x as a function of x. V_x in this case is in reality $V_{x'}$, being formed during the exhaustion of

$D_{x'}$, and is a function, not of x, or of $x - x'$, but of x': hence the condition $M_x = 3T_x$ is to be understood of x' and Qx'.

$V_{6n'-1}$ is either V_{12n+5} or V_{12n+11}, as n' is odd or even.

(D) Prop. $V_{12n+5} = 6n + 4$.

Let Q_{6n+1} be formed: $V_{6n+1} = 0$. Let D_{6n+4} be written out with other $6n + 4$ letters a, b, c, &c., in $6n + 3$ columns, each containing all the $6n + 4$ letters. To these columns, in any order, let the letters $A, B, C. \ldots$, of which Q_{6n+1} is formed, be successively added as initials; then two of the $6n + 3$ columns will remain unemployed, and these $2.(3n + 2)$ duads are V_{12n+5} and $= 6n + 4$; and the $6n + 1$ columns of new triads being added to Q_{6n+1}, compose Q_{12n+5}. That V_{12n+5} is not $> 6n + 4$ is plain, because $V_{6n+1} = 0$; *i.e.* there are no duads made with *ABC*, &c. which are not employed in Q_{6n+1} which forms part of Q_{12n+5}; and each of $A, B, C. \ldots$ is combined, in the formation of the added triads, with each of a, b, c, &c. That V_{12n+5} not $< 6n + 4$ it is difficult to prove, otherwise than by the arguments already adduced, and by induction from trial. In the manner above pointed out of forming Q_{12n+5}, $M_x = T_x = 0$, and in any case, by a little transformation, the condition $M_x = 3T_x > 0$, can be shewn to obtain; but if any one denies that these, along with $V_x = y$, are the conditions of a minimum, and asserts that V_x may be reduced by some other arrangement of triads below $6n + 4$, I confess myself unable to prove the negative. It is easy to shew that V_x, $(x = 2n + 1)$, can always be made equal to y; or, in other words, that V_x can always, by simple transpositions, be made to exhibit as many letters as duads. If it could be established, that *in every case* T_x can be made $= 0$, V_x remaining the same, by dispersion of the triads of the form (abc) into other forms, it could be easily shewn that M_x can be made $= 0$, and it would then follow of necessity by (5) that $V_x = \frac{1}{2}(x + 3)$.

Q_{17} is here formed, by adding to Q_7 7 columns of D_{10}.

ABC								
ADE	*BDF*	*CDG*						
AFG	*BEG*	*CEF*						
Ahi	*Bhk*	*Chl*	*Dhm*	*Ehn*	*Fho*	*Ghp*	*hq*	*hr*
Akq	*Bir*	*Cik*	*Dil*	*Eim*	*Fin*	*Gio*	*ip*	*iq*
Alp	*Blq*	*Cmq*	*Dkr*	*Ekl*	*Fkm*	*Gkn*	*ko*	*kp*
Amo	*Bmp*	*Cnp*	*Dnq*	*Eoq*	*Flr*	*Glm*	*ln*	*lo*
Anr	*Bno*	*Cor*	*Dop*	*Epr*	*Fpq*	*Gqr*	*mr*	*mn*

Here $M_{17} = T_{17} = 0$, $V_{17} = 10 = y_{17}$.

Q_{17} is here differently formed:

ABL								
AFP	*BIc*	*LIr*						
AIk	*BFg*	*LFq*						
AOe	*BPm*	*LOd*	*FOk*	*Pln*				
Acm	*BOr*	*LPe*	*FIm*	*POc*				
Adn	*Bhk*	*Lck*	*Fch*	*Pdr*	*Ioq*			
Ahg	*Bdq*	*Lng*	*Fed*	*Pkg*	*Idg*	*Omg*	*cgr*	*eqm*
Aqr	*Ben*	*Lhm*	*Fnr*	*Pqh*	*Ieh*	*Onh*	*cqn*	*ekr* *kdm*.

Here $M_{17} = 15 = 3T_{17}$, $I_{17} = 2$, $R_{17} = 20$

$$\left\{\begin{array}{cc} \multicolumn{2}{c}{V_{17} = 10 = y_{17}} \\ qg & ge \\ ec & cd \\ dh & hr \\ rm & mn \\ nk & kq \end{array}\right\}$$

And again differently thus:

FIP							
FED	*IBL*	*PLE*					
FBg	*IEO*	*PDr*					
FAm	*IDn*	*POc*	*LOD*	*EBn*			
FOk	*IAk*	*PAn*	*LAr*	*EAh*	*DBh*	*OAq*	
FLq	*Icm*	*PBm*	*Lck*	*Ecg*	*DAg*	*OBr*	
Fch	*Ihg*	*Pkg*	*Lgn*	*Emq*	*Dcq*	*Omg*	*BAc*
Fnr	*Iqr*	*Pqh*	*Lhm*	*Ekr*	*Dkm*	*Onh*	*Bkq*

Here $I_{17} = 6$, $T_{17} = 0$, $M_{17} = 18$, $R_{17} = 18$,

$$\left\{\begin{array}{cc} \multicolumn{2}{c}{V_{17} = 10 = y + 2} \\ gq & kh \\ gr & mn \\ cr & cn \\ mr & kn \\ hr & qn \end{array}\right\}$$

We conclude then, that $V_{12n+5} = V_{6(2n+1)-1} = 3(2n+1) + 1 = 6n + 4$, or that when n is odd, $V_{6n-1} = 3n + 1$.

When n is even let $n = 2^m(2k+1)$, or let $2k + 1$ be the greatest odd factor in n. In that case

(E) $$V_{6n-1} = 6k + 4.$$

The truth of this depends on the truth of the following proposition

$$V_{2x+1} = V_{4x+3}.$$

That this is true when $V_{2x+1} = 0$, we have already (A) seen; for in that case Q_{4x+3} is completed by the addition of D_{2x+2} entire to Q_{2x+1}; and $V_{4x+3} = 0$. The same is certainly true in all cases, if it be allowed that Q_{4x+3} contains all the triads in Q_{2x+1}; for whatever V_{2x+1} may be, it cannot by hypothesis be reduced by the aid of the $2x + 1$ letters $A, B, C. \dots$ in Q_{2x+1}. If now D_{2x+2}, made with the $2x + 2$ letters $a, b, c. \dots$ be placed at our disposal, it is absurd to require more than that D_{2x+2} shall be exhausted as well as all the duads possible with A and $abc. \dots$, B and $abc. \dots$, C and $abc. \dots$, &c. Thus much we can always do; if then we add the D_{2x+2} new triads to Q_{2x+1} already formed, the sum will constitute Q_{4x+3}; and the duads V_{2x+1} are still what they were, and are all that remain to constitute V_{4x+3}. But as it is not in our power to prove that Q_{4x+3} must of necessity contain all the triads in Q_{2x+1} it may be denied, that as many duads must be excluded from the former as from the latter. Any person who thinks that $V_{23} < V_{11} < V_5$, or that $V_{35} < V_{17}$, is earnestly requested to try to shew it so to be; and till that is done, let the following proposition stand, as in all probability true;

$$V_{4x+3} = V_{2x+1} \,.\, (x = 2n + 1).$$

By the repeated application of this principle, it easily follows that $\quad V_{6n-1} = V_{6 \cdot 2^m \cdot (2k+1)-1} = V_{12k+5} = 6k + 4$, ($n$ being even): and ($m \gtreqless 0$),

(E)′ $\quad V_{6n-1} = V_{6 \cdot 2^m . (2k+1)-1} = \quad 6k + 4$, n even or odd, and $= 2^m . (2k + 1)$.

(F) Prop. If n is odd, $V_{6n-2} = 3n - 1 + \frac{1}{2}(3n - 1)$.

Let $n = 2p + 1$; $6n - 2 = 12p + 4$. Let Q_{6p+2} be formed with the $6p + 2$ letters $ABC. .$ Let D_{6p+2} made with other $6p + 2$ letters $abc. \dots \dots$ be written out in $6p + 1$ columns, each containing all the letters $abc. \dots \dots$ Prefix $6p + 1$ letters $ABC. \dots \dots$, as before pointed out; and $(6p + 1)(3p + 1)$ triads are thus completed, which, being added to Q_{6p+2}, will form Q_{12p+4} or Q_{6n-2}. Let S be the one of the $6p + 2$ letters $ABC. \dots$ which is not employed as an initial letter; then the duads Sa, Sb, Sc, &c. are to be added to V_{6p+2} before given, and the sum forms $V_{12p+4} = V_{6p+2} + 6p + 2$; which means $V_{6n-2} = 3n - 1 + \frac{1}{2}(3n - 1)$; since we have already proved that $V_{6p+2} = 3p + 1 = \frac{1}{2}(3n - 1)$. Hence V_{6n-2} is not

greater than $3n - 1 + \frac{1}{2}(3n - 1)$. Neither is it less: for if it can be reduced, this can only be done by the aid of some letters repeated more than once in V_{6n-2}; the only repeated letter is S, for no letter is repeated in V_{6P+2}; and it is plainly impossible with the aid of S to form a new triad. This reasoning, although it may convince most readers, may yet be opposed by any person who refuses to concede that Q_{12P+4} a maximum can comprehend the triads Q_{6P+2}. Concluding, then, (F) that when n is odd, $V_{6n-2} = 3n - 1 + \frac{1}{2}(3n - 1)$, we proceed to show that, when n is even, and if $n = 2^m(2k + 1)$, or if $2k + 1$ be the greatest odd factor in n,

(G) $$V_{6n-2} = 3k + 1 + 3n - 1.$$

This is deduced immediately from the following, which we have now to prove:

$$V_{4n+2} = V_{2n} + n + 1.$$

When $V_{2n} = n$, or when $2n = 6n$ or $= 6n + 2$, this expresses what we have already established. For any value of V_{2n} the proof is easy. Let Q_{2n} be formed of the $2n$ letters ABC, &c. Let D_{2n+2}, made with the $2n + 2$ letters abc &c., be written out as before, in $2n + 1$ columns, each containing all the letters abc, &c. By prefixing the $2n$ letters ABC, &c., $2n$ columns of triads may be completed, and can be added to Q_{2n}. The ABC, &c., are thus each of them combined with each of abc, &c. One column of $n + 1$ duads remains of D_{2n+2} unemployed, and these being added to V_{2n}, compose, with it, V_{4n+2}: and $V_{4n+2} = V_{2n} + n + 1$. For, plainly, V_{4n+2} is not greater than this: neither is it less. For it contains no duads but of the forms (AB) and (cd), of which forms the first cannot be reduced in number, because V_{2n} is a minimum; nor can the second, since no letters are found in that form, except what must perforce be exhibited, viz, each of the letters abc. . . .once.

$$\therefore\ V_{4n+2} = V_{2n} + n + 1.$$

Let $V_{2x} - x = \mu_{2x}$; then if $V_{2n} = n + \mu_{2n}$, $V_{4n+2} = n + n + 1 + \mu_{2n}$; but $V_{4n+2} - (2n + 1) = \mu_{4n+2}$; therefore the above may be expressed thus:

$$\mu_{4n+2} = \mu_{2n}.$$

Having already proved that (F)

$$\mu_{6n-2} = \frac{1}{2}(3n - 1) \text{ when } n \text{ is odd,}$$

by applying the above principle to $\mu_{6\cdot 2^m\cdot(2k+1)-2}$, we easily find

$$\mu_{6\cdot 2^m\cdot(2k+1)-2} = \mu_{6\cdot(2k+1)-2} = \frac{3.(2k+1)-1}{2} = 3k+1.$$

Hence, whether n is odd or even, if $n = 2^m.(2k+1)$, $(m \overline{>} 0)$,

$$\mu_{6n-2} = 3k+1.$$

Also we have before established that, n being even or odd,

$$V_{6n-1} = 6k+4. \quad (\text{E}').$$

Q_x will therefore be found in all cases from the equation

$$3Q_x = D_x - V_x,$$

if for V_x we put $6k+4$ when $x = 6n-1$; $\frac{1}{2}x + 3k + 1$ when $x = 6n-2$; 0, when $x = 6n+1$ or $6n+3$; and $\frac{1}{2}x$ when $x = 6n$ or $6n+2$; n being $2^m(2k+1)$. The theorem in page 192 is therefore established, unless the proof be thought imperfect in the case of $x = 6n-1$.

It is worthy of remark that V_x is always either 0 or $\frac{1}{2}x$, or one of the two simple functions $\frac{1}{2}(x+3)$, $\frac{3}{2}.\frac{1}{2}x$; Q_x being formed by one complete operation; *i.e.* not being of the form $Q_{x'} + D_{x-x'}$.

If a general analytical expression be desired for Q_x, the following is one of the various forms in which it may be written:

$$Q_x = x\,.\,\frac{x - 2^{\cos^2(\frac{1}{2}x\pi)}}{6} -$$

$$\frac{t}{6}\{(t-1).\cos^2(\tfrac{1}{2}m\pi).(6k+4) + (t+1).\sin^2.(\tfrac{1}{2}m\pi).(3k+1)\};$$

where $x = 3m \pm t$, $t < 2$, and $2^r.(2k+1) = m + \sin^2(\frac{1}{2}m\pi)$; x, m, t, r, k being all integers $\overline{>} 0$.

PROB. Required the method of constructing Q_x, x being any number.

First let x be odd; it will either be $6n+1$ or $6n+3$, or $6n-1$. Exclude the latter, and let the two principles in page 196 be thus written.

$$\text{If } V_{2n+1} = 0,\ V_{4n+3} = 0 \text{ and } (v_{4n+1} = C_{4n})$$

$$\text{If } (v_{2n+1} = C_{2n}),\ V_{4n+1} = 0 \text{ and } (v_{4n-1} = C_{4n-2}).$$

If x is $4n+3$, it is necessary that Q_{2n+1} should be first formed, and all that is then required is the addition of D_{2n+2} entire, with the initial letters affixed. If x is $4n+1$, it is required that Q'_{2n+1} be first formed, and that v_{2n+1} shall be appended to it, as in page 195, a circle of $2n$; the obtaining

of which circle depends on our having previously formed Q_{2n+3}; the formation of which will again depend on the question whether $2n+3$ is $4n'+1$ or $4n'+3$. The following rule is safe and easy. In order to find x_1, such that Qx_1 is required to be formed, before Q_x can be completed; if x is $4n+3$, subtract 1 and divide by 2, giving $x_1 = 2n+1$; if x is $4n+1$, add 5 and divide by 2, giving $x_1 = 2n+3$. By the same rule $x_{\prime\prime}$ is to be found from x_1, &c. Thus in order to form Q_{609}, we have the following numbers: 609, $\overline{307}$, 153, $\overline{79}$, 39, 19, 9, $\overline{7}$, 3. The mark (–) over a number shows that we want, not, *e.g.* Q_{79}, but Q'_{77}, in order to complete it into Q_{153}; but it will be found that the easiest way to obtain Q'_{77} is to construct Q_{79}, and then to follow the directions given (A) and (B) p. 193. This mark is placed over a number when the next higher number is of the form $4n+1$.

When x is $6n-1$, it is $12n'+5$ or $12n'+11$. If the former, $Q_{6n'+1}$ must be formed, and then completed into $Q_{12n'+5}$ as page 199 (D) points out.

If x is $12n'+11$, Q_{12k+5} $\{n' = 2^m.(2k+1)\}$ must be formed; and this being Qx', all that is necessary is to add $D_{2x'+1}$, $D_{4x'+3}$, &c, in succession, till Qx is completed.

Q_{2x}, when $2x = 6n$ or $6n+2$, is obtained by erasing any letter from Q_{2x+1}. When $2x = 6n-2$, the mode of construction is apparent from p. 201, and what is said above of the formation of Q_{6n-1}.

Croft Rectory, near Warrington, Dec. 23, 1846.

ON SYMBOLICAL GEOMETRY.

By Sir William Hamilton.

[*Continued from* p. 133.]

Symbolical Expressions and Investigations of some Properties of Cyclic Cones, with reference to their Tangent Planes.

22. If the side b of the cyclic cone be conceived to approach to the side a, and ultimately to coincide with it, the first equation (152) will take this limiting form:

$$\frac{\mathrm{b}''}{\mathrm{a}} = \frac{\mathrm{a}}{\mathrm{a}'}, \ldots\ldots\ldots\ldots\ldots (155);$$

which expresses the known theorem that the side of contact a bisects the angle between the traces a′ and b″ of the

tangent plane on the two cyclic planes; bisecting also the vertically opposite angle between the traces $-a'$ and $-b''$, but being pependicular to the bisector of either of the two other angles, which are supplementary to the two already mentioned, namely the angle between the traces a' and $-b''$, and that between $-a'$ and b''. And if in like manner we conceive the side d to approach indefinitely to the side c, the plane of these two sides will tend to become another tangent plane to the cone; of which plane the traces c' and d'' on the two cyclic planes will satisfy an equation of the same form as that last written, namely the following, which is the limiting form of the third equation (152):

$$\frac{d''}{c} = \frac{c}{c'} \ldots\ldots\ldots\ldots\ldots (156).$$

At the same time, the two secant planes bc and da will tend to coalesce in one secant plane, containing the two sides of contact a and c, with which the two other sides b and d tend to coincide; so that the traces d' and a'' of the latter secant plane, on the two cyclic planes, will ultimately coincide with the traces b' and c'' of the former secant plane on the same two cyclic planes; and the equations (148) (153) become:

$$\frac{a'}{b'} = \frac{b'}{c'}; \quad \frac{b''}{c''} = \frac{c''}{d''} \ldots\ldots\ldots (157);$$

which express that the traces b' and c'' of the one remaining secant plane bisect respectively the angles between the pairs of traces, a', c', and b'', d'', of the two tangent planes, on the two cyclic planes. And the two remaining equations (152) concur in giving this other equation:

$$\frac{c''}{c} = \frac{a}{b'} \ldots\ldots\ldots\ldots\ldots (158):$$

expressing that the rotations in the secant plane from b' to a and from c to c'', that is to say from one trace to one side, and from the other side to the other trace, are equal in amount, and similarly directed; in such a manner that these two traces b' and c'', of the secant plane on the two cyclic planes, are equally inclined to the straight line which bisects the angle between these two sides a and c, along which the plane cuts the cone: all which agrees with the known properties of cones of the second degree.

23. The eight straight lines a, c, a', b', c', b'', c'', d'', being supposed to be equally long, the first of them, which has

been seen to coincide in direction with the bisector of the angle between the third and sixth, can differ only by a scalar (or real and numerical) coefficient from their symbolic sum; because the diagonals of a plane and equilateral quadrilateral figure (or rhombus) bisect the angles of that figure. We have therefore, by (155), and by the supposition of the equal lengths of the eight lines,

$$\mathrm{a}' + \mathrm{b}'' \parallel \mathrm{a}; \quad \text{or,} \quad \mathrm{a}' + \mathrm{b}'' = l\mathrm{a} \, \ldots\ldots (159),$$

l being a numerical coefficient, and the sign of parallelism being designed to include the case of coincidence.

In like manner, by (156), we have

$$\mathrm{d}'' + \mathrm{c}' \parallel \mathrm{c}; \quad \text{or,} \quad \mathrm{d}'' + \mathrm{c}' = l' \, \mathrm{c} \, \ldots\ldots (160),$$

l' being another scalar coefficient. Again, by (157),

$$\left.\begin{array}{ll} \mathrm{c}' + \mathrm{a}' \parallel \mathrm{b}'; & \mathrm{c}' + \mathrm{a}' = m \, \mathrm{b}'; \\ \mathrm{b}'' + \mathrm{d}'' \parallel \mathrm{c}''; & \mathrm{b}'' + \mathrm{d}'' = m' \, \mathrm{c}''; \end{array}\right\} \ldots (161),$$

m and m' being two other scalars. But, by (158),

$$\frac{\mathrm{c}''}{\mathrm{c}} \, \frac{\mathrm{b}'}{\mathrm{a}} = 1 \, \ldots\ldots\ldots\ldots\ldots (162);$$

therefore

$$\frac{\mathrm{b}'' + \mathrm{d}''}{\mathrm{d}'' + \mathrm{c}'} \, \frac{\mathrm{c}' + \mathrm{a}'}{\mathrm{a}' + \mathrm{b}''} = \frac{m'}{l'} \, \frac{m}{l} = V^{-1} 0 \, \ldots\ldots (163);$$

this symbol $V^{-1}0$ denoting generally, in the present system, *any geometrical fraction of which the vector part is zero*, and therefore any positive or negative number (including zero). (Compare the definition and remarks in the 7th article).

By comparing this equation (163) with the first form (150), we see that the four straight lines,

$$- \mathrm{b}'', \mathrm{d}'', - \mathrm{c}', \mathrm{a}' \, \ldots\ldots\ldots\ldots (164),$$

which have been supposed to diverge from one common origin, namely the vertex of the cone, have their terminations on the circumference of one common circle. But these four lines, by supposition, are also equally long; they must therefore be four sides of a new cone, which is not only cyclic, as having a circular base, but is also a *cone of revolution*. The axis of revolution of this new cone is perpendicular to the plane of the circle in which the four lines (164) terminate; and this plane is parallel to the plane of the symbolic differences of those four lines, namely, the following,

$$\mathrm{d}'' + \mathrm{b}'', - \mathrm{c}' - \mathrm{d}'', \mathrm{a}' + \mathrm{c}', - \mathrm{b}'' - \mathrm{a}' \, \ldots (165);$$

but these have been seen to be parallel respectively to the four lines c″, c, b′, a, which are contained in the secant plane of the former cone; consequently the axis of revolution of the new cone is perpendicular to this secant plane. We arrive therefore, by this symbolical process, at a new proof of the known theorem, discovered by M. Chasles,* that two planes, touching a cyclic cone along any two sides, intersect the two cyclic planes in four right lines, which are sides of one common cone of revolution, whose axis of revolution is perpendicular to the plane of the two sides of contact of the former cone.

24. If we conceive the first and fourth of the sides (164) of the cone of revolution to tend to coincide with each other, then the fourth of the sides (165) of the plane quadrilateral inscribed in the circular base of that cone will tend to vanish; consequently the direction of this last mentioned side $-\mathrm{b}'' - \mathrm{a}'$, or the opposite direction of $\mathrm{a}' + \mathrm{b}''$, will become at last tangential to this circular base; and the plane of the two sides previously mentioned, namely $-\mathrm{b}''$ and a′, which plane has been seen to touch the cyclic cone along the side a, will become ultimately tangential also to the cone of revolution, touching it along the line a′, which becomes one trace of the second cyclic plane on the first cyclic plane; the opposite line, $-\mathrm{a}'$, being of course also situated in the intersection of those two planes, so that it may be regarded as the opposite trace of one cyclic plane on the other. Thus, at the limit here considered, the equation (155) and the second equation (157) are replaced by the equations

$$\frac{-\mathrm{a}'}{\mathrm{a}} = \frac{\mathrm{a}}{\mathrm{a}'}, \quad \frac{-\mathrm{a}'}{\mathrm{c}''} = \frac{\mathrm{c}''}{\mathrm{d}''} \dots\dots\dots (166);$$

of which the first expresses that the side a is equally inclined to the two opposite traces, a′ and $-\mathrm{a}'$; while the numerical coefficient l vanishes, and the formula (163) is replaced by this other,

$$V.\frac{\mathrm{d}'' - \mathrm{a}'}{\mathrm{d}'' + \mathrm{c}'}\,\frac{\mathrm{c}' + \mathrm{a}'}{\mathrm{a}} = 0 \dots\dots\dots (167).$$

We see also that the two rectangular but equally long lines a, a′, of which the former is a side of the cyclic cone,

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* See the Translation of Two Geometrical Memoirs by M. Chasles, on the General Properties of Cones of the Second Degree, and on the Spherical Conics; which Translation was published, with an Appendix, by the Rev. Charles Graves, in Dublin, 1841.

while the latter is part of the line of intersection of the two cyclic planes of that cone, are such that their plane is a common tangent to both the cyclic cone and the cone of revolution; which latter cone has also, as sides of the same sheet with a′, the two other of the four lines (164), namely the lines $-$ c′ and d″. Indeed, the formula (167) is sufficient to show, by comparison with the first formula (150), that if the three straight lines a′, d″, $-$ c′ be still supposed to diverge from one common origin, the circle passing through the three points in which they terminate is touched, at the termination of the line a′, by a straight line parallel to the line a; and therefore that the cone of revolution, having these three equally long lines a′, $-$ c′, d″ for sides of one common sheet, is touched along the side a′ by the plane which contains the two rectangular lines aa′; so that we may regard this formula (167) as containing the symbolical solution of the problem, to draw a tangent plane, along any proposed side, to the cone of revolution which passes through that side and through two other sides also given, and belonging to the same sheet as the former. Now if three such sides be connected by three planes, forming three faces of a triangular pyramid, inscribed in a single sheet of a cone of revolution, and having its vertex at the vertex of that cone, while the sheet is touched by a fourth plane along one edge of the pyramid, it follows from the most elementary principles of solid geometry, that the difference between the two exterior angles which the faces meeting at that edge make with the tangent plane to the cone is equal to the difference of the two interior angles which the same two faces make with the third face of the pyramid; the greater exterior angle being the one which is the more remote from the greater interior angle; as may be shown by conceiving three planes to pass through the three edges respectively, and through the axis of revolution of the cone. The same equality between the differences of these two pairs of angles between planes, will become still more evident if, without making use of any formula of spherical trigonometry, we consider a spherical triangle inscribed in a small circle on the sphere, which small circle is touched at one corner of the triangle by a great circle, while arcs are drawn to that and to the two other corners from a pole of the small circle; the only principles required being these: that the base angles of a spherical isosceles triangle are equal, and that the arcs from the pole of a small circle are all perpendicular to its perimeter. If then

we denote by the symbol $\angle$ (a, b, c) the acute or right or obtuse dihedral or spherical angle, at the edge b, between the planes ab and bc, in such a manner as to write, generally,

$$\angle(a, b, c) = \angle(c, b, a) = \angle(-a, b, -c) = \angle(a, -b, c)$$
$$= \pi - \angle(a, b, -c) \ldots\ldots\ldots\ldots\ldots (168);$$

π being the symbol for two right angles, we shall have, in the present question, the equation

$$\angle(a', d'', -c') - \angle(a', -c', d'') = \angle(-a, a', -c') - \angle(a, a', d'') \,..(169);$$

and therefore, by subtracting both members from π,

$$\angle(a', d'', c') + \angle(a', c', d'') = \angle(-a, a', c') + \angle(a, a', d'') \,..(170).$$

We have also here the relation

$$\angle(c', a', d'') = \angle(a, a', c') + \angle(a, a', d'') \,..\, (171),$$

because the plane aa′ is intermediate between the planes a′c′ and a′ d″, or lies *within* the dihedral angle (c′, a′, d″) itself, and not within either of the two angles which are exterior and supplementary thereto; which again depends on the circumstance that both the cyclic planes are necessarily exterior to each sheet of the cyclic cone. Adding therefore the equations (170) and (171), member to member, and subtracting π on both sides of the result, we find for the *spherical excess* of the new triangular pyramid (a′, c′, d″), or for the excess of the sum of the mutual inclinations of its three faces a′c′, a′d″, c′d″, above two right angles, the expression:

$$\angle(a', d'' c') + \angle(a', c', d'') + \angle(c', a', d'') - \pi = 2\angle(a, a', d'') \,..(172).$$

This spherical excess therefore remains unchanged, while the two lines c′, d″, move together on the two cyclic planes, in such a manner that their plane, always passing through the vertex of the cone, continues to touch that cyclic cone; a′ being still a line situated in the intersection of the two cyclic planes, and a being still a side of contact of the cone with a plane drawn through that intersection. And hence, or more immediately from the equation (170), the known property of a cyclic cone is proved anew, that the sum of the inclinations (suitably measured) of its variable tangent plane to its two fixed cyclic planes is constant.

(*To be continued.*)

ON THE DIFFERENTIAL EQUATIONS WHICH OCCUR IN DYNAMICAL PROBLEMS.

By Arthur Cayley.

Jacobi, in a very elaborate memoir, "Theoria novi multiplicatoris systemati æquationum differentialium vulgarium applicandi,"* has demonstrated a remarkable property of an extensive class of differential equations, namely, that when all the integrals of the system except a single one are known, the remaining integral can always be determined by a quadrature. Included in the class in question are, as Jacobi proceeds to shew, the differential equations corresponding to any dynamical problem in which neither the forces nor the equations of condition involve the velocities; *i.e.* in all ordinary dynamical problems when all the integrals but one are known the remaining integral can be determined by quadratures. In the case where the forces and equations of condition are likewise independent of the time, it is immediately seen that the system may be transformed into a system in which the number of equations is less by unity than in the original one, and which does not involve the time, which may afterwards be determined by a quadrature,† and Jacobi's theorem applying to this new system, he arrives at the proposition "In any dynamical problem where the forces and equations of condition contain only the coordinates of the different points of the system, when all the integrals but two are determined, the remaining integrals may be found by quadratures only. In the following paper, which contains the demonstrations of these propositions, the analysis employed by Jacobi has been considerably varied in the details, but the leading features of it are preserved.

§ 1. Let the variables $x, y, z \ldots$ &c. be connected with the variables $u, v, w \ldots$ by the same number of equations, so that the variables of each set may be considered as functions of those of the other set. And assume

$$dx\, dy \ldots = \nabla\, du\, dv \ldots$$

If from the functions which equated to zero express the relations between the two sets of variables we form two

* *Crelle*, tom. XXVII. p. 199, and tom. XXIX. pp. 213 and 333. Compare also the memoir in *Liouville*, tom. X. p. 337.

† For, representing the velocities by $x', y' \ldots$ the dynamical system takes the form $dt : dx : dy \ldots : dx' : dy' \ldots = 1 : x' : y' \ldots : X : Y \ldots$ and the system in question is simply $dx : dy \ldots : dx' : dy' \ldots = x' : y' \ldots : X : Y \ldots$

[redacted]

determinants, the former with the differential coefficients of these functions with respect to u, v,....and the latter with the differential coefficients of the same functions with respect to x, y,......the quotient with its sign changed obtained by dividing the first of these determinants by the second is, as is well known, the value of the function ∇.

Putting for shortness

$$\frac{dx}{du}=\alpha,\quad \frac{dy}{du}=\beta\ldots.\quad \frac{dx}{dv}=\alpha',\quad \frac{dy}{dv}=\beta'\ldots.\ \&c.$$

and
$$\frac{du}{dx}=A,\quad \frac{du}{dy}=B\ldots.\quad \frac{dv}{dx}=A',\quad \frac{dv}{dy}=B'\ldots.$$

∇ is the reciprocal of the determinant formed with A, B...; A', B'......, &c. Or it is the determinant formed with α, β,....α', β'...., &c.

From the first of these forms, *i.e.* considering ∇ as a function of A, B..

$$\frac{d\nabla}{dA}=-\nabla\alpha,\quad \frac{d\nabla}{dB}=-\nabla\beta\ldots.\quad \frac{d\nabla}{dA'}=-\nabla\alpha',\quad \frac{d\nabla}{dB'}=-\nabla\beta',$$

where the quantities α, β,..α', β',..and A, B,..A', B',.. may be interchanged provided $-\nabla$ be substituted for ∇. (Demonstrations of these formulæ or of some equivalent to them will be found in Jacobi's memoir "De determinantibus functionalibus," Crelle, t. XXII).

Hence

$$\frac{1}{\nabla}\,d\nabla+\alpha dA+\beta dB\ldots.+\alpha' dA'+\beta' dB'\ldots.=0.$$

or reducing by

$$\frac{dA}{dy}=\frac{dB}{dx}\ldots.\quad \frac{dA'}{dy}=\frac{dB'}{dx}\ldots.\ \&c.$$

this becomes

$$\left.\begin{aligned}&\frac{1}{\nabla}\,d\nabla+\alpha\left(\frac{dA}{dx}\,dx+\frac{dB}{dx}\,dy+\ldots\right)+\beta\left(\frac{dA}{dy}\,dx+\frac{dB}{dy}\,dy+\ldots\right)..\\&\quad+\alpha\left(\frac{dA'}{dx}\,dx+\frac{dB'}{dx}\,dy+\ldots\right)+\beta'\left(\frac{dA'}{dy}\,dx+\frac{dB'}{dy}\,dy+\ldots\right)..\end{aligned}\right\}=0,$$

Or reducing

$$\frac{1}{\nabla}\,d\nabla+\left(\frac{dA}{du}+\frac{dA'}{dv}+..\right)dx+\left(\frac{dB}{du}+\frac{dB'}{dv}+..\right)dy+..=0\,;$$

whence separating the differentials and replacing A, A',.... B, B',.... by their values

$$\frac{1}{\nabla}\frac{d\nabla}{dx}+\frac{d}{du}\cdot\frac{du}{dx}+\frac{d}{dv}\cdot\frac{dv}{dx}+\ldots=0,$$

$$\frac{1}{\nabla}\frac{d\nabla}{dy}+\frac{d}{du}\cdot\frac{du}{dy}+\frac{d}{dv}\cdot\frac{dv}{dy}+\ldots=0,$$

$\vdots$

ch $-\nabla, u, v \ldots x, y \ldots$ may be substituted for $\nabla, x, y \ldots$ $v \ldots$).

§ 2. Let $X, Y \ldots$ be any functions of the variables $x, y \ldots$ ..d assume

$$U = X\frac{du}{dx}+Y\frac{du}{dy}+\ldots$$

$$V = X\frac{dv}{dx}+Y\frac{dv}{dy}+\ldots$$

$U, V, \ldots$ being expressed in terms of $u, v, \ldots$ Then

$$\ldots+\frac{dV}{dv}+\ldots$$

$$= X\left(\frac{d}{du}\cdot\frac{du}{dx}+\frac{d}{dv}\cdot\frac{dv}{dx}+\ldots\right)+Y\left(\frac{d}{du}\cdot\frac{du}{dy}+\frac{d}{dv}\cdot\frac{dv}{dy}+\ldots\right)\ldots$$

$$+\left(\frac{dX}{du}\cdot\frac{du}{dx}+\frac{dX}{dv}\cdot\frac{dv}{dx}+\ldots\right)+\left(\frac{dY}{du}\cdot\frac{du}{dy}+\frac{dY}{dv}\cdot\frac{dv}{dy}+\ldots\right)\ldots$$

i.e. $\nabla\cdot\left(\frac{dU}{du}+\frac{dV}{dv}+\ldots\right)$

$$= -\left(X\frac{d\nabla}{dx}+Y\frac{d\nabla}{dy}+\ldots\right)+\nabla\cdot\left(\frac{dX}{dx}+\frac{dY}{dy}+\ldots\right).$$

Also, whatever be the value of M,

$$U\frac{dM\nabla}{du}+V\frac{dM\nabla}{dv}+\ldots = X\cdot\frac{dM\nabla}{dx}+Y\frac{dM\nabla}{dy}+\ldots$$

And from these two properties,

$$\frac{dM\nabla U}{du}+\frac{dM\nabla V}{du}+\ldots=\nabla\cdot\left(\frac{dMX}{dx}+\frac{dMY}{dy}+\ldots\right).$$

§ 3. Consider the system of differential equations

$$dx : dy : dz \ldots = X : Y : Z \ldots$$

(where, for greater clearness, an additional letter z has been introduced). From these we deduce the equivalent system

$$du : dv : dw \ldots = U : V : W \ldots$$

Suppose that u and v continue to represent arbitrary functions of x, y, z, but that the remaining function $w, \ldots$ is such as to

satisfy $W=0$,...(so that w,...may be considered as the constants introduced by obtaining all the integrals but one of the system of differential equations in x, y, z...), we have

$$\frac{dM\nabla U}{du}+\frac{dM\nabla V}{dv}=\nabla\cdot\left(\frac{dMX}{dx}+\frac{dMY}{dy}+\frac{dMZ}{dz}+\ldots\right).$$

Also the only one of the transformed equations which remains to be integrated is

$$du : dv = U : V, \quad \text{or} \quad Vdu - Udv = 0,$$

(in which it is supposed that U and V are expressed by means of the other integrals in terms of u and v).

Suppose M can be so determined that

$$\frac{dMX}{dx}+\frac{dMY}{dy}+\frac{dMZ}{dz}+\ldots=0.$$

(M is what Jacobi terms the multiplier of the proposed system of differential equations). Then

$$\frac{dM\nabla U}{du}+\frac{dM\nabla V}{dv}=0,$$

or $M\nabla$ is the multiplier of $Vdu - Udv = 0$, so that

$$\int M\nabla\,(Vdu - Udv) = \text{const.}$$

Hence the theorem :—"Given a multiplier of the system of equations $dx : dy : dz. \ldots = X : Y : Z. \ldots$(the meaning of the term being defined as above), then if all the integrals but one of this system are known, the remaining integral depends upon a quadrature."

Jacobi proceeds to discuss a variety of different systems of equations in which it is possible to determine the multiplier M. Among the most important of these may be considered the system corresponding to the general problem of Dynamics, which may be discussed under three different forms.

§ 4. Lagrange's first form.*

Let the *whole series* of coordinates, each of them multiplied by the square roots of the corresponding masses, be represented by $x, y \ldots$ and in the same way the whole series of forces, each of them multiplied by the square roots of the corresponding masses, by $P, Q \ldots$; then the equations of motion are

$$\frac{d^2x}{dt^2}=X, \quad \frac{d^2y}{dt^2}=Y\ldots$$

* I have slightly modified the form so as to avoid the introduction of the masses, and to allow x for instance to stand for any one of the coordinates of any of the points, instead of a coordinate parallel to a particular axis.

where
$$X = P + \lambda \frac{d\Theta}{dx} + \mu \frac{d\Phi}{dx} \ldots$$
$$Y = Q + \lambda \frac{d\Theta}{dy} + \mu \frac{d\Phi}{dy} \ldots$$
$$\vdots$$

where $\Theta = 0$, $\Phi = 0$ .... are the equations of condition connecting the variables, and λ, μ ...... coefficients to be determined by substituting the values of $\frac{d^2x}{dt^2}$ &c. in the equations $\frac{d^2\Theta}{dt^2} = 0$, $\frac{d^2\Phi}{dt^2} = 0$ &c. It is supposed that as well P, Q ..... as Θ, Φ ..... are independent of the velocities.

In order to reduce these to an analogous form to that previously employed, we have only to write
$$\frac{dx}{dt} = x', \quad \frac{dy}{dt} = y', \ldots$$
which gives
$$dt : dx : dy : dz \ldots : dx' : dy' : dz' \ldots$$
$$= 1 : x' : y' : z' \ldots : X : Y : Z \ldots$$
Supposing that M is independent of x', y', z', ... the equation on which it depends becomes immediately
$$\partial M + M \left(\frac{dX}{dx'} + \frac{dY}{dy'} + \frac{dZ}{dz'} \ldots\right) = 0,$$
where for shortness
$$\partial = \frac{d}{dt} + x' \frac{d}{dx} + y' \frac{d}{dy} + z' \frac{d}{dz} + \ldots$$
To reduce this we must first determine the values of λ, μ .., and for this we have
$$\frac{d^2\Theta}{dt^2} = \delta'\Theta + \frac{d\Theta}{dx} \cdot \frac{d^2x}{dt^2} + \frac{d\Theta}{dy} \cdot \frac{d^2y}{dt^2} + \ldots = 0, \text{ \&c.}$$
$$i.e. \quad \delta'\Theta + P \frac{d\Theta}{dx} + Q \frac{d\Theta}{dy} + \ldots + a\lambda + h\mu + g\nu \ldots = 0,$$
$$\delta'\Phi + P \frac{d\Phi}{dx} + Q \frac{d\Phi}{dy} + \ldots + h\lambda + b\mu + f\nu \ldots = 0,$$
$$\delta'\Psi + P \frac{d\Psi}{dx} + Q \frac{d\Psi}{dy} + \ldots + g\lambda + f\mu + c\nu \ldots = 0.$$
$$\vdots$$

[redacted]

where for greater clearness an additional letter of the series $\Theta, \Phi \ldots$ has been introduced, and where

$$a = \left(\frac{d\Theta}{dx}\right)^2 + \left(\frac{d\Phi}{dy}\right)^2 + \ldots.$$

$$b = \left(\frac{d\Phi}{dx}\right)^2 + \left(\frac{d\Phi}{dy}\right)^2 + \ldots.$$

$$\vdots$$

$$h = \left(\frac{d\Theta}{dx}\cdot\frac{d\Phi}{dx} + \frac{d\Theta}{dy}\cdot\frac{d\Phi}{dy}\right) + \ldots.$$

$$\vdots$$

Hence differentiating with respect to x',

$$2\delta\frac{d\Theta}{dx} + a\frac{d\lambda}{dx'} + h\frac{d\mu}{dx'} + g\frac{d\nu}{dx'} \ldots. = 0,$$

$$2\delta\frac{d\Phi}{dx} + h\frac{d\lambda}{dx'} + b\frac{d\mu}{dx'} + f\frac{d\nu}{dx'} \ldots. = 0,$$

$$2\delta\frac{d\Psi}{dx} + g\frac{d\lambda}{dx'} + f\frac{d\mu}{dx'} + c\frac{d\nu}{dx'} \ldots. = 0.$$

$$\vdots$$

Or representing by K the determinant formed with the quantities $a, h, g \ldots$; $h, b, f \ldots$ $g, f, c \ldots$ and by $A, H, G \ldots.$ $H, B, F \ldots. G, F, C \ldots.$ the inverse system of coefficients, we have

$$2\left(A\delta\frac{d\Theta}{dx} + H\delta\frac{d\Phi}{dx} + G\delta\frac{d\Psi}{dx}\ldots\right) + K\frac{d\lambda}{dx'} = 0,$$

$$2\left(H\delta\frac{d\Theta}{dx} + B\delta\frac{d\Phi}{dx} + F\delta\frac{d\Psi}{dx}\ldots\right) + K\frac{d\mu}{dx'} = 0,$$

$$2\left(G\delta\frac{d\Theta}{dx} + F\delta\frac{d\Phi}{dx} + C\delta\frac{d\Psi}{dx}\ldots\right) + K\frac{d\nu}{dx'} = 0;$$

$$\vdots$$

whence multipling by $\frac{d\Theta}{dx}, \frac{d\Phi}{dx}, \frac{d\Psi}{dx} \ldots.$ and adding

$$A\delta\left(\frac{d\Theta}{dx}\right)^2 + B\delta\left(\frac{d\Phi}{dx}\right)^2 \ldots. + 2H\delta\frac{d\Theta}{dx}\cdot\frac{d\Phi}{dx}\ldots. + K\frac{dX}{dx'} = 0$$

and forming the similar equations with the remaining variables and adding

$$A\delta a + B\delta b + C\delta c \ldots + 2F\delta f + 2G\delta g + 2H\delta h + \ldots$$
$$+ K\left(\frac{dX}{dx'} + \frac{dY}{dy'} + \frac{dZ}{dz'}\ldots\right) = 0;$$

$$\delta K + K\left(\frac{dX}{dx'} + \frac{dY}{dy'} + \frac{dZ}{dz'} + \,..\right) = 0.$$

the equation in M reduces itself to

$$K\delta M - M\delta K = 0,$$

which is satisfied by $M = K$. It may be remarked that K reduces itself to the sum of the squares of the different functional determinants formed with the differential coefficients of Θ, Φ. . . .with respect to the different combinations of as many variables out of the series x, y. . . .

§ 5. Lagrange's second form.

Here the equations of motion are assumed to be

$$\frac{d}{dt}\frac{dT}{dx'} - \frac{dT}{dx} - P = 0,$$

$$\frac{d}{dt}\frac{dT}{dy'} - \frac{dT}{dy} - Q = 0,$$

$$\frac{d}{dt}\frac{dT}{dz'} - \frac{dT}{dz} - R = 0.$$

$$\vdots$$

where $2T$ represents the vis viva of the system, x, y, z,. . . . are the independent variables on which the solution of the problem depends, and x', y', z'...their differential coefficients with respect to the time. It is assumed as before P, Q, R... do not contain x', y', z'. .

Suppose these equations give

$$dt : dx : dy : dz \,.\,. : dx' : dy' : dz' \,.\,.$$
$$= 1 : x' : y' : z' \,.\,. : X : Y : Z \,.\,.$$

Then the equation which determines the multiplier M takes as before the form

$$\delta M + M\left(\frac{dX}{dx'} + \frac{dY}{dy'} + \frac{dZ}{dz'} + \,..\right). = 0.$$

To reduce this equation, substituting for T its value which is of the form

$$T = \tfrac{1}{2}\,.\,(ax'^2 + by'^2 + cz'^2 \,.\,. + 2fy'z' + 2gz'x' + 2hx'y' \,.\,.)$$

and putting for shortness

$$L = ax' + hy' + gz' \,.\,.\,.\,.$$
$$M = hx' + by' + fz' \,.\,.\,.\,.$$
$$N = gx' + fy' + cz' \,.\,.\,.\,.$$
$$\vdots$$

The equations which determine $X, Y, Z \ldots$ are

$$aX + hY + gZ \ldots + \delta L - \frac{dT}{dx} - P = 0,$$

$$hX + bY + fZ \ldots + \delta M - \frac{dT}{dy} - Q = 0,$$

$$gX + fY + cZ \ldots + \delta N - \frac{dT}{dz} - R = 0.$$

$$\vdots$$

Whence, differentiating with respect to x',

$$a\frac{dX}{dx'} + h\frac{dY}{dx'} + g\frac{dZ}{dx'} \ldots + \delta a \qquad\qquad = 0,$$

$$h\frac{dX}{dx'} + b\frac{dY}{dx'} + f\frac{dZ}{dz'} \ldots + \delta h + \frac{dM}{dx} - \frac{dL}{dy} = 0,$$

$$g\frac{dX}{dx'} + f\frac{dZ}{dx'} + c\frac{dZ}{dx'} \ldots + \delta g + \frac{dN}{dx} - \frac{dL}{dz} = 0.$$

$$\vdots$$

Or representing by K the determinant formed with $a, h, g \ldots h, b, f, \ldots g, f, c, \ldots$ and by $A, H, G, \ldots H, B, F, \ldots G, F, C \ldots$ the inverse system of coefficients, we have

$$K\frac{dX}{dx'} + A\delta a + H\delta h + G\delta g \ldots$$
$$+ \quad \cdot \quad + H\left(\frac{dM}{dx} - \frac{dL}{dy}\right) + G\left(\frac{dN}{dx} - \frac{dL}{dz}\right) \ldots = 0,$$

and similarly

$$K\frac{dY}{dy'} + H\delta h + B\delta b + F\delta f \ldots$$
$$+ H\left(\frac{dL}{dy} - \frac{dM}{dx}\right) + \quad \cdot \quad + F\left(\frac{dN}{dy} - \frac{dM}{dz}\right) \ldots = 0,$$

$$K\frac{dZ}{dz'} + G\delta g + F\delta f + C\delta c \ldots$$
$$+ G\left(\frac{dL}{dz} - \frac{dN}{dx}\right) + F\left(\frac{dM}{dz} - \frac{dN}{dy}\right) + \quad \cdot \quad \ldots = 0.$$

$$\vdots$$

Whence, adding,

$$K\left(\frac{dX}{dx'} + \frac{dY}{dy'} + \frac{dZ}{dz'} \ldots\right)$$
$$+ A\delta a + B\delta b + C\delta c \ldots + 2F\delta f + 2G\delta g + 2H\delta h \ldots = 0;$$

have successively as before, though with symbols
ung an ntirely different signification,

$$K\left(\frac{dX}{dx'}+\frac{dY}{dy'}+\frac{dZ}{dz'}+\ldots\right)+\delta K=0\,;$$

thence $K\delta M - M\delta K = 0$, and $M = K$.

The value of K *in this section* may I think be conveniently ·ied "the determinant of the vis viva," with respect ie variables $x, y, z, .$. It may be remarked that "the ·rminant of the vis viva" with respect to any other system 'ariables $u, v, w, . .$is $\nabla^2 K$, ∇ as before).

§ 6. Third form of the equations of motion.

Here writing

$$\frac{dT}{dx'}=\xi,\ \frac{dT}{dy'}=\eta\ldots$$

nd taking $t, x, y, . . \xi, \eta, . .$for the variables of the problem he equations of motion reduce themselves to

$$\begin{cases} \dfrac{d\xi}{dt} = -\dfrac{dT}{dx}+P, & \dfrac{dx}{dt}=\dfrac{dT}{d\xi}, \\ \dfrac{d\eta}{dt} = -\dfrac{dT}{d\eta}+Q, & \dfrac{dy}{dt}=\dfrac{dT}{d\eta}, \\ \vdots & \vdots \end{cases}$$

Or putting for shortness

$$\begin{cases} P-\dfrac{dT}{dx}=\mathrm{X}, & \dfrac{dT}{d\xi}=\Xi, \\ Q-\dfrac{dT}{d\eta}=\mathrm{Y}, & \dfrac{dT}{d\eta}=\mathrm{H}, \\ \vdots & \vdots \end{cases}$$

they become

$$dt : dx : dy : dz . . : d\xi : d\eta : d\zeta . .$$
$$= 1 : \Xi : \mathrm{H} : \Omega . . : X : Y : Z . .$$

and writing the equation in M under the form

$$\delta M + M.\left(\frac{d\Xi}{dx}+\frac{d\mathrm{H}}{dy}+\ldots+\frac{dX}{d\xi}+\frac{dY}{d\eta}+\ldots\right)=0\,;$$

$$\left(\text{where } \delta=\frac{d}{dt}+\Xi\frac{d}{dx}+\mathrm{H}\frac{d}{dy}\ldots+X\frac{d}{d\xi}+Y\frac{d}{dy}+\ldots\right).$$

[redacted]

we see immediately that (P, Q. . being as before independent of the velocities, and consequently of ξ, η, ζ. .),

$$\frac{d\Xi}{dx} + \frac{dX}{d\xi} = 0, \quad \frac{dH}{dy} + \frac{dY}{d\eta} = 0, \text{ \&c.}$$

Hence $\delta M = 0$, which is satisfied by $M = 1$.

58, *Chancery Lane, Feb.* 6, 1847.

ON A MULTIPLE INTEGRAL CONNECTED WITH THE THEORY OF ATTRACTIONS.

By Arthur Cayley.

Mr. Boole has given for the integral with (n) variables

$$V = \int \frac{\phi\left(\dfrac{x^2}{f^2} + \dfrac{y^2}{g^2} + \,.\,.\right) dx\, dy\,.\,.}{[(a-x)^2 + (b-y)^2\,.\,. + u^2]^{\frac{1}{2}n+q}} \quad \ldots\ldots(1);$$

limits $$\frac{x^2}{f^2} + \frac{y^2}{g^2} + \,.\,. \overline{\underline{<}}\, 1,$$

the following formula, or one which may readily be reduced to that form,*

$$V = \frac{fg\,.\,.\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n + q)} \int_\eta^\infty \frac{Ss^{-q-1} ds}{\sqrt{\{(s+f^2)(s+g^2)\,.\,.}} \quad \ldots.(2),$$

where $$S = \frac{(1-\sigma)^{-q}}{\Gamma(-q)} \int_0^1 t^{-q-1} \phi\,\{\sigma + t\,(1-\sigma)\}\, dt \,\ldots.(3);$$

in which $$\sigma = \frac{a^2}{f^2+s} + \frac{b^2}{g^2+s}\,.\,. + \frac{u^2}{s} \ldots\ldots\ldots.(4),$$

and η is determined by

$$1 = \frac{a^2}{f^2+\eta} + \frac{b^2}{g^2+\eta}\,.\,. + \frac{u^2}{\eta}.$$

Suppose $f = g = \ldots. = \infty$; also assume

$$\phi\,(\lambda) = \frac{1}{(f^2\lambda + v^2)^{\frac{1}{2}n+q'}} \ldots\ldots\ldots\ldots(5);$$

so that the integral becomes

$$U = \int \frac{dx\, dy\,.\,.}{(x^2 + y^2\,.\,. + v^2)^{\frac{1}{2}n+q'}\,\{(x-a)^2 + .\, + u^2\}^{\frac{1}{2}n+q}} \ldots(6),$$

the limits for each variable being $-\infty$, ∞.

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* See Note at the end of this paper.

[?]ration with respect to θ; thus establishing the [?]ond doubt.* The integral may evidently be [?]a n. [?]inite terms when either q or $q-\frac{1}{2}$ is integral. [?] for instance in the simplest case of all, or when $q = -\frac{1}{2}$,

$$\frac{\pi^{\frac{1}{2}(n-1)}}{\frac{1}{2}(n+1)}\frac{1}{(j+2uv)^{\frac{1}{2}(n-1)}} = \int_{-\infty}^{\infty}\frac{dx\,dy\,\ldots\,.}{(x^2+y^2\ldots\,.+v^2)^{\frac{1}{2}(n+1)}\{(x-a)^2+\ldots\,.u^2\}^{\frac{1}{2}(n-1)}}\,.$$

[?]rmula of which several demonstrations have already been [?]n the *Journal.*

[?] following is a demonstration, though an indirect one, [?] [?]mula (11): in the first place

$$\frac{\{[?]+4uv)+\sqrt{s}\}^{-2q}+\{\sqrt{(s+4uv)}-\sqrt{s})^{-2q}}{\sqrt{s}\,\sqrt{(s+4uv)}}\,e^{-\theta s}ds$$

$$= \frac{2\Gamma(\frac{1}{2}-q)\,\theta^q e^{2uv\theta}}{\sqrt{\pi}\,(4uv)^{2q}}\int_0^\infty (4u^2v^2+x^2)^{q-\frac{1}{2}}\,e^{i\theta x}dx\ldots.(16),$$

[?]here as usual $i = \sqrt{-1}$) to prove this, we have

$$\int_{-\infty}^{\infty}(4u^2v^2+x^2)^{q-\frac{1}{2}}e^{i\theta x}dx = \frac{1}{\Gamma(\frac{1}{2}-q)}\int_{-\infty}^{\infty}dx\int_0^\infty dt\, t^{-\frac{1}{2}-q}e^{-t(4u^2v^2+x^2)+i\theta x}$$

$$= \frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}-q)}\int_0^\infty dt\, t^{-1-q}e^{-4u^2v^2t-\frac{\theta^2}{4t}}\,.$$

Or, putting $4uv\ \sqrt{t} = \sqrt{(s+4uv)} \pm \sqrt{s}$ (which is a transformation already employed in the present paper), the formula required follows immediately.

Now, by a formula due to M. Catalan, but first rigorously demonstrated by M. Serret,

$$\int_0^\infty \frac{\cos ax\,dx}{(1+x^2)^n} = \frac{\pi}{(\Gamma n)^2}\int_0^\infty e^{-(a+2z)}(z+a)^{n-1}z^{n-1}dz,$$

(*Liouville*, tom. VIII., p. 1), and by a slight modification in the form of this equation

$$\int_{-\infty}^{\infty}(4u^2v^2+x^2)^{q-\frac{1}{2}}e^{i\theta x}dx = \frac{\pi e^{-2uv\theta}(4uv)^{2q}}{\theta^{2q}\Gamma^2(\frac{1}{2}-q)}\int_0^\infty s^{-q-\frac{1}{2}}(s+4uv)^{-q-\frac{1}{2}}e^{-\theta s}ds,$$

which, compared with (16), gives the required equation.

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* A paper by M. Schlömilch "Note sur la Variation des Constantes Arbitraires d'une Integrale definie," *Crelle*, tom. XXXIII. p. 268-280, will be found to contain formulæ analogous to some of the preceding ones.

[redacted]

the upper sign from $s=\infty$ to $s=\dfrac{u}{v}$, and the lower one from $s=\dfrac{u}{v}$ to $s=0$, it is easy to derive

$$\Theta=(2v)^{2q}\int_0^\infty \frac{\{\sqrt{(s+4uv)}+\sqrt{s}\}^{-2q}+\{\sqrt{(s+4uv)}-\sqrt{s}\}^{-2q}}{\sqrt{s}\;\sqrt{(s+4uv)}}\,\phi(s+j+2uv)\,ds \quad \ldots\ldots (10).$$

Now, by a formula which will presently be demonstrated,

$$\int_0^\infty \frac{\{\sqrt{(s+4uv)}+\sqrt{s}\}^{-2q}+\{\sqrt{(s+4uv)}-\sqrt{s}\}^{-2q}}{\sqrt{s}\;\sqrt{(s+4uv)}}\,e^{-\theta s}ds$$
$$=\frac{2\sqrt{\pi}}{\Gamma(\frac{1}{2}-q)}\,\theta^{-q}\int_0^\infty s^{-\frac{1}{2}-q}(s+4uv)^{-\frac{1}{2}-q}e^{-\theta s}ds \quad \ldots\ldots (11);$$

whence
$$\int_0^\infty \frac{\{\sqrt{(s+4uv)}+\sqrt{s}\}^{-2q}+\{\sqrt{(s+4uv)}-\sqrt{s}\}^{-2q}}{\sqrt{s}\;\sqrt{(s+4uv)}}\,Fs.ds$$
$$=\frac{2\sqrt{\pi}}{\Gamma(\frac{1}{2}-q)}\int_0^\infty s^{-\frac{1}{2}-q}(s+4uv)^{-\frac{1}{2}-q}\left(-\frac{d}{ds}\right)^{-q}Fs.ds \quad \ldots\ldots (12).$$

So that by merely changing the function

$$\Theta=\frac{2^{2q+1}v^{2q}\sqrt{\pi}}{\Gamma(\frac{1}{2}-q)}\int_0^\infty s^{-\frac{1}{2}-q}(s+4uv)^{-\frac{1}{2}-q}\left(-\frac{d}{ds}\right)^{-q}\phi(s+j+2uv)\,ds \quad \ldots\ldots (13);$$

and thence in the particular case in question

$$U=\frac{2^{2q+1}v^{2q}\pi^{\frac{1}{2}(n+1)}}{\Gamma(\frac{1}{2}-q)\,\Gamma(\frac{1}{2}n-q)}\int_0^\infty s^{-\frac{1}{2}-q}(s+4uv)^{-\frac{1}{2}-q}(s+j+2uv)^{-\frac{1}{2}n+q}\,ds \quad \ldots\ldots (14),$$

by means of the formula

$$\left(-\frac{d}{ds}\right)^{-q}(s+a)^{-\frac{1}{2}n}=\frac{\Gamma(\frac{1}{2}n+q)}{\Gamma(\frac{1}{2}n)}(s+a)^{-\frac{1}{2}n+q}.$$

But as there may be some doubt about this formula, which is not exactly equivalent either to Liouville's or Peacock's expression for the general differential coefficient of a power, it is worth while to remark that, by first transforming the $\frac{1}{2}n^{\text{th}}$ power into an exponential, and then reducing as above, (thus avoiding the general differentiation), we should have obtained

$$U=\frac{2^{2q+1}v^{2q}\pi^{\frac{1}{2}(n+1)}}{\Gamma(\frac{1}{2}-q)\,\Gamma(\frac{1}{2}n+q)\,\Gamma(\frac{1}{2}n-q)}$$
$$\int_0^\infty d\theta\int_0^\infty ds\;\theta^{\frac{1}{2}n-q-1}e^{-\theta(s+j+2uv)}s^{-\frac{1}{2}-q}(s+4uv)^{-\frac{1}{2}-q}e^{-\theta s},$$

which reduces itself to the equation (14) by simply perform-

[ON THE] EXISTENCE OF ROOTS OF ALGEBRAICAL EQUATIONS.

By the Rev. HARVEY GOODWIN, M.A., Caius College.

[1.] In a memoir printed in the *Cambridge Philosophical [Trans]actions* (Vol. VIII. Part iii.), I have considered the roots [of equa]tions in the following manner. If $f(x)=0$ represent [an al]gebraical equation of n dimensions, the equation

$$z=f(x+y\sqrt{-1}) \dots\dots\dots\dots (1),$$

[rest]ricted to real values of z, will represent a curve of double [cur]vature, the points of intersection of which with the plane [of x]y will determine by their distances from the origin the [root]s of the equation $f(x)=0$. And I have proved that the [cu]rve consists of n infinite branches, which are continuous [fro]m $+\infty$ to $-\infty$, and therefore must cross the plane of xy in [n] points, and therefore determine n roots.

The mode of investigation adopted in the paper alluded to [is a]pplicable to prove the existence of the roots of equations [wit]hout reference to geometrical considerations, and may be [th]en extended without difficulty to a case discussed by Cauchy (*Exercises*, vol. IV.), namely, that in which the coefficients of the equation are imaginary quantities.

2. I must first explain the method which I adopt for representing the equation (1) when z is restricted to real values. We have

$$\begin{aligned} z &= f(x+y\sqrt{-1}) \\ &= e^{y\sqrt{-1}\frac{d}{dx}}f(x) = \left(\cos y\frac{d}{dx}+\sqrt{-1}\sin y\frac{d}{dx}\right)f(x); \end{aligned}$$

which, supposing the coefficients of $f(x)$ to be real, resolves itself into these two equations,

$$z=\left(\cos y\frac{d}{dx}\right)f(x) \dots\dots\dots\dots (2),$$

$$0=\left(\sin y\frac{d}{dx}\right)f(x) \dots\dots\dots\dots (3).$$

The expressions $\cos y\dfrac{d}{dx}$ and $\sin y\dfrac{d}{dx}$ are to be supposed expanded in powers of $y\dfrac{d}{dx}$, and when the differentiations indicated are performed, the equations (2) and (3) will consist of only a finite number of terms.

3. If $P = \left(\cos\, y\dfrac{d}{dx}\right) f(x)$ and $Q = \left(\sin\, y\dfrac{d}{dx}\right) f(x)$, it is easy to see that

$$\frac{dP}{dx} = \left(\cos\, y\frac{d}{dx}\right)f'(x), \quad \frac{dP}{dy} = -\left(\sin\, y\frac{d}{dx}\right)f'(x),$$

$$\frac{dQ}{dx} = \left(\sin\, y\frac{d}{dx}\right)f'(x), \quad \frac{dQ}{dy} = \left(\cos\, y\frac{d}{dx}\right)f'(x),$$

$$\frac{d^2P}{dx^2} = \left(\cos\, y\frac{d}{dx}\right)f''(x), \quad \frac{d^2P}{dx\,dy} = -\left(\sin\, y\frac{d}{dx}\right)f''(x),$$

$$\frac{d^2P}{dy^2} = -\left(\cos\, y\frac{d}{dx}\right)f''(x),$$

$$\frac{d^2Q}{dx^2} = \left(\sin\, y\frac{d}{dx}\right)f''(x), \quad \frac{d^2Q}{dx\,dy} = \left(\cos\, y\frac{d}{dx}\right)f''(x),$$

$$\frac{d^2Q}{dy^2} = -\left(\sin\, y\frac{d}{dx}\right)f''(x),$$

&c. &c.

$$\text{therefore } \frac{dP}{dx} = \frac{dQ}{dy},$$

$$\frac{dQ}{dx} = -\frac{dP}{dy},$$

$$\frac{d^2P}{dx^2} = \frac{d^2Q}{dx\,dy} = -\frac{d^2P}{dy^2},$$

$$\frac{d^2Q}{dx^2} = -\frac{d^2P}{dx\,dy} = -\frac{d^2Q}{dy^2},$$

&c. &c.

4. THEOREM:—*If $f(x) = x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots\ldots + p_n$, where $p_1, p_2 \ldots p_n$, are either real or imaginary, then if x be allowed to assume all values real or imaginary, subject to the condition that $f(x)$ is real, $f(x)$ does not admit of a maximum or minimum value.*

5. For simplicity's sake suppose first the coefficients $p_1, p_2 \ldots$ to be real. Then putting for x, $x + y\sqrt{-1}$, the equation $z = f(x)$ resolves itself into these two:

$$z = \left(\cos\, y\frac{d}{dx}\right)f(x) = P,$$

$$0 = \left(\sin\, y\frac{d}{dx}\right)f(x) = Q.$$

num or minimum we must have $\delta z = 0$;

rrefore
$$\left.\begin{array}{l}\frac{dP}{dx}\delta x + \frac{dP}{dy}\delta y = 0 \\ \frac{dQ}{dx}\delta x + \frac{dQ}{dy}\delta y = 0\end{array}\right\} \ldots\ldots\ldots\ldots (4).$$

Multiplying these equations by $\frac{dP}{dx}$, $\frac{dQ}{dx}$ respectively, and observing the relations established in Art. 3, we have

$$\frac{dP^2}{dx^2} + \frac{dQ^2}{dx^2} = 0,$$

erefore
$$\left.\begin{array}{ll}\frac{dP}{dx} = 0, & \frac{dQ}{dx} = 0 \\ \text{therefore also} \quad \frac{dP}{dy} = 0, & \frac{dQ}{dy} = 0\end{array}\right\} \ldots\ldots\ldots\ldots (5).$$

In order to ascertain whether the values of x and y, given y these equations, correspond to a maximum or minimum value of z, the value of $\delta^2 z$ must be examined. We have

$$\left.\begin{array}{l}1.2.\delta^2 z = \frac{d^2P}{dx^2}\delta x^2 + 2\frac{d^2P}{dx\,dy}\delta x\,\delta y + \frac{d^2P}{dy^2}\delta y^2 \\ \text{and } \; 0 = \frac{d^2Q}{dx^2}\delta x^2 + 2\frac{d^2Q}{dx\,dy}\delta x\,\delta y + \frac{d^2Q}{dy^2}\delta y^2\end{array}\right\} \ldots (6),$$

(since the coefficients of the quantity $\delta^2 y$ are zero).

Let A and B be the values assumed by $\frac{d^2P}{dx^2}$ and $\frac{d^2Q}{dx^2}$, corresponding to the values of x and y given by equations (5). Then, by the relations established in Art 3,

$$1.2.\delta^2 z = A\delta x^2 - 2B\,\delta x\,\delta y - A\delta y^2,$$
$$0 = B\delta x^2 + 2A\,\delta x\,\delta y - B\delta y^2.$$

Let
$$\delta x = \epsilon \cos\phi,$$
$$\delta y = \epsilon \sin\phi;$$

therefore $1.2\,\delta^2 z = \epsilon^2\{A\cos 2\phi - B\sin 2\phi\}$,
$$0 = \epsilon^2\{B\cos 2\phi + A\sin 2\phi\},$$

which equations may be put under the form

$$1.2.\delta^2 z = \epsilon^2 C\cos(2\phi + a),$$
$$0 = \sin(2\phi + a);$$

therefore
$$2\phi + a = 0 \text{ or } \pi,$$

and
$$1.2\,\delta^2 z = \pm\,\epsilon^2 C.$$

Hence $\delta^2 z$ has two values, one positive and the other negative, and therefore the value of z corresponding to the values of x and y, supposed to be obtained, cannot be said to be either a maximum or a minimum.

6. But it is possible that $\delta^2 z$ may vanish; we shall therefore consider the general case in which $\delta^m z$ is the first of the series of quantities δz, $\delta^2 z$, $\delta^3 z$, &c., which does not vanish; and it is not difficult to see that in this case we have

$$1.2\ldots m\,\delta^m z = \left(\delta x\frac{d}{dx} + \delta y\frac{d}{dy}\right)^m P,$$

$$0 = \left(\delta x\frac{d}{dx} + \delta y\frac{d}{dy}\right)^m Q.$$

Let A, B be the the values assumed by $\dfrac{d^m P}{dx^m}$, $\dfrac{d^m Q}{dx^m}$, respectively; then, in consequence of the relations of Art. 3, we shall have

$$1.2\ldots m\delta^m z = A\delta x^m - mB\delta x^{m-1}\delta y - \frac{m.(m-1)}{1.2}A\delta x^{m-2}\delta y^2 + \ldots$$

$$0 = B\delta x^m + mA\delta x^{m-1}\delta y - \frac{m.(m-1)}{1.2}B\delta x^{m-2}\delta y^2 - \ldots$$

Let $\delta x = \epsilon\cos\phi$, $\delta y = \epsilon\sin\phi$; therefore

$$1.2\ldots m\delta^m z = \epsilon^m\left\{A\left(\cos^m\phi - \frac{m.(m-1)}{1.2}\cos^{m-2}\phi\sin^2\phi + \ldots\right)\right.$$
$$\left. - B\left(m\cos^{m-1}\phi\sin\phi\ldots\right)\right\},$$

$$0 = \epsilon^m\left\{B\left(\cos^m\phi - \frac{m.(m-1)}{1.2}\cos^{m-2}\phi\sin^2\phi + \ldots\right)\right.$$
$$\left. + A\left(m\cos^{m-1}\phi\sin\phi\ldots\right)\right\},$$

or
$$1.2\ldots m\delta^m z = \epsilon^m\left\{A\cos m\phi - B\sin m\phi\right\},$$
$$0 = \epsilon^m\left\{B\cos m\phi + B\sin m\phi\right\};$$

which may be put under the form

$$1.2\ldots m\delta^m z = \epsilon^m C\cos(m\phi + a),$$
$$0 = \sin(m\phi + a);$$

therefore $m\phi + a = k\pi$, where k may have any one of the values 0, 1, 2. . . . . . $m - 1$, and

$$1.2\ldots m\delta^m z = (-1)^k\epsilon^m C.$$

Therefore $\delta^m z$ has m values which are alternately positive and negative, and therefore z admits of no maximum or minimum value.

...ust now consider the still more general case, in ... or more of the coefficients $p_1, p_2 \ldots\ldots p_n$ are ...

Let $$p_1 = r_1(-1)^{\frac{a_1}{\pi}} = r_1(\cos a_1 + \sqrt{-1}\sin a_1),$$

$$p_2 = r_2(-1)^{\frac{a_2}{\pi}} = r_2(\cos a_2 + \sqrt{-1}\sin a_2),$$
&c. = &c.

therefore $$f(x) = x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots\ldots$$
$$= x^n + r_1 \cos a_1 x^{n-1} + r_2 \cos a_2 x^{n-2} + \ldots\ldots$$
$$+ \sqrt{-1}\{r_1 \sin a_2 x^{n-1} + r_2 \sin a_2 x^{n-2} + \ldots\ldots\}$$
$$= M + N\sqrt{-1} \text{ suppose.}$$

$$\therefore f(x + y\sqrt{-1}) = \left(\cos y\frac{d}{dx} + \sqrt{-1}\sin y\frac{d}{dx}\right)M$$
$$+\left(\cos y\frac{d}{dx} + \sqrt{-1}\sin y\frac{d}{dx}\right)N\sqrt{-1}$$
$$= \left(\cos y\frac{d}{dx}\right)M - \left(\sin y\frac{d}{dx}\right)N$$
$$+\sqrt{-1}\left\{\left(\sin y\frac{d}{dx}\right)M + \left(\cos y\frac{d}{dx}\right)N\right\}.$$

Hence if $$f(x + y\sqrt{-1}) = P + Q\sqrt{-1},$$

we have $$z = P = \left(\cos y\frac{d}{dx}\right)M - \left(\sin y\frac{d}{dx}\right)N,$$

$$0 = Q = \left(\sin y\frac{d}{dx}\right)M + \left(\cos y\frac{d}{dx}\right)N.$$

Differentiating

$$\frac{dP}{dx} = \left(\cos y\frac{d}{dx}\right)\frac{dM}{dx} - \left(\sin y\frac{d}{dx}\right)\frac{dN}{dx},$$

$$\frac{dP}{dy} = -\left(\sin y\frac{d}{dx}\right)\frac{dM}{dx} - \left(\cos y\frac{d}{dx}\right)\frac{dN}{dx},$$

$$\frac{dQ}{dx} = \left(\sin y\frac{d}{dx}\right)\frac{dM}{dx} + \left(\cos y\frac{d}{dx}\right)\frac{dN}{dx},$$

$$\frac{dQ}{dy} = \left(\cos y\frac{d}{dx}\right)\frac{dM}{dx} - \left(\sin y\frac{d}{dx}\right)\frac{dN}{dx},$$

$$\therefore \frac{dP}{dx} = \frac{dQ}{dy}, \quad \frac{dP}{dy} = -\frac{dQ}{dx};$$

and in like manner we should find that

$$\frac{d^2P}{dx^2} = \frac{d^2Q}{dx\,dy} = -\frac{d^2P}{dy^2},$$

$$\frac{d^2Q}{dx^2} = -\frac{d^2P}{dx\,dy} = -\frac{d^2Q}{dy^2},$$

and so on, as in Art. 3. Hence the relations between the differential coefficients of P and Q being the same as in the case of the quantities $p_1, p_2 \ldots$ being real, the investigation given in that case is equally applicable to the more general one now under consideration.

Hence the theorem enunciated is true.

8. Since then $f(x)$ admits of no maximum or minimum value, we may give x a succession of values of the form $x + y\sqrt{-1}$, which shall cause $f(x)$ to assume all real values intermediate to $+\infty$ and $-\infty$. Let us now examine how many such sets of values can be found.

When x is very large, the equation

$$z = x^n + p_1 x^{n-1} + \ldots$$

degenerates into the following,

$$z = x^n.$$

Let $\quad x = \rho\,(\cos\theta + \sqrt{-1}\sin n\theta),$

therefore $\quad z = \rho^n(\cos n\theta + \sqrt{-1}\sin\theta);$

which is equivalent to these two,

$$z = \rho^n \cos n\theta,$$

$$0 = \sin n\theta;$$

therefore $n\theta = k\pi$, where k may have any one of the values $0, 1, 2 \ldots (2n-1)$, and

$$z = (-1)^k \rho^n.$$

Hence, when $k = 0, 2, 4 \ldots$, z becomes $+\infty$ when ρ is indefinitely increased, and when $k = 1, 3, 5 \ldots$, becomes $-\infty$; and therefore there are n series of values which may be assigned to x, which will make z vary continuously from $+\infty$ and $-\infty$, and therefore n values may be found which will make z vanish, that is, the equation $f(x) = 0$ has n roots.

Cambridge, Feb. 10, 1847.

ON THE FORCES EXPERIENCED BY SMALL SPHERES UNDER MAGNETIC INFLUENCE; AND ON SOME OF THE PHENOMENA PRESENTED BY DIAMAGNETIC SUBSTANCES.

By William Thomson.

The circumstance that a magnet* attracts small pieces of iron, is the phenomenon of magnetism which was first observed; and an analogous action, presented by rubbed amber, first drew attention to the phenomena of electricity. Now it has since been discovered that no mutual attraction or repulsion between two bodies can result from magnetism in one, unless the other be also magnetized, and that no electric force can exist unless each body be electrically excited. Hence it appears that the forces originally observed are the consequences of a temporary magnetic or electric state induced in a neutral body, when placed in the neighbourhood of a magnet or of an electrified body.

In the following paper the law of such phenomena with reference to magnetism† is considered. It is easily shown however that, by taking $i = 1$ in the formulæ obtained below, the corresponding results for small insulated conductors, electrified by influence, may be obtained, although the physical problems are entirely distinct.

1. We may commence by considering the case of a small sphere of soft iron, or of any other substance susceptible of magnetic induction; and it is easily shewn that the formula expressing the results may be applied to the case of a small cube by merely altering the value of a certain coefficient; and in general to the case of a small portion of matter of any form, such that in whatever way it be turned round the resultant axis of magnetization, for the whole mass, shall coincide with the direction of the magnetizing force.

2. It is well known that if a small homogeneous sphere

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* Originally a piece of magnetic iron ore or loadstone. The term may now be applied to any mass possessing permanent magnetism, and may even be extended to a galvanic wire of any form.

† This has not been made the subject of a special investigation by any writer, so far as I am aware, although the nature of the result, in the case of magnetism, appears to be entirely understood by Mr. Faraday. Thus, from § 2418 (quoted below, in the text,) of his *Experimental Researches*, we might infer that a small sphere or cube of soft iron would in some cases be "urged along, and in others obliquely or directly across the lines of magnetic force"; and that all the phenomena would resolve themselves into this, that such a portion of matter, when under magnetic action, tends to move from weaker to stronger places or points of force.

of soft iron, or of any other substance susceptible of magnetic induction, be placed in the neighbourhood of a magnet, it will become uniformily magnetized, throughout its mass, with an intensity numerically expressed by multiplying the magnetizing force, by a coefficient independent of the dimensions of the sphere. Thus if R denote the resultant force of the magnet, or the force that it would exert upon an imaginary unit of magnetism, at the position occupied by the sphere, of which we suppose the dimensions to be so small that R has sensibly the same value and direction throughout; and if κ be the intensity of the induced magnetism; we have

$$\kappa = \frac{4\pi i}{3} R \dots\dots\dots\dots\dots (1),$$

where i is a proper fraction (nearly equal to unity for soft iron) depending on the capacity of the substance for magnetic induction.

3. If the force R were rigorously constant in magnitude and direction throughout the whole space S occupied by the sphere, then there would be no resulting force tending to move the sphere; as, for example, we may conceive it to be, without committing an appreciable error, in the case of a ball of iron of any ordinary dimensions magnetized by the terrestrial force. In the investigation which follows we shall therefore have to consider the small variation of R through the space S but, although considering the effect of this small variation in causing a moving force upon the magnetized sphere, we may neglect the deviation from rigorous uniformity of magnetization which it will produce.

4. Let X, Y, Z be the components of R at the point (x, y, z), which may be taken as the centre of the small sphere. At any point $(x+f)$, $(y+g)$, $(z+h)$, in the sphere, we shall have, for the components of the resultant force due to the magnet,

$$X + \frac{dX}{dx} f + \frac{dX}{dy} g + \frac{dX}{dz} h,$$

$$Y + \frac{dY}{dx} f + \frac{dY}{dy} g + \frac{dY}{dz} h,$$

$$Z + \frac{dZ}{dx} f + \frac{dZ}{dy} g + \frac{dZ}{dz} h.$$

By considering the effects of these forces upon the elements (as for instance thin bars, in the direction of magnetization)

[illegible] the sphere may be supposed to be [illegible], as has also been done by Poisson, [illegible] the components of the resulting force on the sphere are given by the equations

[illegible]

[illegible] the cosines [illegible] direction of magnetization with [illegible] of the force R, [illegible]

[illegible]

[illegible] a closed galvanic circuit, [illegible] necessarily a complete differential, [illegible]

[illegible]

[illegible] by means of these equations, [illegible]

$$F = \frac{4\pi}{3}\sigma\left(X\frac{dX}{dx} + Y\frac{dY}{dx} + Z\frac{dZ}{dx}\right) = \frac{4\pi}{3}\sigma R\frac{dR}{dx}$$

$$G = \frac{4\pi}{3}\sigma\left(X\frac{dX}{dy} + Y\frac{dY}{dy} + Z\frac{dZ}{dy}\right) = \frac{4\pi}{3}\sigma R\frac{dR}{dy} \quad \ldots (4).$$

$$H = \frac{4\pi}{3}\sigma\left(X\frac{dX}{dz} + Y\frac{dY}{dz} + Z\frac{dZ}{dz}\right) = \frac{4\pi}{3}\sigma R\frac{dR}{dz}$$

From these we deduce

$$Fdx + Gdy + Hdz = \frac{4\pi i}{3}\,\sigma\,.\,RdR = d\left(\frac{2\pi i}{3}\,\sigma\,.\,R^2\right)\ldots(5),$$

which expresses fully the result of equations (4).

6. The interpretation of this result shows that a sphere of soft iron is urged in the direction in which the magnetizing force increases most rapidly; the components of the force in different directions being expressible by the differential coefficients of the function $\frac{2\pi i}{3}\,\sigma R^2$. Thus in some cases it may actually be urged across the direction of the magnetizing force. For instance, if a ball of soft iron be placed symmetrically with respect to the two poles of a horse-shoe magnet, and at some distance from the line joining them, it will be urged towards this line in a direction perpendicular to it, although the magnetizing force is parallel to it; or if the magnetizing force be due to a straight galvanic wire, a ball of soft iron will be *attracted* towards the wire, although the force on an imaginary "magnetic point" is perpendicular to a plane through it and the wire.

7. The positions of equilibrium of a small sphere acted upon by the magnetic forces alone, will be points in the neighbourhood of which R^2 is stationary in value, or points where $d\,(R^2) = 0$. This condition is satisfied by either $R = 0$, or $dR = 0$. Hence the sphere will be in equilibrium at points where the resultant magnetizing force vanishes; where it is a maximum or minimum; or where it is stationary in value.

8. A position of stable equilibrium will be such that R^2 diminishes in every direction from it; and hence, if there be any point, external to the magnet, at which the resultant force has a maximum value, it would be a position of stable equilibrium for a small ball of soft iron, and any other position of equilibrium is essentially unstable.

9. According to Mr. Faraday's recent researches, it appears that there are a great many substances susceptible of magnetic induction, of such a kind that for them the value of the coefficient i is negative. These he calls diamagnetic substances, and, in describing the remarkable results to which his experiments conducted him with reference to induction in diamagnetic matter, he says: "all the phenomena resolve themselves into this, that a portion of such matter, when

etic action, tends to move from stronger to weaker points of force."* This is entirely in accordance the result obtained above; and it appears that the law all the phenomena of induction discovered by Faraday reference to diamagnetics may be expressed in the same as in the case of ordinary magnetic induction, by ely supposing the coefficient i to have a negative value.†

10. In the case of a diamagnetic sphere, the consideron of the stability or instability of equilibrium in different itions, is extremely interesting. Thus, at a point where is a minimum, a small sphere of diamagnetic matter be in stable equilibrium; and this is actually the case ny point for which the force vanishes: even if we take account the weight of the sphere, it is readily shewn stable positions of equilibrium may exist. Thus a w cylindrical bar-magnet (if sufficiently powerful), held its axis vertical, would support a small diamagnetic re in a position of stable equilibrium at a point in the , a little below the lower end of the magnet. For, conring different points in the axis, we perceive that there is one below the lower end (at a distance $=\frac{a}{\sqrt{2}}$, if a, the radius of the cylinder, be very great compared with its thickness, and very small compared with its length, and if the distribution of magnetism be uniform) at which the resultant force is a maximum. If, on moving a small diamagnetic sphere upwards from this position, we arrive at a point where the force urging it upwards is greater than the weight, and then let it move freely from rest, it will oscillate about a position of stable equilibrium. It will probably be impossible ever to observe this phenomenon, on account of the difficulty of getting a magnet strong enough, and a diamagnetic substance sufficiently light, as the forces manifested in all cases of diamagnetic induction hitherto examined are excessively feeble.

11. A very curious phenomenon might readily be observed, according to the results given above, by placing two bar-magnets, with similar poles in the neighbourhood of

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* *Experimental Researches*, § 2418.

† The law of induction in a mass of any form, whether of magnetic or diamagnetic matter, may be stated as follows. Let R be the magnetic force upon a point within an infinitely small spherical surface, described round a point P in the mass, resulting from the magnetism of all the matter external to this surface. The intensity of the magnetism at P is equal to $\frac{3}{4\pi} i R$, and the direction is that of the resultant force R.

a ball of soft iron allowed to move in a horizontal straight line (suspended in such a manner that any motion which can take place is in a circle of considerable radius). Thus if a pole, S, of a bar-magnet which we may regard for simplicity as very long and thin, be held in the neighbourhood, the ball will be drawn towards the point A, in which a perpendicular from S meets the line of motion, and A will therefore be a position of stable equilibrium. If now a pole S', of an equally powerful magnet, be presented and held at an equal distance in SA produced, A will become an unstable position; and if the ball be placed in its line of motion, at any distance from A less than $\frac{SA}{\sqrt{2}}$, it will be *repelled* from A, although either magnet alone would cause it to move towards this point.

12. The result obtained above affords the true explanation of the phenomenon observed by Faraday, that a thin bar or needle of a diamagnetic substance, when suspended between the poles of a magnet, assumes a position across the line joining them. For there is no tendency for such a needle to arrange itself across the lines of magnetic force; but, as will be shewn in a future paper, if the needle be very small compared with the dimensions and distance of the magnet (as is the case, for instance, with a bar of any ordinary dimensions, subject only to the earth's influence), the direction it will assume, when allowed to turn freely about its centre of gravity, will be that of the lines of force, whether the material of which it consists be diamagnetic, or magnetic matter such as soft iron. Thus Faraday's result is due to the rapid decrease of magnetic intensity round the poles of the magnet, and to the length of the needle, which is considerable compared with the distance between the poles of the magnet; and is thus explained by the discoverer himself. (§ 2269) "The cause of the pointing of the bar, or any oblong arrangement of the heavy glass, is now evident. It is merely a result of the tendency of the particles to move outwards, or into the positions of weakest magnetic action.* The joint exertion of the action of all the particles brings the mass into the position which, by experiment, is found to belong to it."

St. Peter's College, May 13, 1847.

* The extreme feebleness of the diamagnetic action, on account of which any small sphere or cube of the matter will experience very nearly the same force as if all the rest were removed, seems fully to justify this explanation.

MATHEMATICAL NOTES.

relative to Mr. Newman's paper on Logarithmic f the Second Order.

r these pages had passed through the press, Mr. mentioned to the Editor that nearly the same subject en treated in *Crelle's Journal* (Vol. xxx. 1840), by or Kummer. After a rapid perusal, I can only add is is certainly true, and that *many* of the properties e investigated have bee ... overed, and some others es. Whether some of mi... are not wholly new, I am le to assert positively, by reason of the great difference tation; nevertheless I believe that several of my equa- concerning Λ and ל are not contained in Professor ner's investigations.

tates, that the integral $\int F_1x.\log F_2x.dx$ was treated in the *Journal der Mathematik*, Band III., and the u $\int \log(1 + 2x\cos a + x^2)\,x^{-1}\,dx$, in a separate Latin e, by the same, in 1830. Kummer has enlarged on whose labours he regrets are so little known. It is ous that neither Kummer nor Hill seem to have known of *Spence's* integral, while virtually treating of the same under the form $\int(1+x)^{-1}\log(\pm x).dx$. It appears moreover from Kummer (p. 220), that *Clausen* has actually tabulated my integral ל in p. 298 of *Crelle's Journal*, Vol. VIII., under the form $-\int_0 \log(\pm 2\sin\frac{1}{2}a)\,da$.

Professor Kummer conceives of the general integral under the form $\int F_1x\int F_2x dx.dx$; and he has also extended his views to the third, fourth, fifth, &c. orders of rational integrals (for this appears to be the more appropriate title), and has exhibited in them integrals which are analogous to those of the second order.

F. W. NEWMAN.

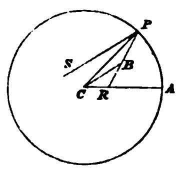
May 8th, 1847.

II. *On the Caustic by Reflection at a Circle.*

[*To the Editor.*]

A paper by Mr. Cayley, under the above title, having been published in the last number of your *Journal*, it appears to me that both M. de St. Laurent and Mr. Cayley have overlooked the admirably symmetrical solution of the problem given by Lagrange in the *Mem. de Turin*. Thinking that some of your correspondents may be interested in it, I beg to send you a translation.

Let B be the luminous point, RBP an incident, and PS a reflected ray; CA a fixed radius $ACP = \alpha$, $ACB = \epsilon$, reciprocal of $CB = c$, reciprocal of $CP = a$. The equations of the incident and reflected ray where $u = \frac{1}{r}$ may be written



$$u = A \sin \theta + B \cos \theta, \text{ incident ray},$$
$$u = A \sin (2\alpha - \theta) + B \cos (2\alpha - \theta), \text{ reflected},$$

the conditions for determining A and B being

$$a = A \sin \alpha + B \cos \alpha,$$
$$c = A \sin \epsilon + B \cos \epsilon;$$

whence $A = \dfrac{a \cos \epsilon - c \cos \alpha}{\sin (\alpha - \epsilon)}$, $B = \dfrac{c \sin \alpha - a \sin \epsilon}{\sin (\alpha - \epsilon)}$.

Substituting these values, the equation of the reflected ray becomes

$$a \sin (2\alpha - \theta - \epsilon) = u \sin (\alpha - \epsilon) + c \sin (\alpha - \theta);$$

from which and its differential with respect to the arbitrary parameter α, the equation to the caustic or envelope of the reflected rays will be found by eliminating α.

In this, α being the only quantity treated as variable in the differentiation, let $2\alpha - \theta - \epsilon = 2\phi$,

therefore $\alpha = \phi + \frac{1}{2}(\theta + \epsilon)$,

and the equation becomes

$$a \sin 2\phi = u \sin \{\phi + \tfrac{1}{2}(\theta - \epsilon)\} + c \sin \{\phi - \tfrac{1}{2}(\theta - \epsilon)\}.$$

Make $$P = \frac{(u + c) \cos \frac{1}{2}(\theta - \epsilon)}{2a},$$
$$Q = \frac{(u - c) \sin \frac{1}{2}(\theta - \epsilon)}{2a}.$$

Also $$x = \frac{1}{\cos \phi}, \quad y = \frac{1}{\sin \phi},$$

and the equation becomes

$$Px + Qy = 1,$$

with the condition $x^{-2} + y^{-2} = 1$.

Hence $$P = \lambda x^{-3},$$
$$Q = \lambda y^{-3}.$$

Multiplying by x and y, and adding, we find $\lambda = 1$;

therefore $x^{-2} = P^{\frac{2}{3}}$, $y^{-2} = Q^{\frac{2}{3}}$.

$$P^{\frac{2}{3}} + Q^{\frac{2}{3}} = 1;$$

restoring the values of P and Q,

$$+ c) \cos \tfrac{1}{2}(\theta - \epsilon)\}^{\frac{2}{3}} + \{(u - c) \sin \tfrac{1}{2}(\theta - \epsilon)\}^{\frac{2}{3}} = (2a)^{\frac{2}{3}},$$

ion of the caustic.

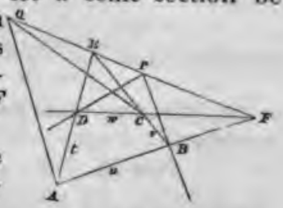
quation, rationalized and transformed to rectangular rdinates, is identical with that of M. de St. Laurent.

PETER SMITH.

British Museum, April 24, 1847.

III. *Solution of a Problem from the Senate-House Papers for* 1847.

LET $ABCD$ be a quadrilateral, and let a conic section be described about it and tangents drawn at A, B, C, D. Let the opposite sides intersect in E, F, and the opposite tangents in P, Q. To prove that P, E, Q, F are in the same straight line.



Let $u = 0$, $v = 0$, $w = 0$, $t = 0$ be the ations to AB, BC, CD, DA, respectively.

The equation to the conic described about $ABCD$ may be expressed by

$$uw = vt \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

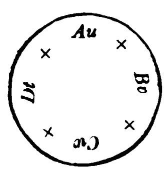
For this will be an equation of the second order, since u, v, w, t are linear in x and y. This equation is also satisfied by $u = 0$, $t = 0$; $u = 0$, $v = 0$; $v = 0$, $w = 0$; $w = 0$, $t = 0$; and therefore the curve passes through $ABCD$. Moreover the general form $ax + by + c$ being taken for each of the functions u, v, w, t, we may conceive each to have been multiplied by an arbitrary constant previously to combination in (1), and therefore the equation (1) will have all the generality possible.

Now let A, B, C, D be four constants such that

$$Au + Bv + Cw + Dt = 0 \text{ identically} \ldots\ldots (2).$$

Then the equation $Au + Cw = 0$ is the same as $Bv + Dt = 0$. But $Au + Cw = 0$ represents a line through F, and $Bv + Dt = 0$ one through E, and hence either equation represents EF.

Arrange the terms of (2) in circular order, thus



Then on taking the combinations u, v; v, w; w, t; t, u, combine each small letter with the adjacent capital, thus

$$\begin{aligned} Du + Cv &= 0, \quad * \\ Av + Dw &= 0, \\ Bw + At &= 0, \\ Ct + Bu &= 0. \end{aligned}$$

These are the equations to the tangents at B, C, D, A, respectively.

Hence for the point P we have simultaneously

$$Du + Cv = 0, \quad Bw + At = 0.$$

Multiply the first by AB and second by CD, and add

$$BD\,(Au + Cw) + AC\,(Bv + Dt) = 0,$$

which is satisfied by either of the identical equations $Au + Cw = 0$, $Bv + Dt = 0$ to EF, and therefore the point P is in EF. Combining the other two, we have the point Q in EF, therefore P, E, Q, F all in a straight line.

Cor. u, v, w, t being proportional to the perpendiculars from any point in the curve on the respective sides of $ABCD$, it appears from the equation $uw = tv$ that the product of perpendiculars on two opposite sides is proportional to the product of perpendiculars on the two other sides.

February 17. G. W. H.

Note. Let O be the intersection of AC, BD. In general if O be any point, and through it there be drawn lines meeting the conic in A, C and B, D if AD, BC meet in E, and AB, CD in F, EF is the polar of O, which is the case therefore in the figure. But P, Q are the poles of BD, AC respectively; hence these points lie in EF, or P, Q, E, F are in the same straight line.

March 27. A. C.

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* That $Du + Cv = 0$ is a tangent may be shewn thus:

$$uw = vt = -\frac{v}{D}(Au + Bv + Cw),$$

$$\text{or} \quad Duw + v\,(Au + Bv + Cw) = 0.$$

Let $u = \mu v$ be the tangent at B,

therefore $$D\mu vw + v\,(Au + Bv + Cw) = 0,$$

or $$v = 0 \text{ and } D\mu w + Au + Bv + Cw = 0.$$

But if $u = \mu v$ be a tangent, it must satisfy the equation *twice*. Hence

$$D\mu + C = 0, \quad \mu = -\frac{C}{D},$$

and then $$(A\mu + B)\,v = 0, \quad \text{or } v = 0 \text{ a second time.}$$

Hence $u = -\frac{C}{D}v$, or $Du + Cv = 0$ is the equation to the tangent at B. Similarly the other equations represent tangents.

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

each into as many detached systems as there are separate belts, will be distributed in a manner exactly similar.

Such is the case when both bounding curves consist of the same number of detached curves: when however the number is not the same for both, then will only some of the detached systems of lines fall under the head already considered, and the others must be referred to some one or other of the following cases.

In the case when one of the two curves on each sheet is imaginary and the other real and single, then obviously the modification from the above general type will be, that one of the two systems of lines will have no cusps and the other will have no envelope, but in other respects their nature and their distribution will be exactly similar, the point or points of contact of each line of the enveloped system, whatever that may happen to be, with the single real curve envelope of that system coinciding always with the cusp or cusps of the corresponding line of the other system, and the two corresponding tangents at every coincident point being always conjugate to each other with respect to the surface; which indeed also is always the case all over the available portions of the envelope where a line of either system has a point in common with or, which is the same thing, intersects a line of the conjugate system.

But, as is much more frequently the case, one of the bounding curves being imaginary, when the one real curve on each sheet consists of two detached curves or of two or more distinct pairs of detached curves, then, for the reasons before explained, will the available intervals between each pair be all continuous and generally, of finite breadth, though returning back into themselves or going off to infinity indifferently as the case may be; the two systems of lines will be divided both into as many detached portions as there are belts of this nature, that is, as there are pairs of curves, and of the portions on each belt every line of one system will touch the two bounding curves of that belt each at least in one point, but often in more, and sometimes in an infinite number, but always both in the same number of points; and every line of the other system will have as many pairs of cusps as the lines of the former have pairs of points of contact, every pair of cuspal points of every line of that system lying on the double curve and coinciding with the corresponding pairs of points of contact of the corresponding line of the former system, and the two corresponding tangents at every individual point of coincidence, being always

conjugate to each other with respect to the surface. If the field, whether closed or going off to infinity, be, as is most frequently the case, of finite breadth, then on every line of both systems a point of inflexion must necessarily exist, for the latter system, between every pair of successive cusps, and for the former system, between every pair of successive points of contact; and whenever the number of points of contact of each line of that system is large or infinite, the lines themselves will obviously be all of an undulating nature, consisting like the contour of a number of alternate elevations and depressions, with an equal number of points of inflexion ranged alternately between, and forming the points of transition from the depressed to the elevated arches, while in the same case the lines of the other system will be of a nature altogether different, consisting certainly each of a number of successive portions all of the same kind, but in place of an undulating appearance presenting two rows of sharp cusps, pointing alternately in opposite directions, and placed between these containing also a row of points of inflexion ranged alternately between every successive pair of opposite cusps, the cusps in each row being equal in number and pointing all outwards, and the number of points of inflexion being equal to the total number of cusps, and therefore to double the number contained in each row.

Finally, in the case when the two separating curves are both altogether imaginary, then will the two sheets of the envelope be both altogether available, and the two systems of lines on each, the lines of contact and the lines of regression, will both be altogether changed in character; neither system will admit of an envelope, and no line of either system will be endowed with a cusp; but both systems of lines will completely cover the whole surface, and will either or both on each sheet, as the case may be, either return back into themselves or extend in both directions and meet at infinity; and at every point taken arbitrarily on either sheet there will cross two distinct lines of each system intersecting at an angle of finite magnitude, the two tangents to the two intersecting lines of either system being respectively the two conjugates with respect to the surface of the two tangents to the two corresponding intersecting lines of the other system.

This last property is generally true, whatever be the nature of the envelope and of the two bounding curves, all over the available regions of that surface; for at all the points of these regions there will obviously cross as many lines of regression

there are tangents which are principal axes, such tangents

giving the directions in which the lines of that system, whatever be their number, diverge from each point: but the cone of principal axes diverging from every point being always of the second order, there will generally diverge from each point two, and there can diverge but two tangents which will be principal axes, and therefore two and but two lines of regression, and consequently also two and but two lines of the conjugate system. Hence, to state the property before us in its greatest generality, we may say, that through every point on each sheet of every surface envelope of a system of principal axes subject to a single condition, including not only the available but also the untouched region or regions of that surface and the separating curve or curves, there pass always two lines of each species, of regression and of contact, the pairs being simultaneously both real, both imaginary, or both coincident, as the case may be.

It not unfrequently happens, especially on envelopes of which the two sheets are altogether available, that points exist for which the two tangent principal axes are conjugate to each other with respect to the surface: at all such points the two diverging lines of each species coincide obviously in direction with each other, and therefore if any envelope be such that the same takes place at every point, then for that surface will the two systems of lines of contact and of regression absolutely coincide with each other, that is, the same lines which form the curves of contact of one of the two component systems of developable surfaces, into which the enveloped system of axes may be resolved, will also form the curves of regression of the other component developable system; and conversely, if either sheet of an envelope possess the property that its two conjugate systems of lines coincide with each other, then at every point of that sheet will the two tangent principal axes be always conjugate to each other with respect to the surface. We shall see as we proceed, that there exists in every body a very extensive class of systems of envelopes, the class containing an infinite number of systems, and each system containing an infinite number of envelopes, all possessing these unusual properties, that both their sheets are entirely available, and that their two systems of lines on each sheet, of contact and of regression, coincide with each other all over the whole extent of the surface. Moreover, if on a surface of this nature, besides crossing at every point of the surface in directions which are always conjugate to each other, the two intersecting sets of lines of the same species intersect everywhere two and two at right

IV. *On a System of Magnetic Curves.*

LET λ be the potential produced by a magnet symmetrical about an axis OX at a point $P(x, y)$. The magnetic curves, or lines of force, being the orthogonal trajectories of the surfaces for which the potential is constant, will lie in planes through OX; and the system in the plane YOX will be the orthogonal trajectory of the system of curves, λ = const. Their equation, as was shewn in a paper "On the Equations of Motion of Heat referred to Curvilinear Coordinates" (vol. IV. p. 40), is

$$\int y\left(\frac{d\lambda}{dy}dx - \frac{d\lambda}{dx}dy\right) = C \quad \ldots\ldots\ldots\ldots (1).$$

As an example, let λ be due to two small needles placed in the line OX, at points M, M'; so that we may take

$$\lambda = \frac{\mu(x-f)}{\{(x-f)^2+y^2\}^{\frac{3}{2}}} + \frac{\mu'(x-f')}{\{(x-f')^2+y^2\}^{\frac{3}{2}}} = \frac{\mu(x-f)}{\Delta^3} + \frac{\mu'(x-f')}{\Delta'^3}.$$

By integration we find, from (1),

$$\frac{\mu y^2}{\Delta^3} + \frac{\mu' y^2}{\Delta'^3} = C,$$

for the equation of the system of magnetic curves.

If we take as a particular case, $C = 0$, we find $y^2 = 0$, which shows that the axis is a line of force; we have also, for another branch, corresponding to the same value of C,

$$\frac{\mu}{\Delta^3} + \frac{\mu'}{\Delta'^3} = 0.$$

As Δ and Δ' are essentially positive in the physical problem, this can only be satisfied if μ and μ' have different signs. For instance, if $\mu = 1$, $\mu' = -m$, we have

$$\Delta' = m^{\frac{1}{3}}\Delta.$$

The locus of this equation is, as is well known, a circle, which may be described thus. Divide MM' in A, and produce it to A_1, so that

$$M'A = m^{\frac{1}{3}}.MA \text{ and } M'A_1 = m^{\frac{1}{3}}.MA_1;$$

on AA_1 as diameter describe a circle.

This result was suggested to me by the solution of a corresponding problem (of much greater interest however) in fluid motion, verbally communicated to me by Mr. Stokes.

WILLIAM THOMSON.

St. Peter's College, May 18, 1847.

ON PRINCIPAL AXES OF A BODY, THEIR MOMENTS OF INERTIA AND DISTRIBUTION IN SPACE.

By Richard Townsend.

(*Continued from* p. 171.)

These properties not only establish to a certain extent the analogy which we conceive to exist between the relation which connects these two unique developable surfaces with the two component developable systems into which every system of principal axes subject to a single condition may be always resolved, and the relation which connects the unique particular solution of an ordinary differential equation with the whole system of particular integrals of the same, but also they enable us to form a tolerably clear conception of the nature and position of the two conjugate systems of curves, the lines of contact and the lines of regression of the two developable systems on each sheet of the envelope, in all the different cases which may variously present themselves; viz. according as the two different curves which separate the available from the untouched regions of either sheet are both real and single or both real and consisting of two or more detached curves, or when one is imaginary and the other real and single or real and consisting of detached curves, or, finally, when they are both altogether imaginary. And here we may observe that it will be only necessary to consider in any particular instance what takes place on one sheet alone, for the corresponding curves of separation on the two sheets are obviously both at the same time real or at the same time imaginary, and if real are both of the same nature with respect to the mere number of separate curves they consist of, and that is all with which we need ever be concerned in the question before us.

First, then, let the separating curves of both species on one of the sheets of the envelope be both real and single; then will they always one or both either return into themselves, or else go off in both directions and meet at infinity: and moreover, since from the above properties both systems of lines, of contact and of regression, are such that every individual line of each has one or more points on both the separating curves, the available zone of the sheet bounded by these two curves must, in order to have that possible, be a continuous portion of the surface; hence, if the sheet be closed, that zone must necessarily be the closed belt occupying the interval between the two curves and not the two separated caps,

but if, on the contrary, it be open and extend to infinity in both directions, the infinite belt consisting of the two zones which, separated by the belt of finite breadth continuous or discontinuous lying between the two curves, extend each in opposite directions and meet at infinity, is not for the same reason excluded from being sometimes the available portion of the sheet; this is evident, for continuity is never broken or even virtually interrupted by mere passage through infinity.

Hence when the two bounding curves on the two sheets of the envelope are both real and single, the systems of lines of both species on each sheet, of contact and of regression, will cover the whole continuous zone included between these two curves: every line of both systems will have always one, often two or more, and sometimes even an infinite number, but in all cases not only for one but for both systems, the same number of cusps of the ramphoid species all placed on one of the two bounding curves, and also will have always the same number, whatever that may be, of points of contact with the other bounding curve, the points of contact placed alternately each on the interval between two successive cusps. These two systems of lines therefore, in all such cases, either will consist both of a number of closed or of open curves having each a cusp of the ramphoid species pointing outwards from the curve, such in the closed case as would be the common cardioid if its cusp were reversed, and in the open case of the same nature as the evolute of the common parabola; or they will consist both of two, of more, or of an infinite nnmber of arches placed side by side with each other and forming a curve of the same nature with the common cycloid: but in all cases the two bounding curves will be each the envelope of the whole system of lines of one species and the locus of the whole system of cusps of the other, the points of contact one or more of each line of either system coinciding always with the cusps one or more of the corresponding line of the other system.

Such being the nature and the distribution of the two systems of lines in the case when the two different separating curves on the two sheets are both real and single, the case when they are both real and consist of two or more detached curves presents no additional difficulty; for then, in case of one belt on which the lines are distributed as above, there will exist on each sheet two or more continuous belts of the same nature upon which the two conjugate systems of lines, both of the same nature as in the former case but divided

each into as many detached systems as there are separate belts, will be distributed in a manner exactly similar.

Such is the case when both bounding curves consist of the same number of detached curves: when however the number is not the same for both, then will only some of the detached systems of lines fall under the head already considered, and the others must be referred to some one or other of the following cases.

In the case when one of the two curves on each sheet is imaginary and the other real and single, then obviously the modification from the above general type will be, that one of the two systems of lines will have no cusps and the other will have no envelope, but in other respects their nature and their distribution will be exactly similar, the point or points of contact of each line of the enveloped system, whatever that may happen to be, with the single real curve envelope of that system coinciding always with the cusp or cusps of the corresponding line of the other system, and the two corresponding tangents at every coincident point being always conjugate to each other with respect to the surface; which indeed also is always the case all over the available portions of the envelope where a line of either system has a point in common with or, which is the same thing, intersects a line of the conjugate system.

But, as is much more frequently the case, one of the bounding curves being imaginary, when the one real curve on each sheet consists of two detached curves or of two or more distinct pairs of detached curves, then, for the reasons before explained, will the available intervals between each pair be all continuous and generally, of finite breadth, though returning back into themselves or going off to infinity indifferently as the case may be; the two systems of lines will be divided both into as many detached portions as there are belts of this nature, that is, as there are pairs of curves, and of the portions on each belt every line of one system will touch the two bounding curves of that belt each at least in one point, but often in more, and sometimes in an infinite number, but always both in the same number of points; and every line of the other system will have as many pairs of cusps as the lines of the former have pairs of points of contact, every pair of cuspal points of every line of that system lying on the double curve and coinciding with the corresponding pairs of points of contact of the corresponding line of the former system, and the two corresponding tangents at every individual point of coincidence, being always

conjugate to each other with respect to the surface. If the belt, whether closed or going off to infinity, be, as is most frequently the case, of finite breadth, then on every line of both systems a point of inflexion must necessarily exist, for the latter system, between every pair of successive cusps, and for the former system, between every pair of successive points of contact; and whenever the number of points of contact of each line of that system is large or infinite, the lines themselves will obviously be all of an undulating nature, consisting like the sinusoid of a number of alternate elevations and depressions, with an equal number of points of inflexion ranged alternately between, and forming the points of transition from the depressed to the elevated arches, while in the same case the lines of the other system will be of a nature altogether different, consisting certainly each of a number of successive portions all of the same kind, but in place of an undulating appearance presenting two rows of sharp cusps, pointing alternately in opposite directions, and placed between these containing also a row of points of inflexion ranged alternately between every successive pair of opposite cusps, the cusps in each row being equal in number and pointing all outwards, and the number of points of inflexion being equal to the total number of cusps, and therefore to double the number contained in each row.

Finally, in the case when the two separating curves are both altogether imaginary, then will the two sheets of the envelope be both altogether available, and the two systems of lines on each, the lines of contact and the lines of regression, will both be altogether changed in character, neither system will admit of an envelope, and no line of either system will be endowed with a cusp, but both systems of lines will completely cover the whole surface, and will either or both on each sheet, as the case may be, either return back into themselves or extend in both directions and meet at infinity; and at every point taken arbitrarily on either sheet there will cross two distinct lines of each system intersecting at an angle of finite magnitude, the two tangents to the two intersecting lines of either system being respectively the two conjugates with respect to the surface of the two tangents to the two corresponding intersecting lines of the other system.

This last property is generally true, whatever be the nature of the envelope and of the two bounding curves, all over the available regions of that surface; for at all the points of these regions there will obviously cross as many lines of regression as there are tangents which are principal axes, such tangents

giving the directions in which the lines of that system, whatever be their number, diverge from each point: but the cone of principal axes diverging from every point being always of the second order, there will generally diverge from each point two, and there can diverge but two tangents which will be principal axes, and therefore two and but two lines of regression, and consequently also two and but two lines of the conjugate system. Hence, to state the property before us in its greatest generality, we may say, that through every point on each sheet of every surface envelope of a system of principal axes subject to a single condition, including not only the available but also the untouched region or regions of that surface and the separating curve or curves, there pass always two lines of each species, of regression and of contact, the pairs being simultaneously both real, both imaginary, or both coincident, as the case may be.

It not unfrequently happens, especially on envelopes of which the two sheets are altogether available, that points exist for which the two tangent principal axes are conjugate to each other with respect to the surface: at all such points the two diverging lines of each species coincide obviously in direction with each other, and therefore if any envelope be such that the same takes place at every point, then for that surface will the two systems of lines of contact and of regression absolutely coincide with each other, that is, the same lines which form the curves of contact of one of the two component systems of developable surfaces, into which the enveloped system of axes may be resolved, will also form the curves of regression of the other component developable system; and conversely, if either sheet of an envelope possess the property that its two conjugate systems of lines coincide with each other, then at every point of that sheet will the two tangent principal axes be always conjugate to each other with respect to the surface. We shall see as we proceed, that there exists in every body a very extensive class of systems of envelopes, the class containing an infinite number of systems, and each system containing an infinite number of envelopes, all possessing these unusual properties, that both their sheets are entirely available, and that their two systems of lines on each sheet, of contact and of regression, coincide with each other all over the whole extent of the surface. Moreover, if on a surface of this nature, besides crossing at every point of the surface in directions which are always conjugate to each other, the two intersecting sets of lines of the same species intersect everywhere two and two at right

angles, then for that surface will these two sets of lines be the opposite systems of lines of curvature: the classes of surfaces to which we have just now alluded possess also that additional property.

We may now, by means of the above properties, find readily upon any surface whatever, given or arbitrarily assumed in a body, the two different bounding curves which separate from each other its regions of real and imaginary contact with the principal axes of the body; and moreover we can then know immediately the nature and distribution of the whole system of curves upon the surface along every one of which the developable circumscribing the surface will have all its edges principal axes, and also the nature and distribution of that system of lines traced out on the same which will all possess the property that their systems of tangents will be all principal axes. For we have but to describe the second sheet of the surface envelope of the system of principal axes of the body which are all tangents to the given surface, that sheet will intersect the surface in one of the curves required; to find the second we have but to describe the developable surface circumscribing the two sheets, that developable will touch the given surface along the other curve required; and then, finally, to find the two conjugate systems of lines we shall merely have to apply the preceding general principles, which, whenever we have the two bounding curves, put us at once in possession of the nature, position, and distribution of the two required systems of lines.

Hence we see that on every surface assumed arbitrarily in a body there exist two remarkable curves loci of two distinct systems of points, for both of which the two tangent principal axes will at every point coincide with each other; that one of these curves is always the envelope of the corresponding system of coincident principal axes, and that the other is always the envelope of the system of tangents conjugate to its corresponding system of coincident principal axes; that through every point of the surface there pass two lines of regression and two lines of contact, all real and different, all imaginary, or two and two coincident, as the case may be; and that these two systems of lines, whose nature and distribution different for different surfaces is in all cases determinable from the nature and positions of the two bounding curves, are on every surface whatever always conjugate to each other.

On every surface upon which the two bounding curves are both real—since at all points on one of them the two

tangent principal axes intersect at an evanescent angle, while at all points on the other they intersect at an angle equal to two right angles—there must always exist on the intervening region of real contact between the two curves a line of points for all of which the two tangent principal axes intersect at right angles: this line, which is by no means confined to that particular class of surfaces, but is also found frequently upon surfaces of the other two classes, and which is in all cases a very remarkable curve, we shall often again have occasion to notice; and in the sequel we shall see moreover that in every body there exists an infinite number of classes of surfaces, for which every surface of each class possesses at every point that peculiar property.

Every system of principal axes subject to a single restricting condition being resolvable, as appeared from the preceding principles, into either of two and of but two component systems of developable surfaces, it is obvious that through any individual axis of the system there can pass one and but one developable of each component system; hence we know respecting every such system of principal axes, that of the infinite number of axes of the system which pass all infinitely near to any one individual axis of the same, four and but four will in general intersect that axis, which four will consist of two distinct pairs of axes, the axes of each pair lying at opposite sides of the original axis and ultimately intersecting it at the same point, so that though there are always four intersecting axes, there are never but two points of intersection. This property is much more general, and holds not only for every such system of principal axes, but equally for every system of right lines in space which are subject to two independent restricting conditions whatever be their nature; for if we retrace the steps by which the preceding principles were established, we shall see that every such system of right lines possesses always an envelope of the same nature as that we have been endeavouring to describe, and that when the envelope is real the system may always be resolved into either of two and of but two different and distinct systems of developable surfaces. Hence deducing from these properties the above general inference for every system of right lines whatever which are subject to two independent restricting conditions, we arrive in substance at a celebrated theorem of the illustrious Monge, much admired by the no less distinguished Professor Chasles, who in the Appendix to his History of Geometry has furnished us with the polar reciprocal correlative property.

Indeed, whatever has been said in the present article respecting a system of principal axes subject to a single restricting condition, holds more generally and with scarcely an exception for every system of right lines in space subject to two independent restricting conditions, the only difference between the particular and the more general case being, that in the former one of the two restricting conditions is given and is always the same, while in the general case they are both variable and arbitrary. It was for this reason that we have dwelt so long on this (which is far from being the most interesting) part of our subject, because that we have all along been implicitly discussing the more general question respecting the management and properties of a system of right lines subject to two conditions, the nature and properties of the different systems of rule surfaces into which such a system of lines may be resolved, the nature and varieties of the surface, their envelope, and the consequent nature, position, distribution, and varieties of the two conjugate systems of generating curves on each sheet of that surface, the lines, namely of contact and of regression of the two component systems of developable surfaces into which every such system of right lines in space may be always resolved.

One property indeed (and it seems to be the only one) requires a different method of establishment, viz. that at every point of the envelope two and but two right lines of the system enveloped can touch that surface, and therefore that through every point on the same there can pass two and but two lines of each system of contact and of regression; for it is not every pair of conditions for which the cone resulting from one or either of them will be always of the second order, and besides that the property itself is not without exception true; that it holds however in the general case also, and holds moreover for the most part though not universally, may be easily shewn as follows: Let a system of right lines subjected to one of the two conditions, whatever they be, be constrained to pass all through a point, they will generate a cone of some order or other, let then another system of right lines subjected to the other condition be constrained to pass all through the same point, they will generate another cone: the intersecting sides of these two cones will evidently be the only lines passing through the point which fulfil at once the two conditions, and it is obvious that in general no three of them, except accidentally or in particular cases, will ever lie in the same plane. Hence we see that even one

right line of a system subject to two independent conditions cannot always be drawn through a given point so as to lie in a plane given or drawn arbitrarily through that point, and that out of the whole system of planes which contain some one of the limited number of lines diverging from that point which fulfil the two conditions, not one will in general ever contain more than two. Again, let a system of right lines subjected to one of the two conditions be constrained to lie all in a plane, they will envelope a curve in that plane of some order or other; let then another system subjected to the other condition be constrained to lie all in the same plane, they will envelope another curve: the common tangents to these two curves will evidently be the only lines lying in that plane which fulfil at once the two conditions, and, as in the other case, it is obvious that in general no three of them, except accidentally or in particular cases, will ever pass through the same point. Hence we see that even one right line of a system subject to two independent conditions cannot always be drawn in a given plane so as to pass through a point given or arbitrarily assumed in that plane, and that out of the whole system of points which lie on some one of the limited number of lines lying in that plane which fulfil the two conditions, through no one will there in general ever pass more than two.

From these two general properties combined we see immediately that between a point and a plane passing through it, or between a plane and a point lying in it, a very close and intimate connexion must exist with respect to a system of right lines subject to two independent conditions, in order that two lines of the system should both at the same time pass through the point and lie in the plane, and also that in general more than two lines of the system could never, except accidentally, at the same time pass all through a point and lie all in a plane. It is obvious that the necessary connexion must exist at every point all over the available regions of the surface which envelopes that system of right lines, between every individual point and the corresponding tangent plane to the surface, and that therefore at every point of the envelope, whatever be its nature, there will generally touch that surface two and but two right lines of the system enveloped, both real or both imaginary as the case may be.

But here we must be cautious, for this result though very generally is by no means universally true: this is obvious from the following considerations, which also explain at the same time the general cause of failure in particular cases.

Suppose that one of the two curves in any plane whose common tangents were the only lines in that plane, fulfilling at once the two restricting conditions had a double point, nodal or conjugate, or more generally a multiple point of any order whatever, then would all the tangents drawn from the multiple point to the other curve come under the head of common tangents to the two curves, and therefore in such cases exceptions would exist to the general rule that no more than two right lines of the system fulfilling the two conditions could at the same time pass through the same point and lie in the same plane. A very general and extensive class of exceptions of this nature is to be found in a system of right lines subject to the two independent conditions of touching two given surfaces; for if we draw any plane whatever intersecting both surfaces in a pair of curves, then will the common tangents to these curves be the lines in that plane which fulfil the two conditions; but in the particular case when the common intersecting plane touches either of the surfaces, then will the point of contact be a double point of the curve in which it intersects that surface, and obviously all tangents from that point to the other curve will fulfil the two conditions. The same manifestly may be said also of every system of right lines subject to the two conditions of having double contact with a single given surface, such being in fact but particular cases of the former, the two given surfaces being conceived as coming together and coinciding in one. Hence in the extensive class of cases where the complete envelope itself is given and where the restricting conditions are in contact with both its sheets, or double contact with its single sheet if it consist of only one, the above general property fails, and more than two tangents at the different points of the surface could in general be drawn fulfilling the two conditions.

In every case when we have a system of principal axes, such as we have been considering, subject to a single condition, the corresponding system of principal planes will, obviously, also envelope a surface; and to find that envelope when the restricting condition is given, we may proceed exactly on the principles already described—for the introduction of an arbitrary condition resolves the system of axes into a multitude of groups each forming a surface gauche or developable as the case may be, and therefore divides the whole corresponding system of planes into an infinite number of smaller systems having each for its envelope a developable surface, and of this system of developables the envelope is of

course that of the whole original system of planes: in order therefore to find its equation we have but to investigate, after the manner indicated in (31), the equation of the developable corresponding to one of the smaller systems of axes, determined by an arbitrarily introduced condition, and from that equation containing the parameter of that system and therefore expressing the whole system of developables, proceed then in the usual way to find the envelope of that system of surfaces, that is, of the whole system of principal planes.

An obvious but very particular example illustrative of the principles discussed in this article is afforded by every system of principal axes which are normals all to the same surface of the second order confocal with the ellipsoid of gyration: here they are restricted by but a single condition; here they may be resolved into a multitude of smaller systems forming each a surface, and this resolution may be performed in an infinite number of different ways, for by simply introducing at random the equation of a surface containing a parameter, we shall then, by varying that parameter, have the original surface of the second order covered with a multitude of curves, and we may take for our resolved systems those groups of axes which pass each through one of these curves: by introducing the equations of the confocal system itself the system of curves will be the lines of curvature on the original surface, the corresponding resolved systems of axes will all form developable surfaces, and there being two and but two different and distinct systems of lines of curvature, each covering the whole original surface, there are therefore two and but two different and distinct systems of developable surfaces into which this system of axes may be resolved. Again, all the different resolved systems will have one and the same envelope, consisting of two different and distinct sheets, each touched by every axis of the system enveloped and generally at different points of contact, the "surface of centres" of the original surface; and, finally, the whole corresponding system of principal planes will also envelope a surface, the original surface itself.

(To be continued.)

NOTES ON DESCRIPTIVE GEOMETRY. No. II.

By T. S. Davies, F.R.S., F.S.A.

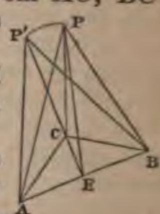
The relative magnitudes, as to greater and less, of an angle and its orthographic projection, have been *assumed* by most if not all writers on the subject of these notes, without investigation. That assumption is, that "the projection is *always* greater than the projected angle." In some cases this is true, in others not: and the object of this note is to examine all the cases that can arise.

The course here employed will have the further advantage of furnishing a more concise and intelligible demonstration of *Euc.* XI. 21, than that which has descended to us from the Greeks, and which is generally employed by modern writers. On this account, the investigation is written in the ordinary language of geometry rather than in the special language of projection.

Problem. *Let* CD *be a line perpendicular to a plane* ABC *meeting the plane in* C; *and let* A, B, *be two points in that plane: it is required to find the point in* CD *to which if lines be drawn from* A *and* B, *they shall contain the greatest angle.*

The problem will divide itself into two general cases, adapted to the circumstances that CE being drawn perpendicular to the line AB, it shall meet AB in E so that E shall be in AB or in AB produced, respectively.

Case 1. Let E lie between A and B. Join AC, BC: then ACB is the greatest angle formed by lines from A, B, to meet in CD.



For let P be any other point in CD, and join PE; make EP′ equal to EP; and join AP′, P′B.

Then, since PCE is a right angle, it is greater than CPE; and hence PE is greater than CE, and P′ lies more remote from E than C does. The point C is therefore within the triangle AP′B; and (*Euc.* I. 21) the angle ACB is greater than AP′B.

Again (*Euc.* XI. 11), PE is perpendicular to AB, or AEP is a right angle. Whence the triangles APE, AP′E have the sides AE, EP equal to the sides AE, EP′, and their included angles AEP, AEP′ also equal: and therefore the angle APE is equal to AP′E.

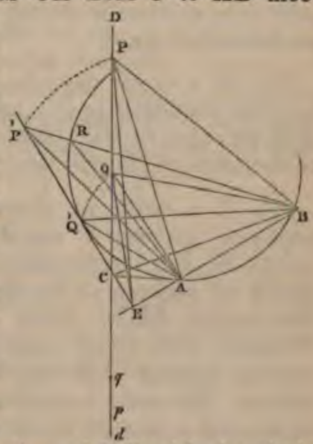
Similarly the angle BPE is equal to BP′E; and consequently the whole angle APB is equal to AP′B.

But ACB is greater than AP′B, and consequently greater than APB.

The same demonstration evidently applies when P is taken on the other side of the plane ACB: and when E falls at A or B, the demonstration becomes still more simple in its details.

Scholium. When PA, PB are the lines forming an angle to be projected on a plane which meets them in A and B, and when the perpendicular from P to the base AB does not fall in AB produced, the projection of the angle is greater than the angle itself. The general assumption referred to above is therefore always true in this case; and it will be seen that this is not the only case in which it is accurate.

CASE 2. Let the perpendicular CE from C to AB meet AB *produced*, in E. Through A and B describe a circle to touch EC *produced* in the point Q′; in CD take Q so that EQ shall be equal to EQ′, and join AQ, QB. Then AQB is the greatest angle that can be formed by lines drawn from A and B to meet in CD, on the same side of the plane ACB.



For, let P be any other point in CD on the same side of the plane ACB; and in EC produced take EP′ equal to EP: the other lines being drawn as in the figure to join the several points already defined, and R being the intersection of P′B with the circle ABQ′.

Then since AQ′B, ARB are angles in the same segment of the circle AQ′B, they are equal; and since in the triangle P′AR the angle ARB is exterior, it is greater than AP′B. Wherefore AQ′B is greater than AP′B.

Again, by reasoning similar to that in the first case, it may be shewn that APB is equal to AP′B, and AQB equal to AQ′B; and hence it follows that AQB is greater than APB.

Scholium. In this case, then, the assumption is erroneous, as the orthographic projection represented by ACB is less than the projected angle AQB. This is the case when P′ is taken at C, and P coincides with P′.

In order to specify the varieties of this general case, one or two remarks may be usefully appended. Let us then suppose the plane ACB to bè horizontal, and CD to be the part of the line *above* ACB; whilst the part *below* ACB is denoted by Cd. Also, let Cq, Cp be equal to CQ, CP. Then,

(α). The angles made by inflecting lines from A and B to points in CD continually increase as the point to which the lines are inflected approach towards Q from D. After passing Q, the angles continually diminish till the point of inflection arrives at C. The angles then again increase till the point arrives at q; and, finally, in passing farther downwards the angles diminish incessantly. The extreme limits of the magnitudes of the angles in both directions is zero.

The problem, then, in this case has two maxima solutions and one minimum—using these terms in their modern mathematical sense.

(β). If a circle described through A, B, C, and cut the line EC produced in P′; and if P be constructed so that EP shall be equal to EP′: then the orthographic projection ACB of the angle APB will be equal to the angle APB itself.

(γ). If a circle be described through A, B to cut CE produced between C and Q′, in H′, and again in K′; and points H, K corresponding to them be taken in CD (these points are omitted in the figure to prevent confusion): then the angles AHB, AKB will be equal to one another. Whence two equal angles situated in different planes passing through AB may have the same orthographic projection. The same is obviously true of two angles AkB, AhB on the other side (or *below*) the plane ACB.

(δ). If the circle through A, B touching CE, touch it in C, the point Q coincides with C; and the angle ACB will be the greatest possible.

In this case, then, the ordinary assumption is correct as to the projection being greater than the projected angle. It is this which gives rise to the limitation expressed in the construction of the second case.

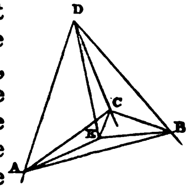
(ϵ). If the circle through A, B touch CE between C and c (c taken equidistant from E with C) the point determined by it is excluded from consideration; since no corresponding point could be constructed in CD, CE being the shortest line that can be drawn from E to CD. In this case, then, the usual assumption is also accurate.

[redacted]

EUCLID XI. 21. ***Every solid angle is contained by plane angles, which are together less than four right angles.***

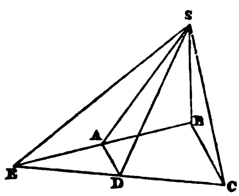
The demonstration of this theorem will only require the property established in the *first case* of the preceding proposition together with the preceding propositions of Euclid. It will be divided into two cases corresponding to Euclid's own division.

(α). Let the angle D be trihedral; take DA, DB, DC all equal; draw the perpendicular DE to the plane ABC; and join EA, EB, EC.



Then, since DEA, DEB, DEC are right angles, it readily follows from (I. 47) and the conditions of our construction, that AE, BE, CE are equal, and hence that E is the centre of the circle about ABC. Whence the perpendiculars from E to the sides of the triangle fall *between* the extremities of those lines, since those perpendiculars bisect the sides.

Wherefore ADB, BDC, CDA are respectively less than AEB, BEC, CEA; and hence their sum less than the four right angles to which AEB, BEC, CEA are equal.



(β). Let the angle S be tetrahedral, and produce two noncontiguous faces to meet in SE; also, let the system of planes be cut by any other plane, ABCDE.

Then, by the preceding case, the angles ESB, BSC, CSE are less than four right angles. But (XI. 20) the angle ASD is less than the two ESA, ESD; and hence the angles ASB, BSC, CSD, DSA are less than ESB, BSC, CSE; and therefore, *à fortiori*, less than four right angles.

(γ). In all other cases the sum of the plane angles may be shewn to become less and less as their number is increased; and hence, that the proposition is true in its most general form, as long as the dihedral angles are salient.

Royal Military Academy, Woolwich,
April 3, 1847.

ON THE THEORY OF ELLIPTIC FUNCTIONS.

By ARTHUR CAYLEY.

ADOPTING the notation of the *Fund. Nova.* except that for shortness $sn\,.\,u$, $cn\,.\,u$, $dn\,.\,u$ are written instead of $\operatorname{sinam}.\,u$, $\operatorname{cosam}.\,u$, $\Delta am\,.\,u$, let the functions $\Theta\,(u)$, $\mathrm{H}\,(u)$ be defined by the equations

$$\Theta u = \sqrt{\left(\frac{2Kk'}{\pi}\right)}\, e^{\frac{1}{2}u^2.\left(1-\frac{E}{K}\right) - k^2.\int_0 du \int_0 du\, sn^2 u} \quad \ldots (1),$$

$$\mathrm{H}u = -\,i\,e^{-\frac{\pi\,(K'-2iu)}{4K}}\;\Theta\,(u+iK)\ldots\ldots\ldots\ldots(2),$$

it is required from these equations to express $sn\,.\,u$ in terms of the functions $\mathrm{H}\,(u)$, $\Theta\,(u)$. To accomplish this we have

$$\frac{d^2}{du^2}\log snu = \frac{1}{snu}\,\frac{d^2}{du^2}\,snu - \frac{1}{sn^2u}\left(\frac{d}{du}\,snu\right)^2$$

$$= -\,(1+k^2) + 2k^2\,sn^2u - \left\{\frac{1}{sn^2u} - (1+k^2) + k^2 sn^2u\right\}$$

$$= k^2 sn^2 u - \frac{1}{sn^2u}\,;$$

whence also

$$\frac{d^2}{du^2}\log snu = k^2 sn^2 u - k^2 sn^2\,.\,(u+iK').$$

If for a moment

$$\psi_{,}\,u = \int_0 du\, sn^2 u, \qquad \psi_{,,}\,u = \int_0 du \int_0 du\, sn^2 u,$$

$$\log snu = k^2\psi_{,,}\,u - k^2\,\psi_{,,}\,(u+iK') + Au + B.$$

Or writing $(-\,u)$ for u and subtracting, $\psi_{,}u$ being an even function,

$$2Au = \pi i - k^2\psi_{,,}\,(iK'-u) + k^2\psi_{,,}\,(iK'+u),$$

or putting $u = K$,

$$2AK = \pi i - k^2\psi_{,,}\,(iK'-K) + k^2\psi_{,,}\,(iK'+K).$$

Now $\qquad sn^2\,(u+K) - sn^2\,(u-K) = 0,$

and therefore $\qquad \psi_{,}\,(u+K) - \psi_{,}\,(u-K) = 2\psi_{,}K,$

$$\psi_{,,}\,(u+K) - \psi_{,,}\,(u-K) = 2u\psi_{,}K\,;$$

or $\qquad \psi_{,,}\,(iK'+K) - \psi_{,,}\,(iK'-K) = 2iK'\psi_{,}K.$

[redacted]

Also $$E(u) = u - k^2\psi_{,}u,$$

or $$E = K - k^2\psi_{,}K, \quad \text{i.e. } \psi_{,}K = \frac{K}{k^2}\left(1 - \frac{E}{K}\right).$$

Hence $$A = iK'\left(1 - \frac{E}{K}\right) + \frac{\pi i}{2K},$$

$$\log snu = k^2\psi_{,,}u - k^2\psi_{,,}(u + iK') + uiK'\left(1 - \frac{E}{K}\right) + \frac{\pi ui}{2K} + B$$

$$= k^2\psi_{,,}u - k^2\psi_{,,}(u + iK') + \tfrac{1}{2}\left[(u + iK')^2 - u^2\right]\left(1 - \frac{E}{K}\right)$$
$$+ \frac{\pi ui}{2K} + B',$$

i.e. $$\log snu = \log\Theta(u + iK') - \log\Theta u + \frac{\pi ui}{2K} + B',$$

or, changing the constant,

$$snu = Ce^{\frac{\pi ui}{2K}}\frac{\Theta(u + iK')}{\Theta u}.$$

Now, to determine C, write $u - iK'$ for u; this gives

$$\frac{1}{ksnu} = Ce^{\frac{\pi i}{2K}(u - iK')}\frac{\Theta u}{\Theta(u - iK')}:$$

and again changing (u) into $(-u)$,

$$-snu = Ce^{-\frac{\pi ui}{2K}}\frac{\Theta(u - iK')}{\Theta u}:$$

whence, multiplying these last two equations,

$$C^2 = -\frac{1}{k}e^{-\frac{\pi K'}{2K}},$$

or $$C = \frac{1}{i\sqrt{k}}e^{-\frac{\pi K'}{4K}};$$

whence $$sn\,u = \frac{1}{i\sqrt{k}}e^{-\frac{\pi(K' - 2iu)}{4K}}\frac{\Theta(u + iK')}{\Theta u},$$

i.e. $$\sqrt{k}\,sn\,u = \frac{\mathrm{H}(u)}{\Theta(u)} \quad \ldots\ldots\ldots\ldots \quad (3);$$

and the equations (1), (2) and (3) may be considered as comprehending the theory of the functions $H(u)$, $\Theta(u)$. The preceding process is, in fact, the converse of that made use of in the *Fund. Nova;* Jacobi having obtained for $\operatorname{sn} u$ an expression in the form of a fraction, takes the numerator of it for $H(u)$ and the denominator for $\Theta(u)$, and thence deduces the equations (1), (2), the intermediate steps of the demonstration being conducted by means of infinite series; the necessity of which is avoided by the preceding investigation.

I proceed to investigate certain results relating to these functions, and to the theory of elliptic functions which have been given by Jacobi in two papers, "Suite des notices sur les fonctions elliptiques," *Crelle*, tom. III. p. 306, and tom. IV. p. 185, but without demonstration.

In the first place, the equation

$$\frac{d^2\Sigma}{du^2} + 2u\left(k'^2 - \frac{E}{K}\right)\frac{d\Sigma}{du} + 2kk'^2\frac{d\Sigma}{dk} = 0 \quad \ldots . \ (4)$$

is satisfied by $\Sigma = \Theta(u)$ or $\Sigma = H(u)$. It will be sufficient to prove this for $\Sigma = \Theta(u)$, since a similar demonstration may easily be found for the other value. The following preliminary formulæ will be required:

$$k\frac{dK}{dk} = \frac{E}{k'^2} - K, \quad k\frac{dE}{dk} = E - K,$$

$$k\frac{dK'}{dk} = -\frac{E'}{k'^2} + \frac{k^2K}{k'^2}, \quad KK' - EK' - E'K = -\frac{\pi}{2},$$

which are all of them known.

Now, writing $\Theta(u)$ under the slightly more convenient form

$$\Theta u = \sqrt{\left(\frac{2Kk'}{\pi}\right)}\, e^{\int_0 du \int_0 du\, dn^2 u - \frac{1}{2}u^2 \frac{E}{K}},$$

we have $$\frac{d\Theta u}{du} = \left(\int_0 du\, dn^2 u - \frac{E}{K}u\right)\Theta u$$

$$= \left\{u\left(k'^2 - \frac{E}{K}\right) + k^2\int_0 du\, cn^2 u\right\}\Theta u,$$

$$\frac{d^2\Theta u}{du^2} = \left[dn^2 u - \frac{E}{K} + \left\{u\left(k'^2 - \frac{E}{K}\right) + k^2\int_0 du\, cn^2 u\right\}^2\right]\Theta u,$$

$$\frac{d\Theta u}{dk} = \left[\frac{1}{2Kk'}\frac{dKk'}{dk} - \tfrac{1}{2}u^2\frac{d}{dk}\frac{E}{K} + \int_0 du\int_0 du\frac{d}{dk}dn^2 u\right]\Theta u.$$

The success of the process depends upon a transformation of the double integral

$$\int_0 du \int_0 du \, \frac{d}{dk}\, dn^2 u.$$

To effect this we have

$$\frac{d}{dk}\, dn^2 u = -2k\, sn u \left(sn u + k\, \frac{d}{dk}\, sn u\right);$$

but, by a known formula,

$$k'^2 \frac{d}{dk}\, sn u = -k\, cn u\, dn u \int_0 cn^2 u\, du + k\, cn^2 u\, sn u;$$

whence $sn u + k\,\frac{d}{dk}\, sn u = \frac{1}{k'^2}\, sn u\, dn^2 u - k^2\, cn u\, dn u \int_0 du\, cn^2 u,$

or $$\frac{d}{dk}\, dn^2 u = -\frac{2k}{k'^2}\left(sn^2 u\, dn^2 u - k^2\, sn u\, cn u\, dn u \int_0 du\, cn^2 u\right)$$

$$= -\frac{2k}{k'^2}\left\{sn^2 u\, dn^2 u + \tfrac{1}{2}k^2 \left(\frac{d}{du}\, cn^2 u\right) \int_0 du\, cn^2 u\right\};$$

whence $\int_0 du \int_0 du \, \frac{d}{dk}\, dn^2 u$

$$= -\frac{2k}{k'^2}\left\{\int_0 du \int_0 du\, sn^2 u\, dn^2 u + \tfrac{1}{2}k^2 \int_0 du \left(cn^2 u \int du\, cn^2 u - \int du\, cn^4 u\right)\right\}$$

$$= -\frac{k}{k'^2}\left\{\int_0 du \int_0 du \left(2 sn^2 u\, dn^2 u - k^2\, cn^4 u\right) + \tfrac{1}{2}k^2 \left(\int_0 du\, cn^2 u\right)^2\right\}.$$

But $$\frac{d^2}{du^2}\, sn^2 u = 2\left(cn^2 u\, dn^2 u - sn^2 u\, dn^2 u - k^2\, sn^2 u\, cn^2 u\right)$$
$$= 2\left(k'^2 - 2 sn^2 u\, dn^2 u + k^2\, cn^4 u\right);$$

or, integrating,

$$sn^2 u = k'^2 u^2 - 2\int_0 du \int_0 du \left(2 sn^2 u\, dn^2 u - k^2\, cn^4 u\right);$$

whence at length

$$\int_0 du \int_0 du \, \frac{d}{dk}\, dn^2 u = -\tfrac{1}{2} k u^2 + \tfrac{1}{2}\,\frac{k}{k'^2}\, sn^2 u + \frac{k^3}{2k'^2}\left(\int_0 du\, cn^2 u\right)^2.$$

Also $$\frac{d}{dk}\, KK' = \frac{E-K}{KK'}, \quad \frac{d}{dk}\,\frac{E}{K} = \frac{1}{kk'^2}\left\{k'^2\left(\frac{2E}{K} - 1\right) - \frac{E^2}{K^2}\right\},$$

so that

$$\frac{d\Theta u}{dk} = \frac{1}{2kk'^2}\left\{\frac{E}{K} - dn^2 u - u^2\left(k'^2 - \frac{E}{K}\right)^2 - k^4\left(\int du\, cn^2 u\right)^2\right\}\Theta u.$$

And substituting these values of $\frac{d}{du}\Theta u$, $\frac{d^2}{du^2}\Theta u$ and $\frac{d}{dk}\Theta u$ in the equation (4) in the place of the corresponding differential coefficients of Σ, all the terms vanish, or the equation is satisfied by $\Sigma = \Theta(u)$, and similarly it would be satisfied by $\Sigma = \mathrm{H}(u)$.

Assume now $\quad \omega = \frac{\pi K'}{K}, \quad v = \frac{\pi u}{2K}.$

Then observing the equation

$$\frac{d}{dk}\frac{K'}{K} = \frac{1}{K^2 kk'^2}(KK' - KE' - K'E) = -\frac{\pi}{2K^2 kk'^2},$$

we have $\quad \frac{d\Sigma}{du} = \frac{\pi}{2K}\frac{d\Sigma}{dv}, \quad \frac{d^2\Sigma}{du^2} = \frac{\pi^2}{4K^2}\frac{d^2\Sigma}{dv^2},$

$$\frac{d\Sigma}{dk} = \frac{v}{kk'^2}\left(k'^2 - \frac{E}{K}\right)\frac{d\Sigma}{dv} - \frac{\pi^2}{2K^2 kk'^2}\frac{d\Sigma}{d\omega};$$

whence, substituting in the equation (4), this becomes

$$\frac{d^2\Sigma}{dv^2} - 4\frac{d\Sigma}{d\omega} = 0 \;\ldots\ldots\ldots\ldots\; (5),$$

which is of course satisfied as before by $\Sigma = \Theta(u)$, or $\Sigma = \mathrm{H}(u)$, an equation demonstrated in a different manner (by means of expansions) by Jacobi in the Memoirs quoted.

Consider next the equation

$$\frac{d^2\Sigma}{du^2} - 2nu\left(k'^2 - \frac{E}{K}\right)\frac{d\Sigma}{du} + 2nkk'^2\frac{d\Sigma}{dk} = 0 \;\ldots\; (6),$$

(n being any positive integer number). Then, by assuming

$$\omega = n\frac{\pi K'}{K}, \quad v = \frac{n\pi u}{K},$$

we should be led as before to the equation (5). Hence, considering Θu or $\mathrm{H}u$ as functions of u and $\frac{K'}{K}$, the equation (6) is satisfied by assuming for Σ a corresponding function of nu and $\frac{nK'}{K}$. Let λ be the modulus corresponding to a transformation of the n^{th} order; then Λ, Λ' being the complete functions corresponding to this modulus,

$\dfrac{\Lambda'}{\Lambda} = n\,\dfrac{K'}{K}$, so that the equation (6) will be satisfied by assuming $\Sigma = \Theta_{,}(nu)$ or $\Sigma = \mathrm{H}_{,}(nu)$, where $\Theta_{,}$, $\mathrm{H}_{,}$ correspond to the new modulus λ.

Assume now in the equation (6),

$$\Sigma = \left(\frac{\pi}{2}\right)^{\frac{n-1}{2}} (Kk')^{-\frac{n-1}{2}}\, \Theta^n u.z.$$

Hence, substituting,

$$\frac{d^2}{du^2}(\Theta^n u.z) - 2nu\left(k^n - \frac{E}{K}\right)\frac{d}{du}(\Theta^n u.z)$$

$$+ 2nkk'^2 (Kk')^{\frac{n-1}{2}} \frac{d}{dk}\left[(Kk')^{-\frac{n-1}{2}}\, \Theta^n u.z\right] = 0:$$

but $(Kk')^{\frac{n-1}{2}} \dfrac{d}{dk}\left[(Kk')^{-\frac{n-1}{2}} \Theta^n u.z\right] = \dfrac{d}{dk}(\Theta^n u.z) - \dfrac{n-1}{2Kk'}\,\dfrac{dKk'}{dk}\,\Theta^n u.z,$

or effecting the differentiation, and eliminating $\dfrac{d\Theta u}{dk}$ by means of the equation obtained from (4) by writing $\Sigma = \Theta u$,

$$(Kk')^{\frac{n-1}{2}} \frac{d}{dk}\left[(Kk')^{-\frac{n-1}{2}}\, \Theta^n u.z\right]$$

$$= \Theta^n u\left[\frac{dz}{dk} - \frac{nz}{2kk'^2\Theta u}\left\{\frac{d^2\Theta u}{du^2} - 2\left(k^n - \frac{E}{K}\right)\frac{d\Theta u}{du}\right\} + \frac{n-1}{2kk'^2}\left(1 - \frac{E}{K}\right)z\right].$$

Substituting in (6) and reducing,

$$\frac{d^2z}{du^2} + 2n\left[\frac{1}{\Theta n}\,\frac{d\Theta u}{du} - u\left(k^n - \frac{E}{K}\right)\right]\frac{dz}{du} + 2nkk'^n\frac{dz}{dk}$$

$$+ n(n-1)\left\{\left[\frac{1}{\Theta^2 u}\left(\frac{d\Theta u}{du}\right)^2 - \frac{1}{\Theta u}\,\frac{d^2\Theta u}{du^2}\right] + \left(1 - \frac{E}{K}\right)\right\} z = 0,$$

i.e. $$\frac{d^2z}{du^2} + 2n\left[\frac{d\log\Theta u}{du} - u\left(k^n - \frac{E}{K}\right)\right]\frac{dz}{du} + 2nkk'^n\frac{dz}{dk}$$

$$+ n(n-1)\left[-\frac{d^2\log\Theta u}{du^2} + \left(1 - \frac{E}{K}\right)\right] z = 0$$

But $$\frac{d\log\Theta u}{du} = u\left(k^n - \frac{E}{K}\right) + k^2 \int_0 du\, cn^2 u,$$

$$\frac{d^2\log\Theta u}{du^2} = 1 - \frac{E}{K} - k^2 sn^2 u;$$

$$du\, cn^2 u)\frac{dz}{du}+2nkk'^2\frac{dz}{dk}+n(n-1)k^2 sn^2 u\,.\,z=0\ldots(7);$$

h is therefore satisfied by

$$z=\left(\frac{2Kk'}{\pi}\right)^{\frac{n-1}{2}}\frac{\Theta_{,}nu}{\Theta^n u},\quad z=\left(\frac{2Kk'}{\pi}\right)^{\frac{n-1}{2}}\frac{H_{,}nu}{\Theta^n u}:$$

each of these values are algebraical functions of $sn\,u$, either rational functions or rational functions multiplied $cn\,u\,dn\,u$). Also, in the transformation of the n^{th} order,

$$\sqrt{\lambda}\, sn_{,}u=\frac{H_{,}(nu)}{\Theta_{,}(nu)};$$

that it is clear that the above values of z may be taken for denominator and numerator respectively of $\sqrt{\lambda}\, sn_{,}u$; *i.e.* ies each of them satisfy the equation (7).

ming $x=\sqrt{k}\, sn\, u$, $a=k+\frac{1}{k}$,

becomes

$$n(n-1)x^2 z+(n-1)(ax-2x^3)\frac{dz}{dx}$$
$$+(1-ax^2+x^4)\frac{d^2z}{dx^2}-2n(a^2-4)\frac{dz}{da}=0\ldots(8);$$

which is therefore satisfied by assuming for z either the numerator or the denominator of $\sqrt{\lambda}\, sn_{,}u$ (the transformation of the n^{th} order), which is the form in which the property is given by Jacobi.

In the case where n is odd, the denominator is of the form

$$B_0+B_{,}x^2\ldots\ldots+B_{\frac{1}{2}(n-1)}x^{n-1},$$

and then the numerator is

$$x(B_{\frac{1}{2}(n-1)}\ldots\ldots+B_{,}x^{n-3}+B_0 x^{n-1}),$$

where $\quad B_0=\sqrt{\left(\frac{\lambda'}{k'M}\right)},\quad B_{\frac{1}{2}(n-1)}=\sqrt{\left(\frac{\lambda\lambda'}{kk'M^3}\right)};$

and all the remaining coefficients may be determined from these, the modular equation being supposed known. But the principal use of the formula is for the multiplication of elliptic functions, which it is well known corresponds to the case where n is a square number. Writing $n=\nu^2$, when ν is odd, the denominator is

$$1+B_2x^4\ldots\ldots+B_{\frac{1}{2}(\nu^2-3)}x^{\nu^2-3}\pm\nu x^{\nu^2-1},$$

(the $\pm$ sign according as $\nu=(4p+1)$ or $(4p-1)$); and the

numerator is obtained from this by multiplying by x and reversing the order of the coefficients. When ν is even the denominator is

$$1 + B_2x^4 \ldots \pm B_2x^{\nu^2-4} \pm x^{\nu^2},$$

(+ or −, according as $\nu = 4p$ or $\nu = 4p + 2$), so that there are only half as many coefficients to be determined; but then the numerator must be separately investigated. In general, by leaving (n) indeterminate, and integrating in the form of a series arranged according to ascending powers of x^2; then, whenever n is a square number, the series terminates and gives the denominator of the corresponding formula of multiplication; but the general form of the coefficients has not hitherto been discovered.

By writing $\dfrac{x}{\sqrt{n}}$ instead of x, and then making n infinite, the equation (8) takes the form

$$x^2z + ax\frac{dz}{dx} + \frac{d^2z}{dx^2} - 2(a^2 - 4)\frac{dz}{da} = 0 \ldots (9):$$

and it is worth while, before attempting the solution of the general case, to discuss this more simple one.*

Assume $z = 1 + C_1\dfrac{x^2}{1.2} \ldots + C_r\dfrac{x^{2r}}{1.2 \ldots 2r} + \ldots;$

then it is easy to obtain

$$C_{r+2} = -(2r+1)(2r+2)C_r - (2r+2)aC_{r+1} + 2(a^2-4)\frac{dC_{r+1}}{da}.$$

The general form may be seen to be

$$C_r = (-)^{r+1}\{2^{2r-3}C_r^1a^{r-2} + 2^{2r-6}C_r^2a^{r-4} + \ldots\},$$

and then

$$C^p_{r+1} - pC_r^p = -r(2r-1)C_{r-1}^{p-1} + 16(r+2-2p)C_r^{p-1}.$$

The complete value of C_r^p (assuming $C_r^0 = 0$) is given by an equation of the form

$$C_r^p = {}^0C_r^p + {}^1C_r^p2^r + {}^2C_r^p3^r \ldots + {}^{p-1}C_r^pp^r,$$

---

* Writing $(\beta + 2)$ for a, and putting $z = e^{+\frac{1}{4}ax^2}\rho$, this becomes

$$\frac{d^2\rho}{dx^2} - \rho = \beta x^2\rho - \beta x\frac{d\rho}{dx} + (8\beta + 2\beta^2)\frac{d\rho}{d\beta};$$

and if $\rho = \Sigma Z_n\beta^n$,

$$\frac{d^2Z_n}{dx^2} - (8n+1)Z_n = \left(x^2 + 2n - 2 - x\frac{d}{dx}\right)Z_{n-1};$$

from which the successive values of Z_0, Z_1, &c. might be calculated.

where ${}^0C_r^p$, ${}^1C_r^p$, ... are algebraical functions of r of the degrees $2p-2$, $2p-4$, &c. respectively; but as I am not able completely to effect the integration, and my only object is to give an idea of the law of the successive terms, it will be sufficient to consider the first or algebraical term ${}^0C_r^p$, which is determined by the same equation as C_r^p, and moreover completely determined by this equation and the single additional relation $C_r^1 = 1$, since the arbitrary constants of the integration affect only the terms multiplied by 2^r, 3^r, &c.

Assume C_r^p

$$= \frac{1}{[p-1]^{p-1}} \{2^{p-1} L^p [r-2]^{2p-2} + 3^{p-1} M^p [r-3]^{2p-3} + \dots 2^{p+1} X^p [r-2p]^p\};$$

and substituting this value,

$$\begin{array}{ll}
(1-p)\,L^p & = (1-p)\,\{L^{p-1} \qquad\qquad\qquad\quad \},\\
(1-p)\,M^p - 2p(2-2p)\,L^p & = (1-p)\,\{M^{p-1} - 11L^{p-1} \qquad\quad \},\\
(1-p)\,N^p - 2p(3-2p)\,M^p & = (1-p)\,\{N^{p-1} - \ \ 7M^{p-1} + 12L^{p-1}\},\\
(1-p)\,O^p - 2p(4-2p)\,N^p & = (1-p)\,\{O^{p-1} - \ \ 3N^{p-1} + 30M^{p-1}\},\\
\vdots & \quad\ \vdots
\end{array}$$

the law of which is obvious, the coefficients on the second side in the qth line being 1, $4q-19$, and $(2q-3)(2q-2)$ respectively. By successive integrations and substitutions

$$\begin{array}{ll}
L^p - L^{p-1} = 0, & L^p = 1,\\
M^p - M^{p-1} = 4p - 11, & M^p = (p-1)(2p-7),\\
N^p - N^{p-1} = -8p^3 + 26p^2 + 49p - 114; & \vdots\\
\vdots &
\end{array}$$

(the constants determined by $M^1 = 0$, $N^1 = 0$, $O^2 = 0$, $P^2 = 0, \dots$ so as to make C_r^p contain positive powers only of r).

The following are a few of the complete values of C_r^p, the constants determined so as to satisfy $C^p_{p+1} = 0$ (except $C_2^1 = 1$), and the factorials being partially developed in powers of r, viz.

$$\begin{aligned}
C_r^1 &= 1,\\
C_r^2 &= (r-3)(2r-7),\\
C_r^3 &= \tfrac{1}{2}(r-4)(r-5)(4r^2 - 24r + 51),\\
C_r^4 &= \tfrac{1}{6}\{(r+5)(r-6)(r-7)(8r^3 - 60r^2 + 286r + 63)\\
&\qquad + 384\,(9r^2 - 93.r + 242 - 2.4^r)\},
\end{aligned}$$

&c.

(it is curious that C_5^4, C_6^4, C_7^4, all three of them vanish). It seems hopeless to continue this investigation any further.

Returning to the equation (8), and assuming for z an expression of the same form as before, we have, corresponding to the equations before found for the coefficients C_r,

$$C_{r+2} = -(2r+1)(2r+2)(n-2r)(n-2r-1)\,C_r - (2r+2)(n-2r-2)\,aC_{r+1} + 2n(a^2-4)\frac{dC_{r+1}}{da}.$$

The case corresponding to the denominator in the multiplication of elliptic functions is that of $C_0 = 1$, $C_1 = 0$. It is easy to form the table—

$$\begin{aligned}
C_0 &= 1,\\
C_1 &= 0,\\
C_2 &= -2\,n(n-1),\\
C_3 &= 8\,n(n-1)(n-4)\,a,\\
C_4 &= -4\,n(n-1)(n-4)[n+75] - 32n(n-1)(n-4)(n-9)\,a^2,\\
C_5 &= 96\,n(n-1)(n-4)(n-9)[n+44]\,a\\
&\quad + 128\,n(n-1)(n-4)(n-9)(n-16)\,a^3,\\
C_6 &= -24\,n(n-1)(n-4)(n-9)[17n^2+403n+9000]\\
&\quad - 960\,n(n-1)(n-4)(n-9)(n-16)[n+41]\,a^2\\
&\quad - 512\,n(n-1)(n-4)(n-9)(n-16)(n-25)\,a^4,\\
C_7 &= 96\,n(n-1)(n-4)(n-9)(n-16)[79n^2+2825n+36180]\,a\\
&\quad + 7168\,n(n-1)(n-4)(n-9)(n-16)(n-25)[n+42]\,a^3\\
&\quad + 2048\,n(n-1)(n-4)(n-9)(n-16)(n-25)(n-36)\,a^5,\\
C_8 &= 48\,n(n-1)(n-4)(n-9)[283n^4 - 26978n^3 + 277827n^2\\
&\qquad - 5491932n + 127764000]\\
&\quad - 3840\,n(n-1)(n-4)(n-9)(n-16)(n-25)\times\\
&\qquad [23n^2 + 1069n + 23436]\,a^2\\
&\quad - 15360\,n(n-1)(n-4)(n-9)(n-16)(n-25)(n-36)\times\\
&\qquad [3n+133]\,a^4\\
&\quad - 8192\,n(n-1)(n-4)(n-9)(n-16)(n-25)(n-36)(n-49)\,a^6,
\end{aligned}$$

&c.

in which of course the coefficient of the highest power of n, in the successive coefficients C_r, is the value of C_r obtained from the equation (8). With regard to the law of these coefficients I have found that

$$\begin{aligned}
C_r = (-)^{r+1}\,&2^{2r-3}\,n(n-1^2)\ldots\{n-(r-1)^2\}\,C_r^{\,1}a^{r-2}\\
+\,&2^{2r-6}\,n(n-1^2)\ldots\{n-(r-2)^2\}\,C_r^{\,2}a^{r-4}\\
+\,&2^{2r-9}\,n(n-1^2)\ldots\{n-(r-3)^2\}\,C_r^{\,3}a^{r-6}\\
+\,&\&c.
\end{aligned}$$

(where however the next term does not contain, as would at first sight be supposed, the factor $n(n-1^2)\ldots\{n-(r-4)^2\}$.) And then

$$C_r^1 = 1,$$

$$C_r^2 = (r-3)\,[n(2r-7)+(r-1)(8r-7)],$$

$$\begin{aligned} C_r^3 = \tfrac{1}{2}(r-4)(r-5)\,[n^2(4r^2-24r+51) & \\ +n(32r^3-220r^2+412r-255) & \\ +2(r-1)(r-2)(32r^2-88r+51)]. & \end{aligned}$$

In conclusion may be given the following results, in which, recapitulating the notation

$$x=\sqrt{k}\,sn\,u,\quad a=k+\frac{1}{k},\quad \Delta x=\sqrt{(1-ax^2+x^4)},$$

$$\sqrt{k}\,sn\,2u=\frac{2x\,\Delta x}{1-x^4},$$

$$\sqrt{k}\,sn\,3u=\frac{x(3-4ax^2+6x^4-x^8)}{1-6x^4+4ax^6-3x^8},$$

$$\sqrt{k}\,sn\,4u=\frac{4x\,\Delta x\,(1-x^4)(1-2ax^2+6x^4-2ax^6+x^8)}{1-20x^4+32ax^6-(26+16a^2)x^8+32ax^{10}-20x^{12}+x^{16}},$$

$$\sqrt{k}\,sn\,5u=$$

$$\frac{x\{5-20ax^2+(62+16a^2)x^4-80ax^6-105x^8+360ax^{10}-(300+240a^2)x^{12}+(368a+64a^3)x^{14}-(125+160a^2)x^{16}+140ax^{18}-50x^{20}+x^{24}\}}{\{1-50x^4+140ax^6-(125+160a^2)x^8+(368a+64a^3)x^{10}-(300+240^2)x^{12}+360ax^{14}-105x^{16}-80ax^{18}+(62+16a^2)x^{20}-20ax^{22}+5x^{24}\}}$$

&c.

Thus, writing $-x^2$ for x^2, $k=1$, and $\therefore a=2$,

$$\tan 3u = x\,\frac{(3+8x^2+6x^4-x^8)}{1-6x^4-8x^6-3x^8}=\frac{x(3-x^2)(1+x^2)^3}{(1-3x^2)(1+x^2)^3}=\frac{x(3-x^2)}{1-3x^2},$$

where $x=\tan u$. (And in general in reducing $\tan nu$ the extraneous factor in the numerator and denominator is $(1+x^2)^{\frac{1}{2}n(n-1)}$.)

58, Chancery Lane, London, May 17, 1847.

(To be continued.)

[redacted]

ON CERTAIN ALGEBRAIC FUNCTIONS.

By James Cockle, M.A., of Trinity College, Cambridge; Barrister-at-Law of the Middle Temple.

I. A homogeneous function of the second degree and of m undetermined quantities ξ', ξ'',., $\xi^{(m)}$, may be written as follows:—

$$\kappa_1'^2\xi'^2 + 2(\kappa_1''\xi'' + \kappa_1'''\xi''' + . + \kappa_1^{(m)}\xi^{(m)})\,\kappa_1'\xi' + f(\xi'', \xi''', .., \xi^{(m)}) \ldots (1);$$

add to, and subtract from, this expression the square of half the coefficient of $\kappa_1'\xi'$, and let

$$\kappa_1'\xi' + \kappa_1''\xi'' + . + \kappa_1^{(m)}\xi^{(m)} = h_1:$$

then (1) may be put under the form

$$h_1^2 + \phi(\xi'', \xi''', .., \xi^{(m)}).$$

In like manner $\phi(\xi'', \xi''', .., \xi^{(m)})$, which is a homogeneous function of the second degree and of $m - 1$ undetermined quantities ξ'', ξ''',., $\xi^{(m)}$, may be written thus:—

$$\kappa_2''^2\xi''^2 + 2(\kappa_2'''\xi''' + \kappa_2^{iv}\xi^{iv} + . + \kappa_2^{(m)}\xi^{(m)})\,\kappa_2''\xi'' + \chi(\xi''', \xi^{iv}, .., \xi^{(m)}),$$

which expression may, by proceeding as before, be reduced to the form

$$h_2^2 + \psi(\xi''', \xi^{iv}, .., \xi^{(m)}),$$

where $$h_2 = \kappa_2''\xi'' + \kappa_2'''\xi''' + . + \kappa_2^{(m)}\xi^{(m)};$$

hence (1) may be represented by

$$h_1^2 + h_2^2 + \psi(\xi''', \xi^{iv}, .., \xi^{(m)});$$

and if we reduce ψ and the corresponding subsequent functions, as we have already done ϕ and the given one, we may put (1) under the form

$$h_1^2 + h_2^2 + . + h_r^2 + . + h_m^2,$$

where $$h_r = \kappa_r^{(r)}\xi^{(r)} + \kappa_r^{(r+1)}\xi^{(r+1)} + .. \kappa_r^{(m)}\xi^{(m)}.$$

II. A homogeneous function of the third degree and of v undetermined quantities Ξ', ξ', ., $\xi^{(v-1)}$ may be written thus:

$$K'^3\Xi'^3 + 3A'K'^2\Xi'^2 + B'K'\Xi' + C' \ldots\ldots (2),$$

where A', B', and C' are free from Ξ; and this expression again may be put under the form

$$(K'\Xi' + A')^3 + (B' - 3A'^2)\,K'\Xi' + C' - A'^3 \ldots . (3).$$

Let $$K'\Xi' + A' = h_1;$$

then, since $B' - 3A'^2$ is a homogeneous function of the second

degree and of $v-1$ undetermined quantities $\xi', \xi'', \ldots, \xi^{(v-1)}$, we may (by the processes of paragraph I.) put (3) under the form

$$\mathrm{h}_1^3 + (h_1^2 + h_2^2 + \ldots + h_{v-1}^2)\, \mathrm{K}'\Xi' + \mathrm{C}' - \mathrm{A}'^3 \ldots\ldots (4);$$

now, the $v-1$ quantities $\xi', \xi'', \ldots, \xi^{(v-1)}$, being perfectly undetermined, we may make

$$h_1^2 + h_2^2 = 0, \quad h_3^2 + h_4^2 = 0, \ldots,$$

or
$$h_1 + (1)^{\frac{1}{2}} h_2 = 0, \quad h_3 + (1)^{\frac{1}{2}} h_4 = 0, \ldots, \ldots\ldots (5);$$

the last of which lower line of equations is, ($v-1$ being supposed even),

$$h^2_{v-2} + h^2_{v-1} = 0.$$

By means of the system (5) of linear equations, eliminate, from $\mathrm{C}' - \mathrm{A}'^3$, (which is free from Ξ) $v-1$ of the ξ's, and let $f^a(b)$ denote a homogeneous function of the a^{th} degree and of b undetermined quantities; then, since the coefficient of $\mathrm{K}'\Xi'$ has been made to vanish, (4) may now be represented by

$$\mathrm{h}_1^3 + f^3\left(\frac{v-1}{2}\right),$$

for v write v_1 and let $v_{s+1} = \dfrac{v_s - 1}{2}$; then, by processes similar to those which we have just employed, we may reduce the given function of the third degree, to the form

$$\mathrm{h}_1^3 + \mathrm{h}_2^3 + \ldots + \mathrm{h}_s^3 + f^3(v_{s+1}).$$

The equation of finite differences which gives v_s is

$$v_{s+1} - \tfrac{1}{2} v_s = -\tfrac{1}{2},$$

of which the solution is

$$v_s = -1 + \mathrm{C}(\tfrac{1}{2})^{s-1} \ldots\ldots\ldots\ldots\ldots (6).$$

Now in order that the given function of the third degree may be reduced to the form of a sum of m cubes, and may, after such reduction, still involve m undetermined quantities, the cubic

$$f^3(v_{m+1}) = 0,$$

to which we shall be conducted when we have arrived at h_m^3, ought to contain *two* undetermined quantities; more than two would be superfluous, hence

$$v_{m+1} = 2 = -1 + \mathrm{C}(\tfrac{1}{2})^m \quad \text{(by 6)},$$

therefore $\quad \mathrm{C} = 3.2^m$, and v_1 (or v) $= 3.2^m - 1$;

also $\quad v_s = 3.2^{m-s+1} - 1$;

and $v_s - 1$ is, of course, even.

III. Adopting a notation employed in the preceding paragraph, let $f^4(u_m)$ denote a homogeneous function of the fourth degree and of u_m undetermined quantities ξ. Then, A', B', C' and D' representing quantities free from ξ', we may (omitting for convenience the multiplier of ξ') make

$$f^4(u_m) = \xi'^4 + 4A'\xi'^3 + B'\xi'^2 + C'\xi' + D'$$
$$= (\xi' + A')^4 + (B' - 6A'^2)\,\xi'^2 + \&c.,$$

which last form of $f^4(u_m)$ we may, continuing and extending the notation of the preceding paragraphs, represent by

$$H_1^4 + f^2(u_m - 1)\,\xi'^2 + f^3(u_m - 1)\,\xi' + f^4(u_m - 1);$$

but, $u_m - 1$ being supposed even, the processes of paragraph I. enable us to make the coefficient of ξ'^2 in this last expression vanish, and, referring to that paragraph, we see that we have the resulting equation

$$f^4(u_m) = H_1^4 + f^3\{\tfrac{1}{2}(u_m - 1)\}\,\xi' + f^4\{\tfrac{1}{2}(u_m - 1)\}\ldots(7).$$

Let u_x be determined from the following equation of finite differences,

$$u_{x+1} = 3.2^{1+2u_x} - 1\ldots\ldots\ldots\ldots\ldots(8);$$

then, substituting for u_m its value in terms of u_{m-1}, we may change (7) into the following:

$$f^4(u_m) = H_1^4 + f^3(3.2^{2u_{m-1}} - 1)\,\xi' + f^4(3.2^{2u_{m-1}} - 1)\ldots(9),$$

but the result of paragraph II. shews that a homogeneous function of the third degree and of $3.2^{2u_{m-1}} - 1$ undetermined quantities may be reduced to the form of a sum of $2u_{m-1}$ cubes involving $2u_{m-1}$ undetermined quantities. Group these cubes two and two together as we did the squares in (II.) the preceding paragraph, and equate each group to zero. We shall then have a number of equations of the form

$$h_r^3 + h_{r+1}^3 = 0, \quad \text{or} \quad h_r + (1)^{\frac{1}{3}}\,h_{r+1} = 0,$$

and if by means of these u_{m-1} equations we eliminate from the last term of (9) u_{m-1} of the quantities still remaining undetermined, we shall have, since the coefficient of ξ' will now be zero,

$$f^4(u_m) = H_1^4 + f^4(u_{m-1}),$$

and by similar processes this last equation may be reduced to

$$f^4(u_m) = H_1^4 + H_2^4 + . + H_x^4 + f^4(u_{m-x});$$

hence, in order that $f^4(u_m)$ may be reduced to the form of the sum of m fourth powers and may, after such reduction,

m undetermined quantities, we must, if we wish uperfluous quantities to determine, make

$$u_{m-m} = u_0 = 2,$$

and this ation, combined with (8), will completely determine u_m.

IV. So if w_v be determined from the equation

$$w_{v+1} = 3.2^{1+2u_{2w_v}} - 1 \ldots\ldots\ldots\ldots (10),$$

in which u has the same meaning, and is determined in the same manner as before, the method of the preceding paragraph shews that we may at once proceed to the following reduction of $f^5(w_m)$; viz.

$$f^5(w_m) = \ldots \xi + f^5(u_{2w_{m-1}});$$

and if, as the precedi also shows we may do, we reduce this last equat n

$$f^5(w_m) = \mathfrak{b}_1^{\,5} + (H_1^{\,4} + \ldots \ldots_{2w_{m-1}})\,\xi + f^5(2w_{m-1}) \ldots (11);$$

and, if we in the same manner in former cases, by means of w_{m-1} equations of the form

$$H_r + (1) \ldots = 0,$$

at which we may readily a ke the coefficient of ξ in (11) vanish, the equation (11) ake the form

$$f^5(w_m) = \mathfrak{b}_1^{\,5} + f^5(w_{m-1});$$

which, by proceeding in the same manner, may be further reduced to

$$f^5(w_m) = \mathfrak{b}_1^{\,5} + \mathfrak{b}_2^{\,5} + . + \mathfrak{b}_s^{\,5} + f^5(w_{m-s});$$

and reasoning in the same manner as in the last paragraph, we infer that the equation (10), combined with the following,

$$w_{m-m} = w_0 = 2,$$

will completely determine w_m. This investigation differs from the preceding ones as follows,—the others give absolute results, but this last leaves us an equation of the fifth degree to solve; so that all that we can say is, that we have made the difficulty of reducing $f^5(u_m)$ to the form

$$\mathfrak{b}_1^{\,5} + \mathfrak{b}_2^{\,5} + . + \mathfrak{b}_m^{\,5}$$

(where $\mathfrak{b}_r$ involves $m - r + 1$ undetermined quantities) depend upon that of solving an equation of the fifth degree. The discussion of the equations of differences above given as well as of that which occurs in the succeeding paragraph must be deferred till another opportunity.

[redacted]

V. For u write ${}_4u$, and for w write ${}_5u$, then if ${}_ru_m$ denote the number of disposable quantities necessary in order that an algebraic function of the r^{th} degree of those quantities may be made to satisfy the condition

$$f^r({}_ru_m) = {}'h_1^{\,r} + {}'h_2^{\,r} + . + {}'h_m^{\,r}$$

(${}'h$ having the same meaning as H, h, h, $\mathfrak{h}$, &c. and ${}'h$, involving $m - s + 1$ undetermined quantities), it will be found that the equation of finite differences

$${}_ru_{s+1} - 3.2^{1 + 2_4u\, 2_5u \,.\,.\, 2_ru_s} + 1 = 0,$$

and the subordinate ones implicitly included in it, combined with ${}_ru_0 = 2$, suffice for the determination of ${}_ru_m$, subject to the solution of an equation of the r^{th} degree. This will be seen if we reflect on the preceding paragraphs.

VI. It is hardly necessary to observe that, if m be even, then whenever we can reduce the left hand side of an algebraic equation, of which the right hand side is zero, to the above form, we may, by grouping the h^r's in pairs, equating the sum of each pair to zero, and making an obvious depression of degree, eliminate $\frac{1}{2}m$ of the quantities ξ', &c. between this equation and another, without introducing any elevation of degree by elimination, and without having to solve any equation of a degree higher than the higher of the two given equations.

VII. I now proceed to inquire what is the number of disposable quantities requisite in order that we may, by means of equations whose degrees shall not exceed the r^{th}, simultaneously satisfy a equations of the r^{th} degree, b of the $(r - 1)^{th}$, c of the $(r - 2)^{th}$, . , β of the second, and α of the first degree between those quantities.

VIII. Call the equations of the r^{th} degree the 1st, 2d, and a^{th}, respectively; then, if we can reduce the $(a - 1)^{th}$ equation to the form

$$h_1^{\,r} + h_2^{\,r} = 0,$$

the a^{th} equation will be solvable without elevation of degree arising from elimination. Now it will be seen that the processes employed in the reductions here treated of do not in any case conduct to an equation of a degree higher than the r^{th}. So that if we had

$${}_ru_2$$

disposable quantities, we might reduce the solution of the

$(s-1)^{th}$ and s^{th} equations to that of two equations of the r^{th}, and others of lower degrees.

IX. Again, in order that we may avoid elevation of degree from the $(s-2)^{th}$ equation, we must reduce it to the sum of $2_r u_s$ powers, and then group the powers two and two and eliminate. But this requires that we should have

$$ {}_r u_{2_r u_s} $$

disposable quantities, and the whole of the equations of the r^{th} degree will require that we should have

$$ {}_r u_{2_r u_{2_r u_{\cdot_{\cdot_{2_r u_s}}}}} $$

disposable quantities; in which expression fully written the letter u would occur $a-1$ times. Call this expression U or $u\left[\begin{smallmatrix} r \\ a \end{smallmatrix}\right]$. Then, from considerations similar to those which have conducted us to the above expression, we find that the b equations of the $(r-1)^{th}$ degree will increase this last expression to

$$ {}_{r-1}u_{2_{r-1}u_{2_{r-1}u_{\cdot_{\cdot_{2_{r-1}u\,U}}}}}; $$

in which expression, when fully written, u would occur b times. Call this last expression

$$ u\begin{bmatrix} r-1, & r \\ b, & a \end{bmatrix}; $$

then these operations must be carried on till we have ascertained how many quantities are required for the reduction of all the equations from the fourth degree upwards. The expression for the number of quantities requisite for this purpose will, on the principle of the above notation, be represented by

$$ u\begin{bmatrix} 4, & 5, & \ldots, & r-1, & r \\ \delta, & \epsilon, & \ldots, & b, & a \end{bmatrix}. $$

Then, if the equations of the second and third degrees be treated by obvious extensions of processes which I have above given and if the equations of the first degree be also taken into consideration, we shall find that the formula which expresses the number of arbitrary quantities necessary in order that we may make the solution of a equations of

the r^{th} degree, b of the $(r-1)^{\text{th}}$, ., γ of the third, β of the second, and a of the first, between the same unknowns depend upon the equations of the r^{th}, $(r-1)^{\text{th}}$, and lower degrees only, is

$$a + 2^{\beta}\left(3.2^{\,3.2^{\,3.2^{\,3.2^{\cdot^{\cdot^{\cdot^{\,3.2^{\,1+2u\left[\begin{matrix}4, & 5, . . , & r-1, & r\\ \delta, & \epsilon, . . , & b, & a\end{matrix}\right]}}}}}-1}-1}-2}-1\right),$$

$$\text{or } \Upsilon;$$

in which formula there are supposed to be γ lines of exponents, γ being the number of equations of the third degree which it is required to satisfy. Were these formulæ incapable of giving illusory results, there would remain but little to be done in the Theory of Algebraic Equations; the formulæ would also have other applications. But in order to ascertain the limits of their application, and to compare particular cases of them with corresponding cases in Mr. Jerrard's method, we must ascertain the results which follow from supposing $\Upsilon - v$ of the quantities ξ, ξ', &c. to be functions of the remaining v of them. This point I shall defer, but I hope to discuss the question at some future time in its most general form and so as to include the isolated results at which I have already arrived in a contemporary periodical.*

This paper contains, I think, everything necessary for the complete development of the method of which I have already given isolated discussions in the *Philosophical Magazine*, and the *Mathematician*.

Postscript. June 14, 1847. The reduction which I suggested in a short note, published at pages 285–286 of vol. I. of the present series of this work, will be found to simplify my discussion at page 105 of vol. III. of the former series.

2, *Church-Yard Court, Temple,*
February 25, 1847.

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* The *London, Edinburgh, and Dublin Philosophical Magazine.*

$$x_1 y_1 + x_2 y_2 - x_3 y_3 = 0 \ldots\ldots\ldots\ldots (16),$$

$$x_1 z_1 + x_2 z_2 - x_3 z_3 = 0 \ldots\ldots\ldots\ldots (17),$$

$$y_1 z_1 + y_2 z_2 - y_3 z_3 = 0 \ldots\ldots\ldots\ldots (18).$$

$$\pm\frac{x_1}{a} = \frac{y_2 z_3 - y_3 z_2}{bc},\quad \pm\frac{x_2}{a} = \frac{y_3 z_1 - y_1 z_3}{bc},\quad \mp\frac{x_3}{a} = \frac{y_1 z_2 - y_2 z_1}{bc} \ldots (19),$$

$$\pm\frac{y_1}{b} = \frac{x_3 z_2 - x_2 z_3}{ac},\quad \pm\frac{y_2}{b} = \frac{x_1 z_3 - x_3 z_1}{ac},\quad \mp\frac{y_3}{b} = \frac{x_2 z_1 - x_1 z_2}{ac} \ldots (20),$$

$$\mp\frac{z_1}{c} = \frac{x_2 y_3 - x_3 y_2}{ab},\quad \mp\frac{z_2}{c} = \frac{x_3 y_1 - x_1 y_3}{ab},\quad \pm\frac{z_3}{c} = \frac{x_1 y_2 - x_2 y_1}{ab} \ldots (21).$$

The preceding equations are analogous to those given in my paper entitled "Investigation of certain Properties of the Ellipsoid," (see *Journal*, New Series, vol. II., pp. 13–19), of which communication this is designed to form a continuation. Most of the theorems deduced in that paper are true of conjugate hyperboloids with some slight modifications, the chief of which arise from the squares of all the lines that refer to the hyperboloid of two sheets (those lines which have 3 subscribed) having a negative sign.

As the investigations of the following properties of conjugate hyperboloids, (a)....(g), are the same as those of the analogous ones (A)....(G), in the paper just mentioned, I shall, for brevity's sake, omit them; also since the enunciations of (b) and (c) would be the same as those of (B) and (C), the latter will, for the same reason, be simply referred to.

(a) If three conjugate points be projected on any diametral plane by lines drawn parallel to the diameter conjugate to this plane, the difference between the square of the line of projection drawn from the point on the hyperboloid of two sheets, and the sum of the squares of the other two lines, is equal to the square of the semidiameter.

(b) The same as (B), see *Journal*, vol. II. p. 15, New Series.

(c) The same as (C), *ib.*

(d) The difference between the square of any diameter of the hyperboloid of two sheets, and the sum of the squares of two conjugate diameters (which appertain to the hyperboloid of one sheet) is constant. Hence, should it happen that $a^2 + b^2 = c^2$, the square of any diameter of the hyperboloid of two sheets will be equal to the sum of the squares of two conjugate diameters.

[redacted]

(e) Each conjugate parallelepiped* is equal to that constructed on the principal diameters.

(f) The difference between the sum of the squares of those two faces of any conjugate parallelepiped which touch the hyperboloid of two sheets, and the sum of the squares of the other four faces (which touch the hyperboloid of one sheet) is constant. Hence, if $a^2b^2 = a^2c^2 + b^2c^2$ or $\frac{1}{c^2} = \frac{1}{a^2} + \frac{1}{b^2}$, the sum of the squares of the two faces will be equal to the sum of the squares of the four.

(g) If perpendiculars be drawn from the centre of conjugate hyperboloids on any three conjugate tangent planes, the difference between the square of the reciprocal of the perpendicular on the tangent plane to the hyperboloid of two sheets, and the sum of the squares of the reciprocals of the other two perpendiculars, is constant. Hence if $\frac{1}{c^2} = \frac{1}{a^2} + \frac{1}{b^2}$, the square of the reciprocal of the former perpendicular will be equal to the sum of the squares of the reciprocals of the other two.

The locus of the intersections of three conjugate tangent planes will be obtained by eliminating x_1, x_2, &c. from (4, 5, 6), and this elimination is at once effected by deducting the square of (6) from the sum of the squares of (4, 5), and reducing by (13)....(18); therefore

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Hence

(h) The locus of the intersections of conjugate tangent planes to conjugate hyperboloids is the hyperboloid of one sheet itself. Hence, also,

(i) Every conjugate parallelepiped to conjugate hyperboloids is inscribed in the hyperboloid of one sheet itself.

The theorems (h, i) which differ widely from the corresponding propositions (H, I) for the ellipsoid, are very remarkable, and, I must confess, these results were totally unexpected by me. Recollecting that conjugate parallelepipeds to conjugate hyperboloids are in some respects analogous to conjugate parallelograms to conjugate hyperbolas,

* The faces of a *conjugate parallelepiped* touch the conjugate hyperboloids and are parallel to conjugate diametral planes.

I imagined that the locus would be the asymptotic cone (3). It is not, however, conjugate parallelepipeds, but conjugate cylinders (to be noticed presently) that are here analogous to conjugate parallelograms.

Theorems in reference to conjugate hyperboloids have now been given analogous to all those contained in the paper on the ellipsoid, except to (K), (L), and (M), and I have not been able to discover any properties similar to these. (In consequence none of the following propositions will be marked (k), (l), or (m).) I shall now introduce a few additional properties of the hyperboloids, several of which have analogues in Plane Geometry.

Referring the conjugate hyperboloids (1, 2) and the asymptotic cone (3) to any conjugate diameters, $A_1'OA_1 = 2a_1$, $B_1'OB_1 = 2b_1$, and $C_1'OC_1 = 2c_1$, their equations will be

$$\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} - \frac{z^2}{c_1^2} = 1 \ldots\ldots\ldots\ldots\ldots (22),$$

$$\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} - \frac{z^2}{c_1^2} = -1 \ldots\ldots\ldots\ldots (23),$$

and

$$\frac{x^2}{a_1^2} - \frac{y^2}{b_1^2} - \frac{z^2}{c_1^2} = 0 \ldots\ldots\ldots\ldots\ldots (24).$$

The equations of the tangent planes at C_1' and C_1 to the hyperboloid of two sheets (23), are $z = -c_1$, and $z = c_1$; and either of these values of z substituted in (24) gives $\frac{x^2}{a_1^2} - \frac{y^2}{b_1^2} = 1$, for the equation to the section of the cone by the tangent plane at C_1' or C_1; now this is the very same equation we should get by putting $z = 0$ in (22); hence

(n) If sections of the asymptotic cone be made by two parallel tangent planes to the hyperboloid of two sheets, each section (which is an ellipse) is equal, similar and similarly posited, to the section of the conjugate hyperboloid made by a parallel diametral plane.

Again, let sections of (22, 23, 24) be made by the plane $z = z_1$, parallel to that of xy. The areas of these sections being denoted by L, M, N, respectively, and the angle A_1OB_1 by ϕ, we shall evidently have

$$\text{L} = \pi a_1 b_1 \left(\frac{z_1^2}{c_1^2} + 1\right) \sin\phi, \qquad \text{M} = \pi a_1 b_1 \left(\frac{z_1^2}{c_1^2} - 1\right) \sin\phi,$$

$$\text{N} = \pi a_1 b_1 \frac{z_1^2}{c_1^2} \sin\phi;$$

[redacted]

therefore L − N = N − M = $\pi a_1 b_1 \sin\phi$ = area of parallel diametral section. It is hence easily seen that

(o) If elliptic sections of two conjugate hyperboloids and the asymptotic cone be made by any plane, the area of each of the two elliptic rings bounded by the curves of section will be equal to that of the parallel diametral section, or, (n), to that of the section of the cone made by a parallel tangent plane. Hence also the elliptic rings are equal in area for all parallel sections.

Def. *A conjugate cylinder* to conjugate hyperboloids has its generators parallel to a diameter of the hyperboloid of two sheets and tangent to the hyperboloid of one sheet, and it is limited by the tangent planes touching at the extremities of the diameter.

It is plain that

(p) The diametral section parallel to the ends of a conjugate cylinder is the locus of the points of contact of the generators with the hyperboloid of one sheet, and hence also, (n), the asymptotic cone is the locus of the perimeters of the ends of all conjugate cylinders.

It is evident that a conjugate cylinder and the corresponding conjugate parallelepiped have the same altitude, and that their bases are in the proportion $\pi : 4$; hence, (e),

(q) All conjugate cylinders are equal to each other, and the volume of each is $2\pi abc$; a, b, c being the principal semidiameters.

Moreover, the portion of the asymptotic cone cut off by a tangent plane to the hyperboloid of two sheets has evidently the same base as the corresponding conjugate cylinder, but only half its altitude; the volume of the former solid is consequently equal to one-sixth of that of the latter. Hence the following remarkable theorem.

(r) The volume of the portion of the asymptotic cone cut off by a tangent plane to an hyperboloid of two sheets is constant and equal to $\frac{1}{3}\pi abc$.

I shall next establish the following theorem.

(s) If from any point in an hyperboloid of two sheets as vertex, a cone be described having its generators parallel to those of the asymptotic cone, the volume of the solid included between the surfaces of these two cones is constant and equal to $\frac{1}{12}\pi abc$.

For the tangent plane at the point will, (r), cut off from the asymptotic cone a solid $U = \frac{1}{3}\pi abc$, and the solid V mentioned in (s) is evidently composed of two solids W similar to U, but of only half the (linear) dimensions; hence $W = \frac{1}{8}U = \frac{1}{24}\pi abc$, and $V = 2W = \frac{1}{12}\pi abc$.

(t) If any straight line be drawn cutting an hyperboloid in U_1, V_1, the conjugate hyperboloid in U_2, V_2, and the asymptotic cone in U, V; then will $UU_1 = VV_1$, $UU_2 = VV_2$, and $UU_1 . U_1V = UV_1 . V_1V = UU_2 . U_2V = UV_2 . V_2V =$ square of the parallel semidiameter.

The truth of (t) may easily be shewn by referring the surfaces to conjugate diameters, one of which shall be parallel to the straight line. It will also be apparent by drawing a diametral plane through the straight line which will cut the hyperboloids in conjugate hyperbolas and the asymptotic cone in their asymptotes; then, applying well-known properties of the hyperbola, we shall have the theorems (t) at once.

In conclusion, I would observe that the consideration of conjugate hyperboloids seems to be as necessary as that of conjugate hyperbolas. We can obtain a clear geometrical conception of many theorems when enunciated as properties of conjugate hyperboloids, of which we have, I think, but an obscure notion when presented as properties of only *one* of these surfaces. In truth, the few properties of the kind here alluded to, that are usually given in works on Analytical Geometry of Three Dimensions, are enunciated with a tacit reference to the ellipsoid, and the student is afterwards merely informed that the squares of certain quantities are negative in the case of either of the hyperboloids. He is thus furnished with analytical expressions, but with only a very confused idea of their geometrical meaning. The introduction of the conjugate hyperboloid, however, completely dispels this obscurity, and enables us to enunciate such theorems with precision.

ADDENDUM. Since the preceding paper was sketched, it has occurred to me that most of the theorems, (o). . . .(t), have analogues in reference to the ellipsoid, while some of them are capable of being enunciated with still greater generality. I shall insert these propositions here, but shall omit the investigations (which indeed are not difficult) in order to save space.

DEF. A *conjugate cylinder* circumscribed about an ellipsoid is a cylinder whose ends touch the ellipsoid at the extremities of the diameter parallel to the generators.

[redacted]

(P) The locus of the perimeters of the ends of conjugate cylinders circumscribed about an ellipsoid is a concentric similar ellipsoid whose principal diameters are to those of the given ellipsoid as $\sqrt{2} : 1$.

(Q) An ellipsoid is two-thirds of each circumscribed conjugate cylinder, and hence all conjugate cylinders circumscribed about the same ellipsoid are equal to one another.

The former part of this proposition is an extension of a property of the sphere.

(O) If elliptic sections of two concentric, similar, and similarly situated surfaces of the second order be made by parallel planes, the elliptic rings bounded by the curves of section will be equal to each other.

(R) Tangent planes to the inner of two concentric, similar, and similarly posited surfaces of the second order cut off equal volumes from the other.

The propositions (O) and (R) hold if the surfaces are either ellipsoids or hyperboloids, and there are analogous properties in respect of two *equal* elliptic paraboloids which have their principal axes in the same straight line and are similarly posited.* The following properties, (T), are true of any of the surfaces of the second order, providing the enunciation be modified as before for the paraboloids.

(T) If there be two concentric, similar, and similarly posited surfaces of the second order and any straight line be drawn cutting the outer surface in U', V', and the other in U'', V''; then $U'U'' = V'V''$, and $U'U''.U''V' = U'V''.V''V'$ is constant for all parallel lines. (When the surfaces are ellipsoids and are related to each other as in (P), $U'U''.U''V' = U'V''.V''V' =$ square of the parallel semidiameter of the inner ellipsoid.)

To be able to perceive that (O), (R), and (T) are extensions of (o), (r), and (t), it must be recollected that a cone is a limiting case of either of the hyperboloids.

Cottenham St., Newcastle-upon-Tyne,
May 4, 1847.

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* That is, providing the equations to the two paraboloids be

$$z + d = \frac{x^2}{p_1} + \frac{y^2}{p_2}, \text{ and } z + d' = \frac{x^2}{p_1} + \frac{y^2}{p_2},$$

the principal axis of each being the axis of z.

NOTES ON HYDRODYNAMICS.

I.—*On the Equation of Continuity.*

By William Thomson.

The following proof of the Equation of Continuity is simpler than that which is generally given in treatises on Hydrodynamics, and it has also the advantage of shewing in a clearer manner the nature of the property of fluid motion expressed.* Thus, instead of considering a portion of the moving fluid and the varying space which the particles composing it occupy at successive instants, as in the ordinary proof, we imagine a space S fixed in the interior of the fluid, and we consider the fluid which flows into this space, across part of the bounding surface, and that which flows out of it, across the remainder in a given interval of time. The equation of continuity is the analytical expression of the fact that the change in the mean density of the fluid in the space S, during the interval of time considered, is due to the difference between the quantities of fluid which, in that interval, flow into it and out of it, or, if the fluid be of invariable density, that these quantities are equal; and its generality, as applied to all cases of fluid motion, is subject to no exception.

Let the space S be an infinitely small parallelepiped, of which the edges α, β, γ are parallel to the axes of coordinates, and let x, y, z be the coordinates of its centre; so that $x \pm \frac{1}{2}\alpha$, $y \pm \frac{1}{2}\beta$, $z \pm \frac{1}{2}\gamma$ are the coordinates of its angular points. Let ρ be the density of the fluid at (x, y, z), or the mean density through the space S, at the time t. The density at the time $t + dt$ will be $\rho + \dfrac{d\rho}{dt}dt$; and hence the quantities of fluid contained in the space S, at the times t, and $t + dt$, are respectively $\rho \,.\, \alpha\beta\gamma$ and $\left(\rho + \dfrac{d\rho}{dt}dt\right)\alpha\beta\gamma$. Hence the quantity of fluid lost (there will of course be an absolute gain if $\dfrac{d\rho}{dt}$ be positive) in the time dt is

$$-\frac{d\rho}{dt}dt \,.\, \alpha\beta\gamma \; \ldots\ldots\ldots\ldots \; (a).$$

* Poisson admits that the proof which he gives is inapplicable to certain conceivable circumstances of fluid motion; but he erroneously concludes that in such cases the equation "of continuity" does not hold. (See Poisson's *Traité de Mécanique*, No. 651.) The proof in the text has been frequently given in lectures at Cambridge, and elsewhere, and it is likely to occur to any one reading Fourier's Theory of Heat; but I am not aware that it has been hitherto published in any work except Duhamel's *Cours de Mécanique* (Deuxième Partie; Paris 1847).

Now let u, v, w be the three components of the velocity of the fluid (or of a fluid particle*) at P. These quantities will be functions of x, y, z, (involving also t, except in the case of "steady motion,") and will in general vary gradually from point to point of the fluid; although the analysis which follows is not restricted by this consideration, but holds even in cases where in certain places of the fluid there are abrupt transitions in the velocity, as may be seen by considering them as limiting cases of motions in which there are very sudden continuous transitions of velocity. If ω be a small plane area, perpendicular to the axis of x, and having its centre of gravity at P, the volume of fluid which flows across it in the time dt will be equal to $u \,.\, \omega \,.\, dt$, and the mass or quantity will be $\rho \,.\, u \,.\, \omega \,.\, dt$. If we substitute $\beta\gamma$ for ω, the quantity which flows across the either of the sides $\beta\gamma$ of the parallelepiped S, will differ from this only on account of the variation in the value of ρu; and therefore the quantities which flow across the two sides $\beta\gamma$ are respectively

$$\left\{\rho u - \tfrac{1}{2} a \frac{d(\rho u)}{dx}\right\} . \beta\gamma \,.\, dt,$$

and

$$\left\{\rho u + \tfrac{1}{2} a \frac{d(\rho u)}{dx}\right\} . \beta\gamma \,.\, dt.$$

* This explanatory clause must be omitted, and a modified definition of fluid velocity must be given, if it be required to include the case, imagined by Poisson, of the motion of two fluids of different densities *through* one another, in which, as he conceives, the "*molécules*" of the lighter fluid will move upwards, between the "*molécules*" of the heavier fluid, which descend. Thus we should define u as the mean velocity of the "*molécules*" parallel to the axis of x, across any very small plane of which the centre of gravity is at P, and we should thus obtain the same equation as that found in the text, which is applicable to this as to every possible case of fluid motion. It is however very doubtful whether this kind of motion can actually exist in nature. In the case, considered by Poisson (Art. 651), of water contained in a vertical cylinder, open above, and heated at its bottom which is supposed to be horizontal, it is certainly true that the regular upward motion of the whole fluid due to the expansion of the lower strata is practically impossible, because unstable; but as far as experience indicates, (by the *streaks* we can see on looking into the vessel, on account of the varying refracting power of the heterogeneous liquid,) we find that the effect of the instability is to disturb the surfaces of equal density from being horizontal planes, and thus to allow finite portions of the lighter fluid to ascend, their places being filled by the heavier fluid descending. The definition, in the text, of the components, u, v, w, is directly applicable to this kind of motion; and the analysis and resulting equation, when interpreted according to the principles of the differential calculus as applied to discontinuous functions, will not be subject to exception even in cases when the ascending and descending portions slide upon one another with finite velocities; cases which might actually occur were there no "friction of fluids in motion."

Hence $\alpha \dfrac{d(\rho u)}{dx} . \beta\gamma . dt$, or $\dfrac{d(\rho u)}{dx} . \alpha\beta\gamma . dt$ is the excess of the quantity of fluid which leaves the parallelepiped across one of the faces $\beta\gamma$ above that which enters it across the other. By considering in addition the effect of the motion across the other faces of the parallelepiped, we find for the total quantity of fluid lost from the space S, in the time dt,

$$\left\{\frac{d(\rho u)}{dx} + \frac{d(\rho v)}{dy} + \frac{d(\rho w)}{dz}\right\} . \alpha\beta\gamma . dt \;\ldots\ldots (b).$$

Equating this to the expression (a), previously found, we have

$$\left\{\frac{d(\rho u)}{dx} + \frac{d(\rho v)}{dy} + \frac{d(\rho w)}{dz}\right\} . \alpha\beta\gamma . dt = - \frac{d\rho}{dt} . dt . \alpha\beta\gamma ;$$

and we deduce

$$\frac{d(\rho u)}{dx} + \frac{d(\rho v)}{dy} + \frac{d(\rho w)}{dz} + \frac{d\rho}{dt} = 0 \;\ldots\ldots\ldots (1),$$

which is the required equation.

If, instead of taking the infinitely small parallelepiped $\alpha\beta\gamma$, and the infinitely small interval of time dt, we consider a finite space S bounded by a fixed surface, and a finite interval of time, from t_1 to t_2, the equation of continuity should, it is clear from the demonstration given above, express this fact; that the mass of fluid in S at the time t_2 is equal to the mass at the time t_1, wanting the total mass which has been taken away by the flux across the surface. This is verified directly by the following analytical process.*

Let ρ_1, ρ_2 be the densities of the fluid at (xyz) at the times t_1, t_2, and let M_1, M_2 be the total masses contained at those times in the space S. We shall have

$$M_1 = \iiint \rho_1 \, dx\, dy\, dz, \quad M_2 = \iiint \rho_2 \, dx\, dy\, dz,$$

and therefore $M_2 = M_1 + \iiint (\rho_2 - \rho_1)\, dx\, dy\, dz$

$$= M_1 + \int_{t_1}^{t_2} dt \iiint \frac{d\rho}{dt}\, dx\, dy\, dz \;\ldots\ldots\ldots (2).$$

But, by the equation of continuity, we find

$$-\iiint \frac{d\rho}{dt}\, dx\, dy\, dz = \iiint \left\{\frac{d(\rho u)}{dx} + \frac{d(\rho v)}{dy} + \frac{d(\rho w)}{dz}\right\} dx\, dy\, dz,$$

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* Compare *Camb. Math. Jour.* vol. III. p. 203; also Poisson, *Théorie de la Chaleur*, p. 177.

[redacted]

and hence, by separating the second member into three terms, and performing the integrations, in the first with respect to x, in the second with respect to y, and in the third with respect to z, and assigning the limits so as to include the whole space S, we have

$$-\iiint \frac{d\rho}{dt}\, dx\, dy\, dz = \iint \rho u . dy\, dz + \iint \rho v . dz\, dx + \iint \rho w . dx\, dy \ldots (3),$$

where the values of xyz in each term of the second member belong to the surface of S. Now let ds be an element of the surface at xyz, and let l, m, n be the direction cosines of a normal: we may take ds such that

$$ds . l = dy\, dz, \quad ds . m = dz\, dx, \quad ds . n = dx\, dy\,;$$

and modifying accordingly the second member of (3), we have

$$-\iiint \frac{d\rho}{dt}\, dx\, dy\, dz = \iint \rho\, (lu + mv + nw)\, ds \ldots . (4).$$

Hence (2) becomes

$$M_2 = M_1 - \int_{t_1}^{t_2} dt \iint \rho\, (lu + mv + nw)\, ds,$$

or $$M_2 = M_1 - \iint ds . \int_{t_1}^{t_2} \rho\, (lu + mv + nw)\, dt \ldots\ldots (5).$$

Now $lu + mv + nw$ is the component of the velocity of the fluid in the direction of the normal at xyx, and therefore $ds . \int_{t_1}^{t_2} \rho\, (lu + mv + nw)\, dt$ is the quantity which flows out of the space S, across the element ds, of the surface, in the interval considered. Hence the total quantity, lost from S in the time $t_2 - t_1$, is equal to the integral in the second member of equation (5), and this equation is therefore the expression of the required result.

If the mass we are considering be a liquid (that is to say, an incompressible fluid), even although it be heterogeneous, the equation of continuity assumes a simpler form. For the density at a point xyz, moving with the fluid, will be invariable, and therefore the differential of ρ, considered as a function of x, y, z, t, will vanish provided we take $dx = udt$, $dy = vdt$, $dz = wdt$. Hence

$$\frac{d\rho}{dt} dt + \frac{d\rho}{dx} . udt + \frac{d\rho}{dy} . vdt + \frac{d\rho}{dz} . wdt = 0.$$

Dividing by dt, then subtracting the first member from that of the equation of continuity, and dividing the result by ρ, we find

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0 \;\ldots\ldots\ldots\ldots\; (6).$$

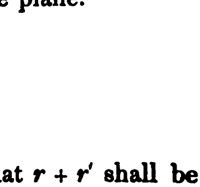
This equation has the same form as in the case when the liquid is homogeneous, as might easily have been proved directly, by considering merely the volume of fluid which flows through the space S, and not its mass, which, when the fluid is incompressible, will enable us to arrive at the equation of continuity.

St. Peter's College, Sept. 29, 1847.

MATHEMATICAL NOTE.

On the Maximum or Minimum Property of Incident and Reflected Rays.

The following is a very simple analytical proof of the proposition, that when a ray of light is reflected at any surface, the length of the path of the ray, measured from a given point in the incident to a given point in the reflected ray, is less than it would be according to any law of reflexion other than the actual law.



Let P be the point of incidence, SP the incident, PH the reflected ray, PG the normal to the surface at P.

Then we have to prove, that if $SP+PH$ is a minimum,

then (1) SP, PH, PG are in the same plane.

(2) $SPG = HPG$.

Let $x\,y\,z$ be the coordinates of P,

$\alpha\,\beta\,\gamma$ .... S,

$\alpha'\beta'\gamma'$ .... H.

$SP=r$, $PH=r'$. Then the condition that $r+r'$ shall be a minimum gives us

$$dr+dr'=0,$$

or $$\frac{dr}{dx}dx+\frac{dr}{dy}dy+\frac{dr}{dz}dz+\frac{dr'}{dx}dx+\frac{dr'}{dy}dy+\frac{dr'}{dz}dz=0 \ldots (A).$$

Now it will be easily seen that the equations of SP, PH, PG, are respectively

$$\frac{x_1 - x}{\frac{dr}{dx}} = \frac{y_1 - y}{\frac{dr}{dy}} = \frac{z_1 - z}{\frac{dr}{dz}},$$

$$\frac{x_1 - x}{\frac{dr'}{dx}} = \frac{y_1 - y}{\frac{dr'}{dy}} = \frac{z_1 - z}{\frac{dr'}{dz}},$$

$$\frac{x_1 - x}{\frac{df}{dx}} = \frac{y_1 - y}{\frac{df}{dy}} = \frac{z_1 - z}{\frac{df}{dz}},$$

f being such a function that $f(x, y, z) = 0$ is the equation of the given surface. In order that these may lie in the same plane, we must have

$$\frac{df}{dx}\left(\frac{dr}{dy}\frac{dr'}{dz} - \frac{dr}{dz}\frac{dr'}{dy}\right) + \frac{df}{dy}\left(\frac{dr}{dz}\frac{dr'}{dx} - \frac{dr}{dx}\frac{dr'}{dz}\right) + \frac{df}{dz}\left(\frac{dr}{dx}\frac{dr'}{dy} - \frac{dr}{dy}\frac{dr'}{dx}\right) = 0 \ldots \text{(B)}.$$

But
$$\frac{df}{dx}dx + \frac{df}{dy}dy + \frac{df}{dz}dz = 0,$$

and this equation and (A) are the only relations between dx, dy, dz; hence we may take

$$\frac{df}{dx} = \lambda\left(\frac{dr}{dx} + \frac{dr'}{dx}\right),$$

$$\frac{df}{dy} = \lambda\left(\frac{dr}{dy} + \frac{dr'}{dy}\right),$$

$$\frac{df}{dz} = \lambda\left(\frac{dr}{dz} + \frac{dr'}{dz}\right).$$

If we substitute these values, (B) assumes the form

$$\left(\frac{dr}{dx} + \frac{dr'}{dx}\right)\left(\frac{dr}{dy}\frac{dr'}{dz} - \frac{dr}{dz}\frac{dr'}{dy}\right) + \left(\frac{dr}{dy} + \frac{dr'}{dy}\right)\left(\frac{dr}{dz}\frac{dr'}{dx} - \frac{dr}{dx}\frac{dr'}{dz}\right) + \left(\frac{dr}{dz} + \frac{dr'}{dz}\right)\left(\frac{dr}{dx}\frac{dr'}{dy} - \frac{dr}{dy}\frac{dr'}{dx}\right) = 0,$$

[redacted]

which being an identical equation, the first part of the proposition is true.

The second follows very simply, for let TPT' be the intersection of the tangent plane with the plane SPH; then if ds be an element of the line TPT',

$$\cos SPT = \frac{dr}{dx}\frac{dx}{ds} + \frac{dr}{dy}\frac{dy}{ds} + \frac{dr}{dz}\frac{dz}{ds},$$

and $$\cos HPT = \frac{dr'}{dx}\frac{dx}{ds} + \frac{dr'}{dy}\frac{dy}{ds} + \frac{dr'}{dz}\frac{dz}{ds};$$

therefore, by the fundamental equation (A),

$$\cos SPT + \cos HPT = 0,$$

or $$SPT = 180° - HPT = HPT',$$

and hence $$SPG = HPG,$$

which is the second part of the proposition.

H. G.

Cambridge, Sept. 29, 1847.

[The following geometrical proof, although not new, may be added, in connection with the preceding.

With S and H as foci, and SH as axis of revolution, describe a prolate spheroid, touching the reflecting surface in P. Then SPH is the course of the incident and reflected ray, since the plane SPH, passing through the axis of the spheroid, is perpendicular to the tangent plane at P, and SP, PH, by the known property of the ellipse, make equal angles with the normal PG. Now the value of $SQ + QH$, for any point Q, without the spheroid, is, as follows from another well-known property of the ellipse, greater than $SP + PH$. Hence if the spheroid is touched externally by the reflecting surface, the actual course of the incident and reflected ray is less than if the point of incidence on the surface were in any other position Q, in the neighbourhood of P. We see also that, in general, the point of incidence, P, on the surface is determined by the maximum or minimum condition; although in some cases $SP + PH$ may be actually a maximum, and in others neutral.]

END OF VOL. II.

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SCIENTIFIC JOURNALS.

Journal de Mathématiques pures et appliquées, ou Recueil Mensuel de Mémoires sur les diverses parties des Mathématiques. **Publié par Joseph Liouville, Membre de l'Académie des Sciences et du Bureau des Longitudes.**

Tome XI. (1846.) No. VII. Remarque sur un point fondamental de la *Mécanique analytique* de Lagrange; par M. *Poinsot*.—Note sur l'emploi d'un symbole susceptible d'être introduit dans les éléments du calcul différentiel; par M. *Ernest Lamarle*.—Lettres sur diverses questions d'analyse et de physique mathématique concernant l'ellipsoïde, adressées à M. P. H. Blanchet, par *J. Liouville*. (Deuxième Lettre.)—Sur la surface des ondes; par M. *A. Cayley*. —No. VIII. Note sur les fonctions de M. Sturm; par M. *A. Cayley*.—Sur les surfaces dont les rayons de courbure sont égaux, mais dirigés en sens opposés; par M. *Michael Roberts*.—Note sur le développement des fonctions en séries ordonnées suivant les puissances ascendantes des variables; par M. *Augustin Cauchy*.

The Mathematician. **Edited by William Rutherford, F.R.S.A., and Stephen Fenwick. Vol. II. No. 4. London, *Dec.* 1846.**

On the Circular Sections of Surfaces of the Second Degree.—Surfaces of the Second Order referred to three Tangent Planes.—Properties of the Ellipse.—On Tangent Planes.—On the Theory of Collineation.—Modern Geometry.—On a New Method of Integrating Linear Differential Equations.—Solutions of Mathematical Exercises.—Mathematical Exercises (*continued*).

(1846) Heft III.—10. De curvis catenariis sphaericis dis- ;eometrica. Auct. Dr. *Christoph. Gudermann*, Math. Prof. ph.—11. Untersuchungen über die analytischen Facultäten. of. *Ottinger* zu Freiburg im Br. (Fortsetzung der Abhand- ersten und No. 7 im vorigen Heft.)—12. Démonstration élémen- proposition générale de la théorie des probabilités. Extrait d'un l'analyse élémentaire de la théorie des probabilités. Par à Moscou.—13. Note sur la varation des constantes arbi- grale définie. Par Mr. le Dr. *O. Schlömilch*, professeur à a.—Fac-simile einer von dem Herrn Professor *Möbius* zu ausgeber gefälligst mitgetheilten Handschrift von *Oriani*.— V.—14. curvis cat sphaericis dissertatio analytico-geometrica. dissert. 10 tom. 33 libr. 3.) Auct. Dr. *Christoph. Gudermann*, Math. rd. Monast. Guestph.—15. Note sur quelques intégrales définies. Par Dr. *O. Schlömilch*, professeur à l'université de Jena.—16. Sur l'intégrale $\int_0^\infty \frac{\partial\theta}{\theta^2 + a^2} e^{-x\theta}$. Par Mr. le Dr. *O. Schlömilch*, professeur à l'université)—17. Nachtrag zu Abschnitt I. und II. der Untersuchungen über die Von dem Herrn Prof. *Ottinger* zu Freyburg in Br.— ntégrales définies, renfermant des fonctions *O. Schlömilch*, professeur à l'université de einer gewissen endlichen Reihe. Von Herrn -20. Über die Anwendung der Sturm'schen Methode chungen. Von Herrn Dr. *Stern* in Göttingen.— Herrn Professor *Steiner* im 1 Hefte 16 Bandes d. J. ellten A . Von dem Herrn Conrector *Fasbender* in Iserlohn. Remarques s ondition d'égalité de deux racines d'une équation algébrique; et sur quelques théorèmes de Géométrie, qui en suivent. Par Mr. *F. Joachimsthal* de Berlin.—Fac-simile einer englischen Handschrift von *J. Newton*, welches das Journal der Güte des Herrn *Cailey* in Cambridge verdankt. Diese Handschrift ist 16 Jahre jünger als die beim 4ten Heft 32ten Bandes mitgetheilte lateinische Handschrift.

(GLASGOW, *May* 19, 1847.)

SCIENTIFIC JOURNALS.

Journal de Mathématiques pures et appliquées, ou Recueil Mensuel de Mémoires sur les diverses parties des Mathématiques. Publié par Joseph Liouville, Membre de l'Académie des Sciences et du Bureau des Longitudes.

TOME XI. (1846.) No. XII. Note sur les surfaces orthogonales; par M. *J. Bouquet.* (Fin.)—Note sur la surface réglée dont les rayons de courbure principaux sont égaux et dirigés en sens contraires; par M. *J.-A. Serret.*—Sur une transformation de l'équation

$$\sin\theta\,\frac{d.\sin\theta\,\frac{d\Phi}{d\theta}}{d\theta}+\frac{d^2\Phi}{d\omega^2}+n\,(n+1)\sin^2\theta.\Phi=0\,;$$

par *J. Liouville.*—Sur la décomposition des fractions rationnelles; par *J. Liouville.*—Sur l'intégrale $\int_0^\infty e^{-x}\,x^n\,dx$; par *J. Liouville.*—Sur une classe d'équations du premier degré; par *J. Liouville.*—Théorèmes de géométrie; par M. *J. Steiner.*—Sur l'intégrale définie $\int_0^{\frac{1}{2}\pi}\frac{\log(1+n\sin^2\phi)\,d\phi}{\sqrt{1-k^2\sin^2\phi}}$; par M. *William Roberts.*—Sur les sommes des puissances semblables des termes d'une progression arithmétique; par M. *Puiseux.*

TOME XII. (1837.) No. I. Sur l'enseignement de la Géométrie supérieure.—Discours d'introduction au Cours de Géométrie supérieure fondé à la Faculté des Sciences de l'Académie de Paris.—Séance d'ouverture, le 22 décembre 1846; par M. *Chasles.*—Généralisation d'une propriété de la lemniscate; par M. *William Roberts.*—No. II. Généralisation d'une propriété de la lemniscate; par M. *William Roberts.* (Fin.)—Développements sur l'équation à l'aide de laquelle on détermine les inégalités séculaires du mouvement des planètes; par M. *C.-W. Borchardt.*—Sur les équations algébriques à plusieurs inconnues; par *J. Liouville.*—Principes d'un nouveau système de moteurs atmosphériques à forces vives, avec ou sans oscillations, avec ou sans soupape; par M. *Anatole de Caligny.*—Extrait d'une Lettre adressée à M. Liouville, par M. *J.-H. Jellett.*—Sur la loi de réciprocité dans la théorie des résidus quadratiques; par *J. Liouville.*

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SCIENTIFIC JOURNALS.

Journal de Mathématiques pures et appliquées, ou Recueil Mensuel de Mémoires sur les diverses parties des Mathématiques. Publié par Joseph Liouville, Membre de l'Académie des Sciences et du Bureau des Longitudes.

Tome XII. (1847.) No. III.—De la vie Descartes, et de sa méthode pour bien conduire sa raison et chercher la vérité dans les sciences; par M. *C.-G.-J. Jacobi.* (Traduit de l'allemand.)—Note sur la détermination des axes principaux d'un corps; par M. *R. Lobatto.*—Note sur le problème des tautochrones; par M. *J. Bertrand.*—Note sur quelques points de la théorie analytique des surfaces; par M. *B. Amiot.*—Extrait d'une Lettre de M. Kummer à M. Liouville.—No. IV. Mémoire sur la résolution, en nombres complexes, de l'équation $A^5 + B^5 + C^5 = 0$; par M. *G. Lamé.*—Mémoire sur la résolution, en nombres complexes, de l'équation $A^n + B^n + C^n = 0$; par M. *G. Lamé.* No. V. Sur les nombres complexes qui sont formés avec les nombres entiers réels et les racines de l'unité; par M. *Kummer.*—Théorèmes généraux sur les systèmes de forces et leurs moments; par M. *Chasles.* No. VI. Théorèmes généraux sur les systèmes de forces et leurs moments; par M. *Chasles.* (Fin.)—Note sur une propriété mécanique du cercle; par M. *A. Rispal.*—Sur quelques formules du calcul integral; par M. *A. Cayley.*—Mémoire sur les surfaces orthogonales; par M. *J.-A. Serret.*—Note au sujet d'un Mémoire de M. Chasles; par *J. Liouville.*—Extraits de deux Lettres adressées à M. Liouville; par M. *W. Thomson.*

The Mathematician. Edited by William Rutherford, F.R.S.A., and Stephen Fenwick. Vol. II., No. 6. London, July 1847.

Properties of the Inscribed and Circumscribed Quadrilaterals to a System of Curves of the Second Order.—Properties of the Centres of Gravity of a Quadrilateral and the Triangles of which it is composed.—Note by the Editors relative to their Elementary Propositions in the Geometry of Coordinates.—Algebraic Demonstration of the Theorem

$$1.2\ldots n = n^n - n(n-1)^n + \frac{n(n-1)}{1.2}(n-2)^n + \ldots etc.$$

—On certain Definite Integrals expressible by means of Elliptic Functions.—Division of the Ellipsoid into Equal Volumes.—Modern Geometry, (*continued*).—On the Ellipse of Minimum Area which can be drawn to pass through Four given points.—On the Peculiar Equation $x^3 - bx = 1$.—Mathematical Notes.—Solutions of Mathematical Exercises.—Mathematical Exercises (*continued*).

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